SUBSPACES OF $c_0(\mathbb{N})$ AND LIPSCHITZ ISOMORPHISMS

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Abstract

We show that the class of subspaces of $c_0(\mathbb{N})$ is stable under Lipschitz isomorphisms. The main corollary is that any Banach space which is Lipschitz-isomorphic to $c_0(\mathbb{N})$ is linearly isomorphic to $c_0(\mathbb{N})$. The proof relies in part on an isomorphic characterization of subspaces of $c_0(\mathbb{N})$ as separable spaces having an equivalent norm with a quantitative form of the weak-star Kadec-Klee property. Quantitative results are shown when the Lipschitz constants of the isomorphisms are small. The quite different non separable theory is also investigated.

1. Introduction

Banach spaces are usually considered within the category of topological vector spaces, and isomorphisms between them are assumed to be continuous and linear. It is however natural to study them from different points of view, e.g. as infinite dimensional smooth manifolds, metric spaces or uniform spaces, and to investigate whether this actually leads to different isomorphism classes. We refer to [J-L-S] and references therein for recent results and description of this field. Some simply stated questions turn out to be hard to answer: for instance, no examples are known of separable Banach spaces X and Y which are Lipschitz isomorphic but not linearly isomorphic. It is not even known if this could occur when X is isomorphic to l_1. The main result of this work is that any separable space which is Lipschitz isomorphic to $c_0(\mathbb{N})$ is linearly isomorphic to $c_0(\mathbb{N})$. Showing it will require the use of various tools from non linear functional analysis, such as the Gorelik principle. New linear results on subspaces of $c_0(\mathbb{N})$ will also be needed.

We now turn to a detailed description of our results. Section 2 contains the main theorems of our article (Theorems 2.7 and 2.8), which contribute to the classification of separable Banach spaces under Lipschitz isomorphisms. These results are non linear. However, their proof requires linear tools such as Theorem 2.2 which provides a characterization of linear subspaces of $c_0(\mathbb{N})$ in terms of existence of equivalent norms with a property of uniform smoothness. This technical property of uniform smoothness, called Lipschitz weak-star Kadec-Klee property (in short, LKK^*), is defined through the dual norm, which is such that the weak* and norm topologies agree quantitatively on the sphere (see Definition 2.1). The main topological argument we need is Gorelik's principle (Proposition 2.5) which is combined with a renorming technique and with Theorem 2.2 for showing (Theorem 2.7) that the class of subspaces of

$c_0(\mathbb{N})$ is stable under Lipschitz-isomorphisms. It follows (Theorem 2.8) that a Banach space is isomorphic to $c_0(\mathbb{N})$ as soon as it is Lipschitz-isomorphic to it. The renorming technique is somewhat similar to "maximal rate of change" arguments which are used for differentiating Lipschitz functions (see [P]).

We subsequently investigate extensions of the separable isomorphic results of section 2 in two directions: what can be said when the Lipschitz constants of the Lipschitz isomorphisms are small? What happens in the non separable case? These questions are answered in sections 3 and 4. For reaching the answers, we have to use specific tools, since the proofs are not straightforward extensions of those from section 2.

Section 3 deals with quantitative versions of Theorem 2.8. These statements are "nearly isometric" analogues, in the case of $c_0(\mathbb{N})$, of Mazur's theorem which states that two isometric Banach spaces are linearly isometric. Indeed we show that a Banach space X is close to $c_0(\mathbb{N})$ in Banach-Mazur distance if there is a Lipschitz-isomorphism U between X and $c_0(\mathbb{N})$ such that the Lipschitz constants of U and U^{-1} are close to 1 (Propositions 3.2 and 3.4). Proposition 3.2 relies on an examination of the proof of Gorelik's principle in the case of $c_0(\mathbb{N})$ and on an unpublished result of M. Zippin ([Z3]), while Proposition 3.4 uses the concept of K_0-space from [K-R].

The non separable theory is studied in section 4. Quite surprisingly, this theory looks very different. Let us first recall that there are spaces which are Lipschitz isomorphic to $c_0(\Gamma)$ with Γ uncountable but not linearly isomorphic to a subspace of that space (see [D-G-Z 2] and Example 4.13). Moreover, the quantitative behaviour of the equivalent LKK^* norms on X, which does not really matter in the separable case, turns out to be crucially important in the non separable situation, where it is closely related with the w.c.g. character of the space X (Lemma 4.2). However we characterize, both isomorphically (Theorem 4.4) and almost isometrically (Theorem 4.5) subspaces of $c_0(\Gamma)$, as well as the space $c_0(\Gamma)$ itself (Proposition 4.8). We also obtain satisfactory classification results for $C(K)$-spaces, when some finite derivative of the compact space K is empty. More precisely, we show that K is an Eberlein compact and K^{ω_0} is empty if and only if $C(K)$ is linearly isomorphic to some space $c_0(\Gamma)$ (Theorem 4.12), while the same equivalence holds with "Lipschitz isomorphic" if we drop the requirement that K is Eberlein (Theorem 4.11). Projectional resolutions of identity play a leading role in this last section.

A similar theory can be developed for uniform homeomorphisms. This is the subject of the forthcoming paper [G-K-L2]. Some results of [G-K-L2] and of the present paper have been announced in [G-K-L1].

Notation: We denote by B_X, respectively S_X, the open unit ball, respectively the unit sphere of a Banach space X. If V is a uniformly continuous map from a Banach space X to a Banach space Y, we denote, for $t > 0$,

$\omega(V,t) = \sup\{\|Vx_1 - Vx_2\|,\ \|x_1 - x_2\| \le t\}$ its modulus of uniform continuity. Two Banach spaces X and Y are Lipchitz isomorphic if there is a bijective map U from X onto Y such that U and U^{-1} are both Lipschitz maps when X and Y are equipped with the metric given by their norm.

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2. SEPARABLE ISOMORPHIC RESULTS

First, we need to introduce some terminology.

Definition 2.1. Let X be a separable Banach space. The norm of X is said to be *Lipschitz weak-star Kadec-Klee* (in short, LKK^*) if there exists c in $(0,1]$ such that its dual norm satisfies the following property: for any x^* in X^* and any weak* null sequence $(x_n^*)_{n\ge 1}$ in X^* ($x_n^* \xrightarrow{w^*} 0$),

$$\limsup \|x^* + x_n^*\| \ge \|x^*\| + c \limsup \|x_n^*\|.$$

If the above property is satisfied with a given c in $(0,1]$, we will say that the norm of X is *c-LKK**. If it is satisfied with the optimal value $c = 1$, we will say that the norm of X is *metric-KK**.

In our first theorem we establish that having an equivalent LKK* norm is an isomorphic characterization of the subspaces of $c_0(\mathbb{N})$. The precise quantitative version of this result is the following.

Theorem 2.2. *Let c in $(0,1]$ and X be a separable Banach space whose norm is c-LKK*. Then, for any $\varepsilon > 0$, there is a subspace E of $c_0(\mathbb{N})$ such that $d_{BM}(X,E) \le \dfrac{1}{c^2} + \varepsilon$; where $d_{BM}(X,E)$ denotes the Banach-Mazur distance between X and E.*

Proof. The following lemma gives a dual formulation of the notion of LKK* norm.

