COUNTEREXAMPLES CONCERNING SECTORIAL OPERATORS

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ABSTRACT

In this paper we give two counterexamples to the closedness of the sum of two sectorial operators with commuting resolvents. In the first example the operators are defined on an L^p-space, with $1 < p \neq 2 < \infty$, and one of them admits bounded imaginary powers. The second example is concerned with operators defined on a Hilbert valued L^p-space; one acts on L^p and admits bounded imaginary powers as the other acts on the Hilbert space. In the last section of the paper we show that the two partial derivations on $L^2(\mathbb{R}^2; X)$ admit a so-called bounded joint functional calculus if and only if X is a UMD Banach space with property (α) (geometric property introduced by G. Pisier).

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1. INTRODUCTION-NOTATIONS

Let X be a complex Banach space and let A and B be two sectorial operators on X, with commuting resolvents and whose respective types ω_A and ω_B satisfy $\omega_A + \omega_B < \pi$ (see definitions below). In [8], Da Prato and Grisvard showed that $A + B$, with domain the intersection of the domains of A and B, is then closable and they derived important consequences for the study of abstract differential equations. It turns out that, even if X is a Hilbert space, this sum is not necessarily closed (see [2] for a counterexample). However, with additional assumptions on A and B, some positive results have been obtained. The most famous is due to Dore and Venni ([9]), who proved that if X is a UMD Banach space (see [3] and [5] for the definition and important characterizations) and if the imaginary

(*) This work was completed while the author was visiting the University of Missouri in Columbia.

powers of A and B satisfy: $\|A^{is}\| \leq Ce^{\mu_A|s|}$, $\|B^{is}\| \leq Ce^{\mu_B|s|}$, for any s in $\mathbb{R}$, with $\mu_A + \mu_B < \pi$; then $A + B$ is closed.

The purpose of Section 2 is to show the optimality of two other positive results. The first one was also obtained by Dore and Venni [9], who showed the closedness of $A+B$ when X is a Hilbert space and A admits bounded imaginary powers. We will show that this is false if X is an L^p-space, with $1 < p \neq 2 < +\infty$.
For the second, consider H a Hilbert space, (Ω, μ) a measure space and $1 < p < \infty$. Let A be a closed operator on $L^p(\Omega)$ whith a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some sector $\Sigma_\theta = \{z \in \mathbb{C}^* \ : \ |\arg z| < \theta\}$ and B a sectorial operator of type ω_B, with $\omega_B + \theta < \pi$. In a joint work with F. Lancien and C. Le Merdy [12], we proved that $\mathcal{A}+\mathcal{B}$ is a closed operator on $L^p(\Omega; H)$, where $\mathcal{A}$ and $\mathcal{B}$ are the respective closures of $A \otimes I_H$ and $I_{L^p} \otimes B$. Here, we show that this is false, if we only assume that A admits bounded imaginary powers.

In Section 3 we study the notion of joint functional calculus. Let us first recall that it follows from Dore and Venni's theorem that if X is a UMD Banach space and if A and B admit respectively a bounded $H^\infty(\Sigma_{\theta_A})$ and a bounded $H^\infty(\Sigma_{\theta_B})$ functional calculus on X, with $\theta_A + \theta_B < \pi$, then $A + B$ is closed. In [12] it is shown that if X enjoys some "local unconditionality", namely property (α) introduced by Pisier [16], then $A+B$ admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus for any $\mu > \max\{\theta_A, \theta_B\}$. This type of stability result, already obtained for bounded imaginary powers in UMD Banach spaces ([9],[19]), is crucial if one wants to iterate the application of Dore and Venni's theorem. In fact a stronger property is proved in [12]: if X has property (α), if A and B admit respectively a bounded $H^\infty(\Sigma_{\theta_A})$ and a bounded $H^\infty(\Sigma_{\theta_B})$ functional calculus, then (A, B) admits a bounded joint functional calculus for any (μ, ν) in $(\theta_A, \pi) \times (\theta_B, \pi)$. Although property (α) is not comparable with the UMD property, it must be kept in mind that the UMD property is crucial for the boundedness of the functional calculi associated with many differential operators. For instance, it is known that, if $\mu > \frac{\pi}{2}$ and $1 < p < +\infty$, then the first derivation operator on $L^p(\mathbb{R}; X)$, with domain $W^{1,p}(\mathbb{R}; X)$, admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus if and only if X is a UMD Banach space (see [18]). It is then clear that the above theorem is essentially applicable in UMD spaces with property (α). In Section 3, we show the optimality of this theorem in quite a strong way. More precisely, we consider U (resp. V) the derivation with respect with the first (resp. second) variable on $L^2(\mathbb{R}^2; X)$ and we prove that (U, V) admits a bounded joint functional calculus if and only if X is UMD with property (α). Along the proof, we also show that this is equivalent to the validity of some vector valued multiplier theorems due to Zimmermann [22].