Lemma 2.3. *Let c in $(0,1]$ and X be a separable Banach space with a c-LKK* norm, then*

$$\max\Big(\|x\|, \frac{1}{2-c}\limsup\|x_n\|\Big) \le \limsup\|x+x_n\| \le \max\Big(\|x\|, \frac{1}{c}\limsup\|x_n\|\Big),$$

whenever (x_n) is a weakly null sequence in X ($x_n \xrightarrow{w} 0$). Let us call $m_\infty(c)$ this property.

Proof. Let x in X and $(x_n) \subset X$ with $x_n \xrightarrow{w} 0$. Without loss of generality, we may assume that $\lim \|x_n\|$ and $\lim \|x+x_n\|$ exist with $\lim \|x_n\| > 0$.
We will first prove the right hand side inequality. For $n \geq 1$, pick y_n^* in X^* so that $\|y_n^*\| = 1$ and $y_n^*(x+x_n) = \|x+x_n\|$. Passing to a subsequence, we may assume that $y_n^* \xrightarrow{w^*} y^*$ and $\lim \|y_n^* - y^*\|$ exists. Then, it follows from our assumption that

$$c \lim \|y_n^* - y^*\| \leq 1 - \|y^*\|.$$

Notice now that $\|x+x_n\| = y_n^*(x+x_n) = y_n^*(x) + y^*(x_n) + (y_n^* - y^*)(x_n)$. So we have

$$\lim \|x+x_n\| \leq \|y^*\| \, \|x\| + \frac{(1-\|y^*\|) \lim \|x_n\|}{c} \leq \mathrm{Max}\, (\|x\|, \frac{1}{c} \lim \|x_n\|).$$

For the left hand side inequality, we only need to show that $\lim \|x+x_n\| \geq \dfrac{1}{2-c} \lim \|x_n\|$. So we select now x_n^* in X^* with $\|x_n^*\| = 1$ and $x_n^*(x_n) = 1$ and we assume that $x_n^* \xrightarrow{w^*} x^*$ and $\lim \|x_n^* - x^*\|$ exists. Again , we have

$$c \lim \|x_n^* - x^*\| \leq 1 - \|x^*\|.$$

Since $(x_n^* - x^*)(x_n) \to \lim \|x_n\|$, we also obtain $\lim \|x_n^* - x^*\| \geq 1$ and therefore $\|x^*\| \leq 1 - c$. We can write $x_n^*(x+x_n) = \|x_n\| + (x_n^* - x^*)(x) + x^*(x)$. So, passing to the limit we obtain $\lim \|x+x_n\| + (1-c)\|x\| \geq \lim \|x_n\|$. Then we conclude by using the fact that $\|x\| \leq \lim \|x+x_n\|$. □

Remark. The best constant $\dfrac{1}{2-c}$ is not crucial for the proof of Theorem 2.2 that will be achieved with the trivial value $\dfrac{1}{2}$. However it will be used in the proof of Proposition 3.2 and it helps us to relate this with Theorem 3.2 in [K-W] which states, in the particular case $p = \infty$, that a space satisfying the property $m_\infty = m_\infty(1)$ embeds almost isometrically into $c_0(\mathbb{N})$.

Our next Lemma is the analogue of Lemma 3.1 in [K-W].

Lemma 2.4. *(i) If F is a finite dimensional subspace of X and $\eta > 0$, then there is a finite dimensional subspace U of X^* such that*

$$\forall (x,y) \in F \times U_\perp \;\; (1-\eta)\, \mathrm{Max}(\|x\|, \frac{1}{2}\|y\|) \leq \|x+y\| \leq (1+\eta)\, \mathrm{Max}(\|x\|, \frac{1}{c}\|y\|).$$

(ii) If G is a finite dimensional subspace of X^* and $\eta > 0$, then there is a finite dimensional subspace V of X such that

$$\forall (x^*, y^*) \in G \times V^\perp \;\; (1-\eta)(\|x^*\| + c\|y^*\|) \leq \|x^* + y^*\| \leq \|x^*\| + \|y^*\|.$$

Proof. Since the norm of X is LKK*, X^* is separable. Then the proof is identical with the proof of Lemmma 3.1 in [K-W]. □

We will now proceed with the proof of Theorem 2.2, which is only a slight modification of the proof of Theorem 3.2 in [K-W]. So let $0 < \delta < \dfrac{1}{3}$ and pick a positive integer t such that $t > \dfrac{6(1+\delta)}{c^3\delta}$. Let also $(\eta_n)_{n\geq 1}$ be a sequence of positive real numbers satisfying

$$0 < \eta_n < \frac{\delta}{2}, \quad \prod_{n\geq 1}(1-\eta_n) > 1-\delta \text{ and } \prod_{n\geq 1}(1+\eta_n) < 1+\delta.$$

Finally, let $(u_n)_{n\geq 1}$ be a dense sequence in X. Following the ideas of Kalton and Werner, we then construct subspaces $(F_n)_{n\geq 1}$, $(F'_n)_{n\geq 1}$ of X^* and $(E(m,n))_{1\leq m\leq n}$ of X so that:

(a) $\dim F_n < \infty$, $\dim E(m,n) < \infty$ for all $m \leq n$.

(b) $F'_n \subseteq [u_1,..,u_n]^\perp \cap \bigcap_{j\leq k<n} E(j,k)^\perp$ is weak*-closed and $X^* = F_1 \oplus .. \oplus F_n \oplus F'_n$.

(c) $F'_n = F_{n+1} \oplus F'_{n+1}$.

(d) If $x^* \in F_1 + .. + F_n$ and $y^* \in F'_{n+1}$, then

$$(1-\eta_n)(\|x^*\| + c\|y^*\|) \leq \|x^* + y^*\| \leq \|x^*\| + \|y^*\|.$$

(e) If $x \in (F_1 + .. + F_n)_\perp$ and $y \in \sum_{j\leq k<n} E(j,k)$, then

$$(1-\eta_n)\operatorname{Max}(\|x\|, \frac{1}{2}\|y\|) \leq \|x+y\| \leq (1+\eta_n)\operatorname{Max}(\|x\|, \frac{1}{c}\|y\|).$$

(f) $(F_1 + .. + F_{m-1} + F'_n)_\perp \subseteq E(m,n)$ and $E(m,n) \subseteq (F_1 + .. + F_{m-2})_\perp$ if $1 \leq m \leq n$.

(g) If $x^* \in F_m + .. + F_n$, then there exists $x \in E(m,n)$ so that $\|x\| \leq 1$ and $x^*(x) \geq c(1-\delta)\|x^*\|$.