Let us now fix some notations and terminology. In this paper, all Banach spaces will be complex ones. We denote by $\mathcal{L}(X)$ the algebra of all bounded operators on the Banach space X. For any θ in $(0, \pi)$ (resp. (θ, θ') in $(0, \pi) \times (0, \pi)$), $H^\infty(\Sigma_\theta)$ (resp. $H^\infty(\Sigma_\theta \times \Sigma_{\theta'})$) will denote the Banach algebra of all bounded analytic functions on Σ_θ (resp. $\Sigma_\theta \times \Sigma_{\theta'}$), equipped with the supremum norm. For a linear operator A on a Banach space X, $D(A)$, $R(A)$, $\sigma(A)$ and $\rho(A)$ will denote respectively its domain, range, spectrum and resolvent set. The sum of two linear operators will always be meant with domain $D(A + B) = D(A) \cap D(B)$ and the product with domain $D(AB) = \{x \in D(B),\ Bx \in D(A)\}$. Let

$0 < \omega < \pi$, A is said to be sectorial of type ω if A is closed, injective, with dense range and domain and if it satisfies the following spectral conditions:

$$\sigma(A) \subset \overline{\Sigma_\omega} \qquad \text{and} \qquad \forall\, \theta \in (\omega, \pi), \quad \sup_{\lambda \notin \overline{\Sigma_\theta}} \left\| \lambda(\lambda - A)^{-1} \right\| < \infty. \tag{1}$$

If for any ω in $(0, \pi)$, A is sectorial of type ω, we will say that A is sectorial of type 0.
Let A be a sectorial operator of type ω and let θ in (ω, π). We refer the reader to [19] for the definition of the bounded imaginary powers of A and to [15] for the construction of the $H^\infty(\Sigma_\theta)$ functional calculus associated with A. We will just recall the construction of McIntosh's joint functional calculus.
Consider A and B two sectorial operators with commuting resolvents, of respective types ω and ω'. For (μ, μ') in $(\omega, \pi) \times (\omega', \pi)$, let

$$H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'}) = \{f \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'}) \ ; \ \exists\, s > 0 \ / \ \Phi^{-s} f \in H^\infty(\Sigma_\mu \times \Sigma_{\mu'})\},$$

where $\Phi(z, z') = \dfrac{zz'}{(1+z)^2(1+z')^2}$. Then, for any F in $H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$, we set

$$F(A, B) = -\frac{1}{4\pi^2} \int_{\Gamma_\theta \times \Gamma_{\theta'}} F(\lambda, \lambda')(\lambda - A)^{-1}(\lambda' - B)^{-1} d\lambda \, d\lambda' \tag{2}$$

where $(\theta, \theta') \in (\omega, \mu) \times (\omega', \mu')$ and Γ_θ is the oriented contour defined by:

$$\Gamma_\theta(t) : \begin{cases} -te^{i\theta} & \text{if } -\infty < t \leq 0 \\ te^{-i\theta} & \text{if } 0 \leq t < +\infty \end{cases}$$

We have in particular that $\Phi(A, B) = A(I+A)^{-2} B(I+B)^{-2}$ which is a bounded one to one operator with a dense range and we denote by $\Phi(A, B)^{-1}$ its inverse defined on $R(\Phi(A, B))$. Then, for F in $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$, $F(A, B) := \Phi(A, B)^{-1}(F\Phi)(A, B)$ is closed and densely defined. If this extends to a bounded algebra homomorphism from $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ into $\mathcal{L}(X)$, we say that (A, B) admits a bounded $H^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ joint functional calculus (see [1] and [12] for details). Notice that (2) makes sense for F in $H_0^\infty(\Sigma_\mu \times \Sigma_{\mu'})$ even if A and B are not injective or with dense range.
We also introduce $(r_i)_{i \geq 1}$, the sequence of Rademacher functions on $I = [0, 1]$ and we say that a Banach space X has property (α) if there exists a constant $C > 0$ such that for every integer n and every choice of $(\alpha_{i,j})$ in $\mathbb{C}^{n^2}$ and $(x_{i,j})$ in X^{n^2}.

$$\Big\| \sum_{1 \leq i,j \leq n} \alpha_{i,j} (r_i \otimes r_j) x_{i,j} \Big\|_{L^2(I \times I; X)} \leq C \sup_{1 \leq i,j \leq n} |\alpha_{i,j}| \Big\| \sum_{1 \leq i,j \leq n} (r_i \otimes r_j) x_{i,j} \Big\|_{L^2(I \times I; X)}. \tag{α}$$

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2. COUNTEREXAMPLES

Before to proceed with the the construction of our counterexamples, we need to state two well known lemmas, that can essentially be found in [2] or [21] for instance.