Now, as in [K-W], we define, for $0 \leq s \leq t-1$

$$T_s : Y_s = c_0(E(4(n-1)t + 4s + 4, 4nt + 4s + 1)_{n\geq 0}) \to X$$

and

$$R_s : Z_s = c_0(E(4nt + 4s + 2, 4nt + 4s + 3)_{n\geq 0}) \to X$$

by $T_s((y_n)_{n\geq 0}) = \sum y_n$ and $R_s((z_n)_{n\geq 0}) = \sum z_n$. And also

$$T : Y = \ell_\infty((Y_s)_{s=0}^{t-1}) \to X \quad \text{and} \quad R : Z = \ell_\infty((Z_s)_{s=0}^{t-1}) \to X$$

by

$$T(\xi_0,..,\xi_{t-1}) = \frac{1}{t}\sum_{s=0}^{t-1} T_s\xi_s \quad \text{and} \quad R(\xi_0,..,\xi_{t-1}) = \sum_{s=0}^{t-1} R_s\xi_s.$$

Then we get

$$\forall \xi \in Y_s, \ \frac{1-\delta}{2}\|\xi\| \leq \|T_s\xi\| \leq \frac{1+\delta}{c}\|\xi\|,$$

$$\forall \xi \in Z_s,\ \frac{1-\delta}{2}\|\xi\| \le \|R_s\xi\| \le \frac{1+\delta}{c}\|\xi\|,$$

$$\|T\| \le \frac{1+\delta}{c} \ \text{ and } \ \|R\| \le \frac{1+\delta}{c}.$$

Still following [K-W] we can also show that if x^* in X^* satisfies $R_s^*x^* = 0$, then $\|T_s^*x^*\| \ge c(1-\delta)\|x^*\|$. Then a Hahn-Banach argument yields

$$\forall x^* \in X^* \ \|T_s^*x^*\| \ge c(1-\delta)\|x^*\| - 2(c + \frac{1+\delta}{c(1-\delta)})\|R_s^*x^*\|.$$

Since $\delta < \frac{1}{3}$ and $c \le 1$, we have

$$\forall x^* \in X^* \ \|T_s^*x^*\| \ge c(1-\delta)\|x^*\| - \frac{6}{c}\|R_s^*x^*\|.$$

Therefore

$$\forall x^* \in X^* \ \|T^*x^*\| \ge c(1-\delta)\|x^*\| - \frac{6}{ct}\|R^*x^*\| \ge \left[c(1-\delta) - \frac{6(1+\delta)}{c^2t}\right]\|x^*\|.$$

Thus, for our initial choice of t we obtain

$$\forall x^* \in X^* \ \|T^*x^*\| \ge c(1-2\delta)\|x^*\|.$$

Since we have on the other hand that $\|T\| \le \frac{1}{c}(1+\delta)$, we get that

$$d(X, Y/\mathrm{ker}T) < \frac{1+\delta}{c^2(1-2\delta)}.$$

As a c_0-sum of finite dimensional spaces, Y embeds almost isometrically into $c_0(\mathbb{N})$. Then, by Alspach's theorem [Al], so does $Y/\mathrm{ker}T$. This concludes our proof. □

We now turn to non linear theory. First we state a slight modification of the Gorelik Principle as it is presented in [J-L-S].

Proposition 2.5. (Gorelik's Principle) - *Let E and X be two Banach spaces and U be a homeomorphism from E onto X with uniformly continuous inverse. Let b and d two positive constants and let E_0 be a subspace of finite codimension of E. If $d > \omega(U^{-1}, b)$, then there exists a compact subset K of X such that*

$$bB_X \subset K + U(2dB_{E_0}).$$

Proof. We recall a fundamental lemma due to E. Gorelik [G] and that can also be found in [J-L-S].

Lemma 2.6. *For every $\varepsilon > 0$ and $d > 0$, there exists a compact subset A of dB_E such that, whenever Φ is a continuous map from A to E satisfying $\|\Phi(a) - a\| < (1-\varepsilon)d$ for any a in A, then $\Phi(A) \cap E_0 \ne \emptyset$.*

Now, fix $\varepsilon > 0$ such that $d(1-\varepsilon) > \omega(U^{-1}, b)$. Let $K = -U(A)$, where A is the compact set obtained in Lemma 2.6. Consider now x in bB_X and the map Φ from A to E defined by $\Phi(a) = U^{-1}(x + Ua)$. It is clear that for any a in A, $\|\Phi(a) - a\| < (1-\varepsilon)d$. Then, it follows from Lemma 2.6 that there exists $a \in A$ so that $U^{-1}(x + Ua) \in 2dB_{E_0}$. This concludes our proof. $\square$

We can now state and prove a crucial result.

Theorem 2.7. *The class of all Banach spaces that are linearly isomorphic to a subspace of* $c_0(\mathbb{N})$ *is stable under Lipschitz isomorphisms.*

Proof. Let U be a Lipschitz isomorphism from a subspace E of c_0 onto the Banach space X. Theorem 2.2 indicates that we need to build an equivalent LKK* norm on X. This norm will be defined as follows. For x^* in X^*, set:

$$|||x^*||| = \sup\{\frac{|x^*(Ue - Ue')|}{||e-e'||};\ (e,e') \in E \times E,\ e \neq e'\}.$$

Since U and U^{-1} are Lipschitz maps, $|||\ |||$ is an equivalent norm on X^*. It is clearly weak* lower semicontinuous and therefore is the dual norm of an equivalent norm on X that we will also denote $|||\ |||$.

Consider $\varepsilon > 0$, $x^* \in X^*$ and $(x_k^*)_{k\geq 1} \subset X^*$ such that $x_k^* \xrightarrow{w^*} 0$ and $||x_k^*|| \geq \varepsilon > 0$ for all $k \geq 1$. Fix $\delta > 0$ and then e and e' in E so that

$$\frac{x^*(Ue - Ue')}{||e-e'||} > (1-\delta)|||x^*|||.$$

By using translations in order to modify U, we may as well assume that $e = -e'$ and $Ue = -Ue'$. Since E is a subspace of c_0, it admits a finite codimensional subspace E_0 such that

$$\forall f \in ||e||B_{E_0},\quad ||e+f|| \vee ||e-f|| \leq (1+\delta)||e||. \tag{2.1}$$

Let C be the Lipschitz constant of U^{-1}. By Proposition 2.5, for every $b < \dfrac{||e||}{2C}$ there is a compact subset K of X such that $bB_X \subset K + U(||e||B_{E_0})$. Since (x_k^*) converges uniformly to 0 on any compact subset of X, we can construct a sequence $(f_k) \subset ||e||B_{E_0}$ such that:

$$\liminf x_k^*(-Uf_k) \geq \frac{\varepsilon||e||}{2C}.$$

We deduce from (2.1) that $x^*(Uf_k + Ue) \leq (1+\delta)||e||\,|||x^*|||$ and therefore $x^*(Uf_k) \leq 2\delta||e||\,|||x^*|||$. Using again the fact that $x_k^* \xrightarrow{w^*} 0$, we get that:

$$\liminf(x^* + x_k^*)(Ue - Uf_k) \geq (1-3\delta)||e||\,|||x^*||| + \frac{\varepsilon||e||}{2C}.$$

Since δ is arbitrary, by using the definition of $|||\ |||$ and (2.1), we obtain $\liminf |||x^* + x_k^*||| \geq |||x^*||| + \dfrac{\varepsilon}{4C}$. This proves that $|||\ |||$ is LKK^*. ☐

When dealing with $c_0(\mathbb{N})$ itself, we obtain a more precise theorem, which is the main result of this article.