Lemma 2.1. *Let X be a Banach space with a Schauder basis $(x_n)_{n\geq 0}$ and let $(x_n^*)_{n\geq 0}$ be the sequence of corresponding coordinate functionals. For a fixed sequence $(a_n)_{n\geq 0}$ of complex numbers, we set*

$$D(A) = \{x \in X : \sum_{n\geq 0} a_n x_n^*(x) x_n \text{ converges in } X\}$$

and for x in $D(A)$, we define $Ax = \sum_{n\geq 0} a_n x_n^*(x) x_n$. Then:
(i) A is closed and densely defined.
(ii) If for any $n \geq 0$ $a_n \neq 0$, then A is injective and has a dense range.
(iii) If $(a_n)_{n\geq 0}$ is a non decreasing sequence of positive real numbers, then A is invertible and sectorial of type 0.

Let $\mathbb{T}$ be the unit circle equipped with its Haar measure. For n in $\mathbb{Z}$ and t in $(-\pi, \pi)$ (identified with $\mathbb{T}$), let $e_n(t) = e^{int}$. The next lemma is a slight modification of Lemma 2.1, describing the sectorial Fourier multipliers on $L^p(\mathbb{T})$, for $p \in (1, +\infty)$.

Lemma 2.2. *Let $1 < p < \infty$ and let $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$. We define*

$$D(A) = \{f \in L^p(\mathbb{T}) : \sum_{-N\leq n\leq N} a_n \hat{f}(n) e_n \text{ converges in } L^p(\mathbb{T}) \text{ as } N \text{ tends to} + \infty\}$$

and for f in $D(A)$, $Af = \| \ \|_{L^p} - lim \sum_{-N}^{N} a_n \hat{f}(n) e_n$. Then:
(i) A is closed and densely defined.
(ii) If for any $n \in \mathbb{Z}$ $a_n \neq 0$, then A is injective and has a dense range.
(iii) If $(a_n)_{n\in\mathbb{Z}}$ is non decreasing from $\mathbb{Z}$ into $(0, +\infty)$, then A is sectorial of type 0.

Proof. We will only show that A satisfies the sectoriality estimates of (1), in the setting of Lemma 2.2. For the rest of the proof we refer to [21].
Let $\lambda = |\lambda| e^{i\theta}$ with $0 < |\theta| \leq \pi$ and $|\lambda| > 0$. For any n in $\mathbb{Z}$:

$$\frac{1}{|\lambda - a_n|} \leq \frac{1}{\text{dist}(\lambda, \mathbb{R}_+)} = \frac{C_\theta}{|\lambda|}, \tag{3}$$

where C_θ is a constant depending only on θ.
Moreover,

$$\sum_{n\in\mathbb{Z}} \Big| \frac{1}{\lambda - a_n} - \frac{1}{\lambda - a_{n+1}} \Big| = \frac{1}{|\lambda|} \sum_{n\in\mathbb{Z}} \Big| \varphi(\frac{a_{n+1}}{|\lambda|}) - \varphi(\frac{a_n}{|\lambda|}) \Big|,$$

where $\varphi(x) = \dfrac{1}{1 - xe^{-i\theta}}$, for $x \geq 0$. Since $(a_n)_{n\in\mathbb{Z}}$ is non decreasing, we have

$$\sum_{n\in\mathbb{Z}} \Big| \frac{1}{\lambda - a_n} - \frac{1}{\lambda - a_{n+1}} \Big| \leq \frac{1}{|\lambda|} \int_0^\infty \frac{d\,x}{|1 - xe^{-i\theta}|^2} = \frac{C'_\theta}{|\lambda|}, \tag{4}$$

where C'_θ depends only on θ. Then, in view of Marcinkiewicz's multiplier theorem, it follows clearly from (3) and (4) that A is sectorial of type 0, with

$$\forall \lambda \in \mathbb{C} \setminus [0, +\infty),\ \forall f \in L^p(\mathbb{T}) :\ (\lambda - A)^{-1} f = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda - a_n} \hat{f}(n) e_n. \qquad \diamond$$

Until the end of this section, A will denote the multiplier on $L^p(\mathbb{T})$, for $1 < p < \infty$, associated as in Lemma 2.2 with the sequence $(a_n)_{n\in\mathbb{Z}} = (2^n)_{n\in\mathbb{Z}}$. From Lemma 2.2, it follows that A is sectorial of type 0. Moreover, it is clear that for any s in $\mathbb{R}$, A^{is} is the Fourier multiplier associated with the sequence $(2^{ins})_{n\in\mathbb{Z}}$, which is a translation operator. In particular, for any s in $\mathbb{R}$, $\|A^{is}\| = 1$.
On the other hand, it is worth noticing already that if $1 < p \neq 2 < \infty$, then A fails to admit a bounded $H^\infty(\Sigma_\pi)$ functional calculus. Indeed, $(2^n)_{n\in\mathbb{Z}}$ is a so-called Carleson interpolating sequence. More precisely, there exists a positive constant M such that for any bounded sequence of complex numbers $(\alpha_n)_{n\in\mathbb{Z}}$, there is f in $H^\infty(\Sigma_\pi)$ satisfying $\|f\|_{H^\infty(\Sigma_\pi)} \leq M \sup|\alpha_n|$ and $f(2^n) = \alpha_n$ for every n in $\mathbb{Z}$ (see [11] for instance). Then, the conclusion follows from the fact that for $1 < p \neq 2 < +\infty$, there are bounded scalar sequences whose associated Fourier multipliers are unbounded on $L^p(\mathbb{T})$.
We can now state the main result of this section.