Theorem 2.8. *A Banach space is linearly isomorphic to $c_0(\mathbb{N})$ if and only if it is Lipschitz isomorphic to $c_0(\mathbb{N})$.*

Proof. We only need to prove the "if" part. So let X be a Banach space which is Lipschitz isomorphic to $c_0(\mathbb{N})$. Theorem 2.7 asserts that X is linearly isomorphic to a subspace of $c_0(\mathbb{N})$. Besides, it is known that the class of all $\mathcal{L}^\infty$ spaces is stable under uniform homeomorphisms ([H-M]) and that a $\mathcal{L}^\infty$ subspace of $c_0(\mathbb{N})$ is isomorphic to $c_0(\mathbb{N})$ ([J-Z]). This establishes our result. ☐

Note that although Theorems 2.7 and 2.8 are non linear results, it is critically important that the Banach space X is Lipschitz isomorphic to a *linear subspace* of $c_0(\mathbb{N})$. In fact, given any separable Banach space Y, there is a bi-Lipschitz map between Y and a *subset* of $c_0(\mathbb{N})$ ([Ah]).

3. QUANTITATIVE RESULTS

Recall that for $\lambda \geq 1$, a Banach space X is said to be $\mathcal{L}^\infty_\lambda$ if for every finite dimensional subspace E of X, there is a finite dimensional subspace F of X, containing E and such that $d_{BM}(F, \ell_\infty^{\dim F}) \leq \lambda$. If X is $\mathcal{L}^\infty_\lambda$ for some $\lambda \in [1, +\infty)$, then it is said to be $\mathcal{L}^\infty$. We already used the fact ([J-Z]) that a subspace of $c_0(\mathbb{N})$ is isomorphic to $c_0(\mathbb{N})$ if and only if it is $\mathcal{L}^\infty$. Combining this with Theorem 2.2, we get that if a separable $\mathcal{L}^\infty$ space admits a LKK* norm, then it is isomorphic to $c_0(\mathbb{N})$. The following statement gives a quantitative estimate on the linear isomorphism.

Proposition 3.1. *There exists a function $F : [1, +\infty) \times (0, 1] \to [1, +\infty)$ such that if X is a separable $\mathcal{L}^\infty_\lambda$ space with a c-LKK* norm, then*

$$d_{BM}(X, c_0(\mathbb{N})) \leq F(\lambda, c).$$

Moreover $F(1, 1) = 1$ and F is continuous at $(1, 1)$.

Proof. Let us first mention that for values of λ and c close to 1, the result follows directly from a work of M. Zippin [Z3], who proved that if X is a $\mathcal{L}^\infty_\mu$ subspace of c_0 with $\mu < 7/6$, then

$$d_{BM}(X, c_0(\mathbb{N})) \leq \frac{\mu^2}{\mu^2 - 2\mu^3 + 2}.$$

Then, it is easily checked that in our setting, if we assume moreover that $\lambda/c^2 < 7/6$, we get

$$d_{BM}(X, c_0(\mathbb{N})) \leq \frac{\lambda^2}{2c^6 + \lambda^2 c^2 - 2\lambda^3}.$$

For the general case we do not have an explicit function F. We will just reproduce an argument by contradiction used in ([G-L], p.261). Indeed, if there is no such function, then there exist (λ, c) in $[1, +\infty) \times (0, 1]$ and a sequence (X_n) of separable $\mathcal{L}^\infty_\lambda$ spaces with a c-LKK* norm such that, for all $n \geq 1$, $d_{BM}(X_n, c_0(\mathbb{N})) \geq n$. But the space $Y = \left(\sum \oplus X_n\right)_{c_0}$ is $\mathcal{L}^\infty$ with a LKK* norm and thus by Theorem 2.2 and [J-Z] it is isomorphic to $c_0(\mathbb{N})$. So, the X_n's being uniformly complemented in Y, their Banach-Mazur distance to $c_0(\mathbb{N})$ should be bounded, a contradiction. $\square$

We will now give two quantitative versions of Theorem 2.8.

Proposition 3.2. *There exists a function $F : (1, +\infty) \to (1, +\infty)$ such that $\lim\limits_{\lambda \to 1^+} F(\lambda) = 1$ and such that if $U : X \to c_0(\mathbb{N})$ is a bi-Lipschitz map with $\mathrm{Lip}(U) \cdot \mathrm{Lip}(\mathrm{U}^{-1}) = \lambda$, then $d_{BM}(X, c_0(\mathbb{N})) \leq F(\lambda)$.*

Proof. Let $E_n = \{x = (x(i))_{i \geq 0} \in c_0(\mathbb{N});\ x(i) = 0 \text{ if } i > n\}$. We set $A = B_{E_n}$. It is easily seen that if $\Phi : A \to c_0(\mathbb{N})$ is a continuous map such that $\|a - \Phi(a)\| \leq 1$ for all $a \in A$, then there exists $a_0 \in A$ such that $\Phi(a_0)(i) = 0$ for all $i \leq n$. Indeed, if $\pi : c_0(\mathbb{N}) \to E_n$ is the natural projection and $F(a) = a - \pi(\Phi(a))$, then $F(A) \subseteq A$ and by Brouwer's theorem, there is $a_0 \in A$ with $F(a_0) = a_0$. Hence $|\Phi(a_0)(j)| \leq 1$ for all $j > n$, and thus $\Phi(a_0) \in B_{F_n}$, where

$$F_n = \{x \in c_0;\ x(j) = 0 \text{ if } j \leq n\}.$$

If we now reproduce the proof of Gorelik's Principle (Proposition 2.5), using the compact set A and the space F_n defined above (with an appropriate choice of n), we find in the notation of the proof of Theorem 2.7 that for any $b < \dfrac{\|e\|}{\mathrm{Lip}(U^{-1})}$, there is a compact subset K of X such that

$$bB_X \subset K + U(\|e\| B_{F_n})$$

and it follows that the norm $|||\ .\ |||$ is λ^{-1}-LKK*. Now Theorem 2.2 shows that the distance from $(X, |||\ .\ |||)$ to the subspaces of $c_0(\mathbb{N})$ is at most λ^2. Since the distance between the original norm $\|\ .\ \|$ of X and $|||\ .\ |||$ is less than λ, it follows that the Banach-Mazur distance from $(X, \|\ .\ \|)$ to the subspaces of $c_0(\mathbb{N})$ is at most λ^3.

We now observe the following

Fact 3.3. *There is a function $F_0 : (1, +\infty) \to (1, +\infty)$ with $\lim\limits_{\lambda \to 1^+} F_0(\lambda) = 1$, and such that if X satisfies the assumptions of the proposition, then X is an $\mathcal{L}^\infty_{F_0(\lambda)}$ space.*

Proof. By the ultrapower version of the local reflexivity principle, X^{**} is isometric to a 1-complemented subspace of some ultrapower $(X)_{\mathcal{U}}$. We set $Z = (c_0)_{\mathcal{U}}$. Clearly, there is a bi-Lipschitz map $\tilde{U} : (X)_{\mathcal{U}} \to Z$ with $\mathrm{Lip}(\tilde{U}) \cdot \mathrm{Lip}(\tilde{U}^{-1}) = \lambda$. It follows that there are maps $f : X^{**} \to Z$ and $g : Z \to X^{**}$ with $\mathrm{Lip}(f) \cdot \mathrm{Lip}(g) = \lambda$ and $g \circ f = Id_{X^{**}}$. By ([H-M], Lemma 2.11.), there is $\tilde{g} : Z^{**} \to X^{**}$ extending g and such that $\mathrm{Lip}(\tilde{g})$=$\mathrm{Lip}(g)$. The space Z^{**} is isometric to the dual of an L^1-space, hence it is a $\mathcal{P}_1$ space (see [L-T], p.162). Since $\tilde{g} \circ f = Id_{X^{**}}$, it follows that if M is a metric space, N a subspace of M and $\psi : N \to X^{**}$ a Lipschitz map, there exists a Lipschitz extension $\overline{\psi} : M \to X^{**}$ with $\mathrm{Lip}(\overline{\psi}) \le \lambda \, \mathrm{Lip}(\psi)$. In particular, X^{**} is isometric to a linear subspace Y of $l_\infty(\Gamma)$ on which there exists a Lipschitz projection P with $\mathrm{Lip}(P) \le \lambda$. Since X^{**} is 1-complemented in its own bidual, it follows from ([Li], Corollary 2 to Theorem 3) that there exists a linear projection $\pi : l_\infty(\Gamma) \to Y$ with $\|\pi\| \le \lambda$. Therefore, X^{**} is a $\mathcal{P}_\lambda$ space.

By ([L-R], see p. 338), X^{**} is therefore a $\mathcal{L}^\infty_{10\lambda}$ space, and so is X. Moreover ([Z1],[Z2] and [B] Th. 13), when F is a finite dimensional $\mathcal{P}_{1+\varepsilon}$ space with $\varepsilon < 17^{-8}$, then if we let $\nu = \varepsilon^{1/8}$ (see [B])

$$d_{BM}(F, l_\infty^{\dim(F)}) \le \frac{1 + 6\nu}{(1 - 6\nu)(1 - 17\nu)}.$$

In the above notation, any finite dimensional subspace of Y is contained, up to $\delta > 0$ arbitrary, in a space $\pi(G)$, where G is isometric to a finite dimensional l_∞. Such an F is $\mathcal{P}_\lambda$; therefore ([B], Theorem 13) guarantees the requirement $\lim\limits_{\lambda \to 1^+} F(\lambda) = 1$. □

We now proceed with the proof of proposition 3.2. We know that $(X, \| \,.\, \|)$ is a $\mathcal{L}^\infty_{F_0(\lambda)}$ space whose Banach-Mazur distance to the subspaces of $c_0(\mathbb{N})$ is at most λ^3. Any $\mathcal{L}^\infty_\mu$ subspace H of $c_0(\mathbb{N})$ is isomorphic to $c_0(\mathbb{N})$ ([J-Z]) and by using contradiction (see [G-L], p.261) we show the existence of a function $F_1(\mu)$ such that $d_{BM}(H, c_0) \le F_1(\mu)$. Finally according to ([Z3]), there is such a function F_1 which satisfies $\lim\limits_{\mu \to 1^+} F_1(\mu) = 1$ (see proof of Corollary 3.1 above). For any $\varepsilon > 0$, X is $(\lambda^3 + \varepsilon)$ isomorphic to a subspace G_ε of $c_0(\mathbb{N})$ which is a $\mathcal{L}^\infty_\mu$ space with $\mu = (\lambda^3 + \varepsilon) F_0(\lambda)$; the existence of F as claimed in the proposition clearly follows. □

Using the techniques from [K-O] and the notion of K_0-space ([K-R]), we can actually extend Proposition 3.2. to arbitrary equivalent renormings of $c_0(\mathbb{N})$.

Proposition 3.4. *Let Y be a Banach space which is linearly isomorphic to $c_0(\mathbb{N})$. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that if X is a Banach space and $U : X \to Y$ is a bi-Lipschitz onto map with $Lip(U) \cdot Lip(U^{-1}) < 1 + \delta$, then $d_{BM}(X, Y) < 1 + \varepsilon$.*

Proof. The proof relies heavily on [K-O], from which we take the following notation: if $d_M(E, F)$ denotes the Hausdorff distance between two subsets E and F of a metric space M, the Kadets distance $d_K(X, Y)$ between two Banach spaces X and Y is

$$d_K(X, Y) = \inf\{d_Z(U(B_X), V(B_Y))\},$$

where the infimum is taken over all linear isometric embeddings U, V of X, Y into an arbitrary common Banach space Z.

The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of the Hausdorff distances $d_M(B_X, B_Y)$ over all isometric embeddings of X and Y into an arbitrary common metric space M. By ([K-O], Th. 2.1), we have

$$d_{GH}(X, Y) \leq \sup\{\frac{1}{2}\|\Phi(x) - \Phi(x')\|_Y - \|x - x'\|_X;\ x, x' \in B_X\}$$

where $\Phi : B_X \to B_Y$ is a bijective map.

It follows easily that for any $\eta > 0$, there is $\delta > 0$ such that if there is $U : X \to Y$ a Lipschitz isomorphism with $\mathrm{Lip}(U) \cdot \mathrm{Lip}(U^{-1}) < 1 + \delta$, then $d_{GH}(X, Y) < \eta$. Obviously, one has $d_{GH}(X, Y) \leq d_K(X, Y)$ (and in general these two distances are not equivalent: for instance ([K-O]), $\lim\limits_{p\to 1^+} d_{GH}(\ell_p, \ell_1) = 0$ while $d_K(\ell_p, \ell_1) = 1$ for all $p > 1$). We recall that a Banach space E is a K_0-space ([K-R]) if there exists $K_0 > 0$ such that whenever $f : E \to \mathbb{R}$ is a homogeneous function which is bounded on B_E and satisfies

$$|f(x + x') - f(x) - f(x')| \leq \|x\| + \|x'\|$$

then there exists $x^* \in E^*$ such that

$$|f(x) - x^*(x)| \leq K_0\|x\|,\ \forall x \in E.$$

It is shown in [K-R] that $c_0(\mathbb{N})$ is a K_0-space. By ([K-O], Theorem 3.7.), if E is a K_0-space and (E_n) is such that $\lim d_{GH}(E_n, E) = 0$, then $\lim d_K(E_n, E) = 0$.

Since Y is a K_0-space as isomorphic to $c_0(\mathbb{N})$, for any $\alpha > 0$ there is $\eta > 0$ such that $d_{GH}(X, Y) < \eta$ implies $d_K(X, Y) < \alpha$. Let Z be a Banach space which contains isometric copies of X and Y with $d_Z(B_X, B_Y) < \alpha$. We may and do assume that Z is separable. By Sobczyk's theorem, Y is linearly complemented in any separable super-space Z, and the norm of the projection π_Z is bounded independently of Z. It easily follows that given $\varepsilon > 0$, there is $\alpha > 0$ such that if $d_Z(B_X, B_Y) < \alpha$ then $d_{BM}(X, Y) < 1 + \varepsilon$. Indeed, the restriction to X of π_Z provides the required linear isomorphism. This concludes the proof. $\square$

4. NON SEPARABLE THEORY

We now consider non separable spaces. It turns out that the non separable theory looks quite different; in fact, there are spaces (even $C(K)$-spaces) which have a LKK* norm and are Lipschitz-isomorphic to $c_0(\Gamma)$ but are not linearly isomorphic to subspaces of $c_0(\Gamma)$ since they fail to be w.c.g. (see Theorems 4.11. and 4.12. and Example 4.13. below). However metric-KK* norms will provide positive results.