Theorem 2.3. *Let A be defined as above.*

1) *If $1 < p \neq 2 < +\infty$, then there exists an operator B on $L^p(\mathbb{T})$, sectorial of type 0, resolvent commuting with A and such that $A + B$ is not closed.*

2) *If $p > 2$, then there exists an operator C on $H = L^2(\mathbb{T})$, sectorial of type 0 and such that $\mathcal{A} + \mathcal{C}$ is not closed on $L^p(\mathbb{T}; H)$, where $\mathcal{A}$ and $\mathcal{C}$ denote respectively the closures of $A \otimes I_H$ and $I_{L^p(\mathbb{T})} \otimes C$.*

Proof of 1). It is well known that, for $1 < p \neq 2 < +\infty$, $(e_n)_{n\geq 1}$ is a conditional basic sequence in $L^p(\mathbb{T})$. So there exists a choice of signs ε in $\{-1, 1\}^{\mathbb{N}}$ such that the operator S_ε defined on span$\{e_n; n \geq 1\}$, the vector space spanned by $(e_n)_{n\geq 1}$, as follows:

$$S_\varepsilon \Big(\sum_{n=1}^N x_n e_n \Big) = \sum_{n=1}^N \varepsilon(n) x_n e_n$$

is unbounded for the norm of $L^p(\mathbb{T})$.
For $n \leq 0$, we set $b_n = 1$; for $n \geq 1$ we set $b_n = 2^n$ if $\varepsilon(n) = 1$ and $b_n = 2^{n-1}$ if $\varepsilon(n) = -1$. $(b_n)_{n\in\mathbb{Z}}$ is non decreasing from $\mathbb{Z}$ into $(0, +\infty)$. Therefore, by Lemma 2.2,

the corresponding Fourier multiplier B is sectorial of type 0. B admits clearly a bounded inverse. It is then well known (see [8,9] for instance) that $A+B$ is closed if and only if there is a constant $K>0$ such that:

$$\forall x \in D(A)\cap D(B) \quad \|Ax\| \leq K\|Ax+Bx\|. \tag{5}$$

From the definition of A and B (see Lemma 2.2), it follows clearly that $D(A)\cap D(B)$ contains $\text{span}\{e_n; n\geq 1\}$. Now, if (5) was fulfilled we would have that for any finite sequence $(x_n)_{n=1}^N$ in $\mathbb{C}$:

$$\Big\|\sum_{n=1}^{N} 2^n x_n e_n\Big\| \leq K\Big\|\sum_{n=1}^{N}(2^n+b_n)x_n e_n\Big\|.$$

And therefore, replacing x_n by $\dfrac{x_n}{2^n+b_n}$,

$$\Big\|\sum_{n=1}^{N} \frac{2^n}{2^n+b_n} x_n e_n\Big\| \leq K\Big\|\sum_{n=1}^{N} x_n e_n\Big\|.$$

But, for $n\geq 1$, $\dfrac{2^n}{2^n+b_n}$ is either $\dfrac{1}{2}$ or $\dfrac{2}{3}$. Precisely, the restriction to $\text{span}\{e_n; n\geq 1\}$ of the corresponding multiplier is $\dfrac{7}{12}Id - \dfrac{1}{12}S_\varepsilon$, which is unbounded. Therefore $A+B$ is not closed. $\diamond$

Remark. The following question seems to be open: does there exist such a counterexample, on an L^p-space or more generally on a UMD Banach space, with B sectorial but A admitting a bounded H^∞ functional calculus?

Proof of 2). We first need to introduce some notations. Let $0<\beta<\dfrac{1}{2}$. For n in $\mathbb{Z}$ and t in $(-\pi,\pi)$, we let $f_n(t)=|t|^{-\beta}e^{int}$. It is known that $(f_0, f_{-1}, f_1, ...)$ is a conditional basis of H (see [20, p. 428]). We need the following lemma:

Lemma 2.4. *For $n\geq 0$, let $u_n = e_n\otimes f_n \in L^p(\mathbb{T};H)$.*
If $\dfrac{1}{2}\Big(\dfrac{1}{2}+\dfrac{1}{p}\Big)<\beta<\dfrac{1}{2}$, then $(u_n)_{n\geq 0}$ is a conditional basic sequence in $L^p(\mathbb{T};H)$.

End of proof of 2). Pick ε in $\{-1,1\}^{\mathbb{N}}$ such that the operator T_ε defined on $\text{span}\{u_n; n\geq 0\}$ by:

$$T_\varepsilon\Big(\sum_{0\leq n\leq N} x_n u_n\Big) = \sum_{0\leq n\leq N}\varepsilon(n)x_n u_n$$

is unbounded for the norm of $L^p(\mathbb{T};H)$. Let now C be the multiplier on the basis $(f_0, f_{-1}, f_1, ...)$ associated with the sequence $(c_n)_{n\in\mathbb{Z}}$ defined by:

$$\begin{cases} \text{if } n\geq 0 \text{ and } \varepsilon(n)=1 \text{ then } c_n = 2^n \\ \text{if } n\geq 0 \text{ and } \varepsilon(n)=-1 \text{ then } c_n = 2^{n-1} \\ \text{if } n<0 \text{ then } c_n = c_{|n+1|} = c_{|n|-1}. \end{cases}$$

The sequence $(c_0, c_{-1}, c_1, ...)$ is non decreasing. So by Lemma 2.1, C is sectorial of type 0 on H. On the other hand, for any $n \geq 0$, $\dfrac{2^n}{2^n + c_n} = \dfrac{7}{12} - \dfrac{1}{12}\varepsilon(n)$. Then, using the same arguments as above, it follows from the unboundedness of T_ε and the inclusion $\mathrm{span}\{u_n; n \geq 0\} \subseteq D(\mathcal{A}) \cap D(\mathcal{C})$ that $\mathcal{A} + \mathcal{C}$ is not closed. $\diamond$

Proof of Lemma 2.4. Let us first assume that $(u_n)_{n\geq 0}$ is an unconditional basic sequence. So there exists $K_1 > 0$ so that, for any $N \geq 0$ and any $(a_n)_{n=0}^N \subset \mathbb{C}$:

$$K_1^{-1}\Big\|\sum_{n=0}^N a_n r_n \otimes u_n\Big\|_{L^1(I;L^p(\mathbb{T};H))} \leq \Big\|\sum_{n=0}^N a_n u_n\Big\|_{L^p(\mathbb{T};H)} \leq K_1\Big\|\sum_{n=0}^N a_n r_n \otimes u_n\Big\|_{L^1(I;L^p(\mathbb{T};H))}.$$

Then it follows from a generalization of the classical Khintchine inequality (due to Maurey [14] in the most general case of a Banach lattice with finite cotype, see also [13, p. 49-50]) that there exists $K_2 > 0$ such that, for any $N \geq 0$ and any $(a_n)_{n=0}^N \subset \mathbb{C}$:

$$K_2^{-1}\Big\|\big(\sum_{n=0}^N |a_n u_n|^2\big)^{1/2}\Big\|_{L^p(\mathbb{T};H)} \leq \Big\|\sum_{n=0}^N a_n u_n\Big\|_{L^p(\mathbb{T};H)} \leq K_2\Big\|\big(\sum_{n=0}^N |a_n u_n|^2\big)^{1/2}\Big\|_{L^p(\mathbb{T};H)}.$$

But

$$\Big\|\big(\sum_{n=0}^N |a_n u_n|^2\big)^{1/2}\Big\|_{L^p(\mathbb{T};H)} = \Big\|\big(\sum_{n=0}^N |a_n f_n|^2\big)^{1/2}\Big\|_{H}.$$

Therefore, since H is 2-concave and 2-convex (see [13] for definitions), there exists $K_3 > 0$ such that, for any $N \geq 0$ and any $(a_n)_{n=0}^N \subset \mathbb{C}$:

$$K_3^{-1}\big(\sum_{n=0}^N |a_n|^2\big)^{1/2} \leq \Big\|\sum_{n=0}^N a_n u_n\Big\|_{L^p(\mathbb{T};H)} \leq K_3\big(\sum_{n=0}^N |a_n|^2\big)^{1/2}.$$

So it is enough to show that $(u_n)_{n\geq 0}$ is not equivalent to the canonical basis of l_2. For that purpose we will construct a sequence of scalars (α_n) such that $\sum_{n\geq 0}|\alpha_n|^2 < +\infty$ and $\sum_{n\geq 0}\alpha_n u_n$ diverges in $L^p(\mathbb{T}; H)$.