As will be clear in the sequel, the techniques that we develop are separably determined. So we adopt the following definition:

Definition 4.1. Let X be a Banach space. The norm $\| \ \|$ of X is *c-LKK** if its restriction to any separable subspace of X is c-LKK*.
If $c = 1$, we say again that the norm is *metric-KK**.

We refer to ([D-G-Z], Chapter VI) or ([Di]) for definitions and basic properties of projectional resolutions of identity (P.R.I.'s). A P.R.I. (P_α) is said to be shrinking when (P_α^*) is a P.R.I on X^*. With this notation, one has

Lemma 4.2. *Let X be a Banach space. If $\| \ \|$ is a metric-KK* norm on X, then $(X, \| \ \|)$ has a shrinking P.R.I., and thus X is w.c.g.*

Proof. By ([F-G], Th. 3), it suffices to show that $(X, \| \ \|)$ is an M-ideal in its bidual. But ([H-W-W], Cor.III.1.10), asserts that to be an M-ideal in its bidual is a separably determined property. So let Y be a separable subspace of X and $\pi : Y^{***} \to Y^*$ be the canonical projection. Pick $t \in Y^{***}$ with $\|t\| = 1$, and write $t = y^* + s$ with $s \in \operatorname{Ker} \pi = Y^\perp$. We need to show that

$$\|t\| = \|y^*\| + \|s\|. \tag{4.1}$$

There is a net (y_α^*) in B_{Y^*} such that $t = \lim y_\alpha^*$ in (Y^{***}, w^*) and then $y^* = \lim y_\alpha^*$ in (Y^*, w^*). Since $\|t - y^*\| = \|s\|$, we have

$$\liminf \|y_\alpha^* - y^*\| \geq \|s\|$$

and since $\| \ \|$ is metric-KK*, this implies that $\|y^*\| \leq 1 - \|s\| = \|t\| - \|s\|$. This shows (4.1) since $\|t\| \leq \|y^*\| + \|s\|$ by the triangle inequality. □

Lemma 4.3. *Let X be a Banach space with a c-LKK* norm. For every $x \in X$, there exists a separable subspace E of X^* such that if $y \in E_\perp \subset X$, one has*

$$\max(\|x\|, \frac{\|y\|}{2-c}) \leq \|x + y\| \leq \max(\|x\|, \frac{\|y\|}{c}).$$

Proof. It clearly suffices to show that for any $\varepsilon > 0$, there is $F \subset X^*$ separable such that if $y \in F_\perp$

$$(1 - \varepsilon) \max(\|x\|, \frac{\|y\|}{2-c}) \leq \|x + y\| \leq (1 + \varepsilon) \max(\|x\|, \frac{\|y\|}{c}).$$

Assume, for instance, that for any separable F, there is $y \in F_\perp$ such that

$$\|x+y\| > (1+\varepsilon)\max(\|x\|, \frac{\|y\|}{c}).$$

We construct inductively an increasing sequence (F_n) of separable subspaces of X^*, and (y_n) in X such that for all $n \geq 1$,
(i) if $u \in \operatorname{span}\{x, y_1, .., y_n\}$, then $\|u\| = \sup\{|f(u)|;\ \|f\| \leq 1,\ f \in F_n\}$.
(ii) $y_{n+1} \in (F_n)_\perp$.
(iii) $\|x + y_{n+1}\| > (1+\varepsilon)\max(\|x\|, \frac{\|y_{n+1}\|}{c})$.
We let $G = \overline{\operatorname{span}}\{x, (y_j)_{j\geq 1}\}$. Since the weak* and norm topologies coincide on S_{G^*}, it follows from (i) that $D = \cup_{n\geq 1}(F_n)_{|G}$ is dense in $(G^*, \|\ \|)$. Then (ii) implies that $y_n \xrightarrow{w} 0$. But now (iii) contradicts Lemma 2.3. This proves the lemma, since we can clearly proceed along the same lines with the left hand side of the inequality. $\square$

We now state and prove a first analogous result to Theorem 2.2 for non separable spaces. To avoid dealing with singular cardinals, we limit ourselves to the case where the density character of X, denoted by $\operatorname{dens}(X)$, is equal to ω_1. It is plausible that this restriction is irrelevant.

Theorem 4.4. *Let X be a Banach space such that* $\operatorname{dens}(X) = \omega_1$. *Then X is w.c.g. and X has an equivalent LKK* norm if and only if X is isomorphic to a subspace of $c_0(\Gamma)$, where $|\Gamma| = \omega_1$.*

Proof. The natural norm of $c_0(\Gamma)$ is metric-KK* and every subspace of $c_0(\Gamma)$ is w.c.g. ([Jo-Z], see also [D-G-Z], Chapter VI).

Conversely, if X is w.c.g. and has a c-LKK* norm, then X is a w.c.g. Asplund space and thus ([F], see also [D-G-Z], Th VI.4.3) X has a shrinking P.R.I. $(P_\alpha)_{\alpha\leq\omega_1}$. Using Lemma 4.3, we construct by induction on α, ordinals $\lambda_\alpha < \omega_1$ such that $\lambda_\alpha < \lambda_{\alpha+1}$ and such that if $P_{\lambda_\alpha}(x) = x$ and $P_{\lambda_\alpha}(y) = 0$, then

$$\max(\|x\|, \frac{\|y\|}{2-c}) \leq \|x+y\| \leq \max(\|x\|, \frac{\|y\|}{c}).$$

If we let $X_\alpha = (P_{\lambda_{\alpha+1}} - P_{\lambda_\alpha})(X)$, then X is isomorphic to $(\sum \oplus X_\alpha)_{c_0}$. By Theorem 2.2, the spaces X_α are (uniformly in α) isomorphic to subspaces of $c_0(\mathbb{N})$; this concludes the proof. $\square$

It follows from Lemma 4.2 and Theorem 4.4 that any space X with $\operatorname{dens}(X) = \omega_1$ which has a metric-KK* norm is isomorphic to a subspace of $c_0(\Gamma)$ with $|\Gamma| = \omega_1$. However, a much better result is available, namely:

Theorem 4.5. *Let X be a Banach space. The following assertions are equivalent:*
(i) X has a metric-KK* norm.

(ii) For any $\varepsilon > 0$, there is a subspace X_ε of $c_0(\Gamma)$, with $|\Gamma| = \mathrm{dens}(X)$, such that $d_{BM}(X, X_\varepsilon) < 1 + \varepsilon$.

Proof. (ii) $\Rightarrow$ (i) easily follows from the fact that the natural norm of $c_0(\Gamma)$ is metric-KK*.