Let $\dfrac{1}{2} + \dfrac{1}{p} - \beta < \alpha < \dfrac{1}{2}$ and let f in $L^2(\mathbb{T})$ defined by $f(u) = \dfrac{1}{|u|^\alpha}$ for $u \in (-\pi, \pi) \setminus \{0\}$. For $n \geq 0$, we set $\alpha_n = \hat{f}(n)$. Since f is in $L^2(\mathbb{T})$, we have $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 < \infty$. Moreover, f is even, so for any n in $\mathbb{Z}$, $\hat{f}(-n) = \hat{f}(n)$. In order to conclude our proof, it is therefore sufficient to show that the sequence $\big(\sum_{-N}^N \hat{f}(n)u_n\big)_{N\geq 0}$ does not converge in $L^p(\mathbb{T}; H)$.

Let us assume that this sequence is convergent and notice the following obvious identity:

$$\forall (s,t) \in (-\pi,\pi)^2,\ \sum_{-N}^N \hat{f}(n)e_n(s)f_n(t) = |t|^{-\beta}\sum_{-N}^N \hat{f}(n)e^{in(s+t)}.$$

Then, we have that ψ, defined by

$$\psi(s) = \Big(\int_{-\pi}^{\pi} \frac{|f(s+t)|^2}{|t|^{2\beta}} \Big)^{1/2} \text{ for } a.e.\ s \in (-\pi, \pi)$$

belongs to $L^p(-\pi,\pi)$. But it is easy to check that this not the case for our choice of α and β. $\diamond$

3. JOINT FUNCTIONAL CALCULUS FOR PARTIAL DERIVATIONS

Let X be a Banach space. For (s,t) in $\mathbb{R}^2$ we denote by $\tau_{(s,t)}$ the translation operator of vector (s,t) on $L^2(\mathbb{R}^2;X)$. Let U and V be respectively the generators of the C_0-groups $(\tau_{(s,0)})_{s\in\mathbb{R}}$ and $(\tau_{(0,t)})_{t\in\mathbb{R}}$. Then it is well known that U and V are sectorial of type $\frac{\pi}{2}$ and that, for $\frac{\pi}{2} < \mu < \pi$, the following are equivalent [9,18]:
1) U admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus.
2) V admits a bounded $H^\infty(\Sigma_\mu)$ functional calculus.
3) X is a UMD Banach space.

We now recall a vector valued multiplier theorem due to Zimmermann [22], that we will state only on $L^2(\mathbb{T}^2;X)$ and in a simpler form sufficient for our purpose. We shall need the notation: $I_1 = \{0\} \subset \mathbb{Z}$ and for $n \in \mathbb{N}$, $I_n = \{k \in \mathbb{Z} :\ 2^{n-2} \le |k| < 2^{n-1}\}$.

Theorem 3.1 (Zimmermann [22]). *Let X be a UMD Banach space with property (α). Then, every bounded sequence $a = (a_{(k,l)})_{(k,l)\in\mathbb{Z}^2}$ which is constant on each set $I_n \times I_p$, $(n,p) \in \mathbb{N}^2$, defines a bounded $L^2(\mathbb{T}^2;X)$-Fourier multiplier M_a. Moreover, there exists $C > 0$ so that, for all such sequences a, we have*

$$\|M_a\| \le C sup \{|a_{(k,l)}|,\ (k,l) \in \mathbb{Z}^2\}.$$

Remark. For $1 < p < +\infty$ the Schatten space S_p is UMD [4], but if moreover $p \neq 2$, it fails property (α) [16]. It is also shown in [22] that for $1 < p \neq 2 < +\infty$ Theorem 3.1 is false on $X = S_p$.

Our result is the following:

Theorem 3.2. *Let X be a Banach space and $\frac{\pi}{2} < \mu < \pi$. Consider U and V defined as above. The following assertions are equivalent:*
(i) (U,V) admits a bounded $H^\infty(\Sigma_\mu \times \Sigma_\mu)$ joint functional calculus.
(ii) X is a UMD space with property (α).
(iii) The conclusion of Theorem 3.1 is satisfied.

Proof.

(ii) is equivalent to (iii). Theorem 3.1 exactly states that *(ii)* implies *(iii)*, so let us assume that *(iii)* is true. Then it clearly follows that the Riesz projection is bounded on $L^2(\mathbb{T};X)$, which is one characterization of the UMD spaces. Thus we only have to show that X has property (α). Let n in $\mathbb{N}$, $(\alpha_{i,j})_{1\leq i,j\leq N} \in \mathbb{C}^{N^2}$ and $(x_{i,j})_{1\leq i,j\leq N} \in X^{N^2}$; recall that $\forall z \in \mathbb{T}\ \forall k \in \mathbb{Z}\ e_k(z) = z^k$. It follows from *(iii)* that:

$$\Big\| \sum_{1\leq i,j\leq n} \alpha_{i,j}(e_{2^i} \otimes e_{2^j})x_{i,j}\Big\|_{L^2(\mathbb{T}^2;X)} \leq C \sup_{1\leq i,j\leq n} |\alpha_{i,j}| \Big\| \sum_{1\leq i,j\leq n} (e_{2^i} \otimes e_{2^j})x_{i,j}\Big\|_{L^2(\mathbb{T}^2;X)}.$$

It remains to prove that the sequence $(r_i)_{i\geq 1}$ can be replaced by the sequence $(e_{2^i})_{i\geq 1}$ in the definition of property (α). This is well known in one variable and the results of G. Pisier on Sidon sets [17] imply the existence of a constant $K > 0$ such that for every Banach space Y and every finite sequence $(y_1, .., y_n)$ in X:

$$\frac{1}{K}\Big\| \sum_{1\leq i\leq n} e_{2^i} y_i\Big\|_{L^2(\mathbb{T};Y)} \leq \Big\| \sum_{1\leq i\leq n} r_i y_i\Big\|_{L^2(I;Y)} \leq K\Big\| \sum_{1\leq i\leq n} e_{2^i} y_i\Big\|_{L^2(\mathbb{T};Y)}.$$

Applying this to $Y = L^2(\mathbb{T};X)$ or $L^2(I;X)$, we obtain that for every Banach space X and every family $(x_{i,j})_{1\leq i,j\leq n}$ in X:

$$\begin{aligned}\frac{1}{K^2}\Big\| \sum_{1\leq i,j\leq n} (e_{2^i} \otimes e_{2^j})x_{i,j}\Big\|_{L^2(\mathbb{T}^2;X)} &\leq \Big\| \sum_{1\leq i,j\leq n} (r_i \otimes r_j)x_{i,j}\Big\|_{L^2(I^2;X)} \\ &\leq K^2\Big\| \sum_{1\leq i\leq n} (e_{2^i} \otimes e_{2^j})x_{i,j}\Big\|_{L^2(\mathbb{T}^2;X)}.\end{aligned}$$

(ii) implies (i). Since X is UMD, we already know that for every μ in $(\frac{\pi}{2}, \pi)$, U and V admit a bounded $H^\infty(\Sigma_\mu)$ functional calculus. X has property (α) and therefore, so does $L^2(\mathbb{T}^2;X)$. Then *(i)* follows from [12, Theorem 3.3].

(i) implies (ii). Assume that (U,V) admits a bounded $H^\infty(\Sigma_\mu \times \Sigma_\mu)$ joint functional calculus. It is then clear that U and V admit a bounded $H^\infty(\Sigma_\mu)$ functional calculus and consequently that X is UMD. In order to prove property (α) for X, we introduce U_0 and V_0 the respective generators of the C_0-groups $(T_{(s,0)})_{s\in\mathbb{R}}$ and $(T_{(0,t)})_{t\in\mathbb{R}}$, where

$$\forall (s,t) \in \mathbb{R}^2\ \forall f \in L^2(\mathbb{T};X),\ (T_{(s,t)}f)(e^{iu}, e^{iv}) = f(e^{i(u+s)}, e^{i(v+t)}).$$

Lemma 3.3. *There is a constant $K > 0$ such that*

$$\forall F \in H_0^\infty(\Sigma_\mu \otimes \Sigma_\mu)\ \ \|F(U_0, V_0)\|_{\mathcal{L}(L^2(\mathbb{T}^2;X))} \leq K\|F\|_{H^\infty(\Sigma_\mu\otimes\Sigma_\mu)}.$$

Proof. Let $\mathcal{R}_\mu$ be the algebra of all rational functions, belonging to $H_0^\infty(\Sigma_\mu)$. For any f in $\mathcal{R}_\mu$ There exists k in $L^1(\mathbb{R}_+)$ such that:

$$\forall\ z \in \Sigma_{\frac{\pi}{2}} \ \ : \quad f(z) = \int_0^\infty e^{-sz} k(s)\, ds\,.$$

It follows that for any F in $\mathcal{R}_\mu \otimes \mathcal{R}_\mu$, there is K in $L^1(\mathbb{R}_+) \otimes L^1(\mathbb{R}_+)$ so that:

$$\forall\ (z,z') \in \Sigma_{\frac{\pi}{2}} \times \Sigma_{\frac{\pi}{2}} \ \ : \quad F(z,z') = \int_0^\infty \int_0^\infty e^{-sz} e^{-tz'} K(s,t)\, dsdt\,.$$

Then, for such an F:

$$F(U,V) = \int_0^\infty \int_0^\infty \tau_{(s,t)} K(s,t)\, dsdt \ \text{ and } F(U_0,V_0) = \int_0^\infty \int_0^\infty T_{(s,t)} K(s,t)\, dsdt\,.$$

Therefore Calderon's results on transference (see [6]) and assumption *(i)* yield the existence of a constant $K_1 > 0$ so that

$$\forall F \in \mathcal{R}_\mu \otimes \mathcal{R}_\mu \ \ \|F(U_0,V_0)\|_{\mathcal{L}(L^2(\mathbb{T}^2;X))} \leq K_1 \|F\|_{H^\infty(\Sigma_\mu \times \Sigma_\mu)}. \tag{6}$$