(i) $\Rightarrow$ (ii) relies on

Fact 4.6. *If X has a metric-KK* norm, there exists a P.R.I. (P_α) on X such that for any $\alpha < \mathrm{dens}(X)$, if $(x, y) \in X^2$ are such that $P_\alpha(x) = x$ and $P_\alpha(y) = 0$, then $\|x + y\| = \max(\|x\|, \|y\|)$.*

Indeed by Lemma 4.2 we know that X is w.c.g. Then Lemma 4.3 shows that for all $x \in X$, there is $E_x \subset X^*$ a separable subspace such that $\|x+y\| = \max(\|x\|, \|y\|)$ for every $y \in (E_x)_\perp$. We now use the technique of ([D-G-Z], Lemma VI.2.3): using the same notation, we prove along the same lines that if $A \subset X$ and $B \subset X^*$ are subsets with density $\leq \aleph$, there exist norm closed subspaces $[A] \subset X$ and $[B] \subset X^*$ such that

(i) $A \subset [A]$, $B \subset [B]$.

(ii) $\mathrm{dens}([A]) \leq \aleph$, $\mathrm{dens}([B]) \leq \aleph$.

(iii) For all $x \in [A]$, $\|x\| = \sup\{f(x);\ f \in [B],\ \|f\| \leq 1\}$.

(iv) For all $x \in [A]$, $E_x \subset [B]$.

(v) For all $f \in [B]$, for all $s \in S$, $\sup\limits_{L_s}|f| = \sup\limits_{L_s \cap [A]} |f|$.

Note that ([D-G-Z], Lemma VI.2.4) shows that $X = [A] \oplus [B]_\perp$, while the choice of E_x and (iv) shows that $\|x + y\| = \sup(\|x\|, \|y\|)$ for all $x \in [A]$ and $y \in [B]_\perp$. Fact 4.6 now follows by a simple transfinite induction argument, as in ([D-G-Z], Theorem VI.2.5).

Finally, Theorem 4.5 follows immediately by transfinite induction from Fact 4.6, since Theorem 2.2 proves it in the separable case and allows us to start the induction. $\square$

Remarks 4.7. 1) It is clear that any quotient space of $c_0(\Gamma)$ has a metric-KK* norm, namely the quotient norm. Therefore Theorem 4.5 shows that Alspach's theorem [Al] extends to arbitrary $c_0(\Gamma)$ spaces.

2) Since Lemma 4.3 only uses separable subspaces of X, the proofs of theorems 4.4 and 4.5 provide: let X be a Banach space. If for every separable subspace Y of X, $d_{BM}(Y, \{\text{subspaces of } c_0(\mathbb{N})\}) = 1$, then

$$d_{BM}(X, \{\text{subspaces of } \mathrm{c}_0(\Gamma)\}) = 1.$$

Now consider a w.c.g. space X with $\mathrm{dens}(X) = \omega_1$ and such that every separable subspace of X is isomorphic to a subspace of $c_0(\mathbb{N})$. An argument by contradiction shows the existence of an upper bound $M > 0$ for the Banach-Mazur distance of any separable subspace of X to the subspaces of $c_0(\mathbb{N})$.

Then we get that X is isomorphic to a subspace of $c_0(\Gamma)$. Example 4.13 below shows that we cannot dispense with the assumption "X w.c.g." in this case.

3) An easy saturation argument shows that if X with a metric-KK* norm is $\mathcal{L}^\infty_\lambda$, the P.R.I. from Fact 4.6 can be constructed in such a way that $(P_{\lambda_{\alpha+1}} - P_{\lambda_\alpha})(X)$ is $\mathcal{L}^\infty_\lambda$ for all α. Starting from Corollary 3.1, a transfinite induction then shows that a $\mathcal{L}^\infty$ space which has a metric-KK* norm is isomorphic to $c_0(\Gamma)$ for some set Γ.

This last remark extends to the LKK* case as follows:

Proposition 4.8. *Let X be a Banach space such that* $\operatorname{dens}(X) = \omega_1$. *The following assertions are equivalent:*
(i) X is linearly isomorphic to $c_0(\Gamma)$, with $|\Gamma| = \omega_1$.
(ii) X is a $\mathcal{L}^\infty$ space with an equivalent metric-KK* norm.
(iii) X is a weakly compactly generated $\mathcal{L}^\infty$ space with an equivalent LKK* norm.

Proof. The equivalence between (i) and (ii) is a special case of Remark 4.7.3) and (i) implies (iii) is obvious.
(iii) implies (i): We use the notation from the proof of Theorem 4.4. Through an easy separable exhaustion argument we can ensure that the spaces X_α are (uniformly in α) $\mathcal{L}^\infty$ spaces. By restriction, they have (uniformly in α) LKK* norms. Hence by Proposition 3.1 they are uniformly isomorphic to $c_0(\mathbb{N})$. This clearly implies (i). $\square$

Our next statement provides an extension of ([G-L], Th. IV.1.; see also [H-W-W] p. 134) to non separable spaces.

Proposition 4.9. *Let X be a $\mathcal{L}^\infty$ space with* $\operatorname{dens}(X) = \omega_1$ *which is isomorphic to an M-ideal in its bidual equipped with its bidual norm. Then X is isomorphic to $c_0(\Gamma)$ where $|\Gamma| = \omega_1$.*

Proof. Since X is an M-ideal in X^{**}, it is w.c.g. and it admits a shrinking P.R.I. $(P_\alpha)_{\alpha \leq \omega_1}$ by ([F-G], Th. 3). Let $\lambda \in \mathbb{R}$ be such that X is $\mathcal{L}^\infty_\lambda$. For any sequence (x_n^*) in X^* with $\|x_n^*\| = 1$ and $w^* - \lim x_n^* = 0$, there exists $\alpha < \omega_1$ such that:
(a) $P_\alpha^*(x_n^*) = x_n^*$ for every $n \geq 1$.
(b) $P_\alpha(X)$ is a $\mathcal{L}^\infty_\lambda$ space.
Since $P_\alpha(X)$ is a separable $\mathcal{L}^\infty_\lambda$ space which is M-ideal in its bidual, we have by ([G-L], Remark 1, p. 261) that $d_{BM}(P_\alpha(X), c_0(\mathbb{N})) \leq M$, where $M = M(\lambda)$ depends only upon λ. It follows that there exists a cluster point to the sequence (x_n^*) in (X^{***}, w^*), say G, such that $d(G, X^*) \geq A > 0$, where A depends only on M (that is, on λ).

If now $(x_n^*) \subset B_{X^*}$ and $w^* - \lim x_n^* = x^*$, with $\|x_n^* - x^*\| \geq \varepsilon$, there is, by the above, G in $B_{X^{***}}$ with $d(G, X^*) \geq A\varepsilon$ and $G = w^* - \lim\limits_{\mathcal{U}} (x_n^* - x^*)$ in (X^{***}, w^*).

Since $G + x^* = w^* - \lim\limits_{\mathcal{U}} x_n^*$, one has $1 \geq \|G + x^*\| = \|G\| + \|x^*\|$ and it follows that $\|x^*\| \leq 1 - A\varepsilon$. Recapitulating, we have shown that any separable subspace of X is A-LKK*. Finally, Proposition 4.8 yields the conclusion. □

Remarks 4.10. 1) An alternative approach to show Proposition 4.9. consists into proving (with the same notation) that the sequence $(x_n^* - x^*)$ has a cluster point G in (X^{***}, w^*) with $d(G, X^*) \geq A\varepsilon$ for some constant $A > 0$, by extracting first a subsequence which is $(\varepsilon/2)$-separated, then a further subsequence which is $(K\varepsilon)$-equivalent to the unit vector basis of ℓ_1 for some constant $K > 0$. Indeed, by [L-S], X^* is isomorphic to $\ell_1(\Gamma)$ and thus it has the strong Schur property. Now we can pick a w^*-cluster point G to that subsequence in (X^{***}, w^*) to reach our conclusion. The interest of this alternative route lies in the fact that in the separable case, it provides a proof of ([G-L], Th. IV.1) which relies on Proposition 3.1 instead of using Zippin's converse to Sobczyck's theorem ([Z4]).

2) It is not difficult to show (using an argument from [A]) that if X has an equivalent LKK* norm, then there is an equivalent norm on X^{**} such that X is an M-ideal in X^{**}. But this norm is in general not the bidual norm of its restriction to X: indeed it follows from [La] that for any K scattered compact set with $K^{(\omega_0)} = \emptyset$, $C(K)$ has an equivalent LKK* norm; but such spaces are not in general w.c.g. (see Example 4.13 below).

3) Our statements proved under the assumption $\mathrm{dens}(X) = \omega_1$ (Theorem 4.4, Proposition 4.8, Proposition 4.9) can be extended with similar proofs to the case $\mathrm{dens}(X) < \aleph_{\omega_0}$. We do not know if this restriction is necessary.

Theorem 2.2 shows in particular that a separable Banach space has an equivalent LKK* norm if and only if it has an equivalent metric-KK* norm, hence the distinction between the two notions is purely isometric for separable spaces. Our last two statements show that this is not so in the non separable case.

Theorem 4.11. *Let K be a compact space. The following assertions are equivalent:*
(i) The Cantor derived set of order ω_0 of K is empty.
(ii) $C(K)$ is Lipschitz isomorphic to $c_0(\Gamma)$, where Γ is the density character of $C(K)$.
(iii) $C(K)$ admits an equivalent LKK* norm.

Proof. (i) $\Rightarrow$ (ii) was proved in [D-G-Z 2] and the argument for the converse can be found in [J-L-S] (Theorem 6.3). The equivalence between (i) and (iii) follows easily from the proof of [La], Theorem 3.8). □

Our next statement provides the topological condition which allows "linearizing" the Lipschitz isomorphism from Theorem 4.11.

Theorem 4.12. *Let K be a compact space. The following assertions are equivalent:*
(i) K is an Eberlein compact and its Cantor derived set of order ω_0 is empty.
(ii) $C(K)$ is linearly isomorphic to $c_0(\Gamma)$, where Γ is the density character of $C(K)$.
(iii) $C(K)$ admits an equivalent metric-KK* norm.

Proof. (i) implies (ii): Since K is Eberlein, $C(K)$ is w.c.g. (see [D-G-Z], Chapter VI). By compactness, $K^{(\omega_0)} = \emptyset$ implies that there is n in $\mathbb{N}$ such that $K^{(n)} = \emptyset$. We proceed by induction on n. If $n = 1$, K is finite and the implication is obvious. Assume it holds when $L^{(n)} = \emptyset$ and pick K such that $K^{(n+1)} = \emptyset$. We let $L = K'$ and $X = \{f \in C(K) : \ f_{|L} = 0\}$. The space X is clearly isometric to $c_0(K \setminus L)$; while $C(K)/X$ is isometric to $C(L)$, and thus isomorphic to a $c_0(\Gamma)$ space by our assumption. We observe now that X is complemented in $C(K)$, since any $c_0(I)$ space is 2-complemented in any w.c.g. space. For checking this, let us call Y a subspace isometric to $c_0(I)$ of a w.c.g. space X. Using the notation of ([D-G-Z], VI.2), we can choose the map $\varphi : X^* \to X^{\mathbb{N}}$ from ([D-G-Z], Lemma VI.2.3) in such a way that for any $s \in S$ and any $f \in X^*$:
(i) $\sup_{L_s}|f| = \sup\{|f(x)|;\ x \in \varphi(f) \cap L_s\}$.
(ii) $\sup_{Y\cap L_s}|f| = \sup\{|f(x)|;\ x \in \varphi(f) \cap L_s \cap Y\}$.
(iii) $\overline{\operatorname{span}\varphi(f) \cap Y} = \{x \in c_0(I);\ \operatorname{supp}(x) \subseteq I_f\}$, where I_f is a countable subset of I.

Then ([D-G-Z], Lemma VI.2.4 and Th VI.2.5) provide a P.R.I. (P_α) on X such that for all $\alpha \leq \operatorname{dens}(X)$:
1) $P_\alpha(Y) \subseteq Y$.
2) There exists $I_\alpha \subseteq I$ such that $P_\alpha(x) = \mathbb{1}_{I_\alpha} x$ for all $x \in Y$.
By Sobczyk's theorem, $c_0(\mathbb{N})$ is 2-complemented in any separable super-space. Then we proceed by induction on $\operatorname{dens}(X)$: if it is true for all w.c.g. Z with $\operatorname{dens}(Z) < \operatorname{dens}(X)$, we consider (P_α) which satisfies 1) and 2) above. Since $(P_{\alpha+1} - P_\alpha)(c_0(I)) = c_0(I_{\alpha+1} \setminus I_\alpha)$, there is a projection

$$\Pi_\alpha : P_{\alpha+1}(X) \to (P_{\alpha+1} - P_\alpha)(c_0(I))$$

such that $\|\Pi_\alpha\| \leq 2$. Let $\Pi'_\alpha = \Pi_\alpha P_{\alpha+1}$ and $\Pi = \sum \Pi'_\alpha$. It is easily checked that Π is the required projection from X onto Y with $\|\Pi\| \leq 2$.

To conclude the proof of (i) $\Rightarrow$ (ii), we simply observe that since X is complemented in $C(K)$, we have that

$$C(K) \simeq X \oplus C(L) \simeq c_0(K \setminus L) \oplus c_0(\Gamma).$$

(ii) implies (iii) is clear since the natural norm on $c_0(\Gamma)$ is metric-KK*.

(iii) implies (i): By Lemma 4.2, any Banach space which has a metric-KK* norm has a shrinking P.R.I. and thus is w.c.g. The condition $K^{(\omega_0)} = \emptyset$ follows immediately from Theorem 4.11. □

Example 4.13. There exist ([C-P]; see [D-G-Z], VI.8) compact spaces such that $K^{(3)} = \emptyset$ (hence $C(K)$ is Lipschitz isomorphic to $c_0(\Gamma)$) but there is no continuous one-to-one map from $(B_{C(K)}, w)$ to $(B_{c_0(\Gamma)}, w)$ and thus no linear continuous injective map from such a $C(K)$ to any $c_0(\Gamma)$. Therefore Theorems 2.7 and 2.8 do not extend to the non separable case.

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