Let now F in $H_0^\infty(\Sigma_\mu \otimes \Sigma_\mu)$ and fix $\varepsilon > 0$. For $z, z' \neq \dfrac{-1}{\varepsilon}$, we set $g_\varepsilon(z) = \dfrac{\varepsilon + z}{1 + \varepsilon z}$ and $G_\varepsilon(z,z') = (g_\varepsilon(z), g_\varepsilon(z'))$. We denote $\Omega_\varepsilon = g_\varepsilon(\Sigma_\mu)$, $\overline{\Omega_\varepsilon} = g_\varepsilon(\overline{\Sigma_\mu})$ is a compact subset of Σ_μ. For an open subset U of $\mathbb{C}$ or $\mathbb{C}^2$, $A(U)$ is the algebra of all complex valued functions, continuous on $\overline{U}$ and holomorphic on U. Since $\Omega_{\varepsilon/2}$ is a conformal image of the unit disk, there is a sequence (G_n) in $A(\Omega_{\varepsilon/2}) \otimes A(\Omega_{\varepsilon/2})$ such that $G_n \to F$ uniformly on $\Omega_{\varepsilon/2} \times \Omega_{\varepsilon/2}$. Then Runge's theorem implies that there is a sequence (F_n) in $\tilde{\mathcal{R}}_\mu \otimes \tilde{\mathcal{R}}_\mu$ with $F_n \circ G_\varepsilon \to F \circ G_\varepsilon$ uniformly on $\Sigma_\mu \times \Sigma_\mu$, where $\tilde{\mathcal{R}}_\mu$ is the algebra of all rational functions of nonpositive degree and with poles outside $\overline{\Sigma_\mu}$.
Since $F_n \circ G_\varepsilon$ is not in H_0^∞ and U_0 and V_0 are not injective, we need to use an other standard approximation argument. So, for z, z' in Σ_μ and m in $\mathbb{N}$, let now $\varphi_m(z) = \dfrac{m^2 z}{(m+z)(1+mz)}$ and $\Phi_m(z,z') = \varphi_m(z)\varphi_m(z')$. Note that for any $0 < \mu < \pi$, $(\varphi_m) \subset H_0^\infty(\Sigma_\mu)$ and there exists $C_\mu > 0$ such that for every m, $\|\varphi_m\|_{H^\infty(\Sigma_\mu)} \leq C_\mu$. Then, it follows from (6) that

$$\forall m > 0 \ \forall n > 0 \ \ \|[(F_n \circ G_\varepsilon)\Phi_m](U_0,V_0)\| \leq K_1 \|\Phi_m\|_{H^\infty(\Sigma_\mu \times \Sigma_\mu)} \|F_n \circ G_\varepsilon\|_{H^\infty(\Sigma_\mu \times \Sigma_\mu)}.$$

Therefore

$$\forall m > 0 \ \ \|[(F \circ G_\varepsilon)\Phi_m](U_0,V_0)\| \leq K_1 C_\mu^2 \|F \circ G_\varepsilon\|_{H^\infty(\Sigma_\mu \times \Sigma_\mu)}.$$

On the other hand, Lebesgue's dominated convergence theorem implies

$$\lim_{\varepsilon \to 0} [(F \circ G_\varepsilon)\Phi_m](U_0,V_0) = F(U_0,V_0)\Phi_m(U_0,V_0).$$

Finally, using the so-called Convergence Lemma ([7], or [12] for a two variables version) as m tends to $+\infty$, we get the desired inequality. $\diamond$

Lemma 3.4. *There is a constant $M > 0$ such that for every $(\alpha_{n,p})_{1\leq n,p\leq N}$ in $\mathbb{C}^{N^2}$, there exists F in $H^\infty(\Sigma_\pi \times \Sigma_\pi)$ satisfying:*

$$\forall (n,p) \quad F(i2^n, i2^p) = \alpha_{n,p} \quad \text{and} \quad \|F\|_{H^\infty(\Sigma_\pi\times\Sigma_\pi)} \leq M \sup_{1\leq n,p\leq N} |\alpha_{n,p}|.$$

Proof. We only mention that it relies on the fact that $(i2^n)$ is a Carleson interpolating sequence in Σ_π ([11], see also [12, section 3] for further details). $\diamond$

End of proof of Theorem 3.2. Let $(x_{n,p})_{1\leq n,p\leq N} \subset X$. By the Cauchy Residue Formula,

$$\forall G \in H_0^\infty(\Sigma_\mu \times \Sigma_\mu),\ G(U_0, V_0) \sum (e_{2^n} \otimes e_{2^p}) x_{n,p} = \sum G(i2^n, i2^p)(e_{2^n} \otimes e_{2^p}) x_{n,p}.$$

Therefore, it follows from Lemmas 3.3 and 3.4 that for every $(\alpha_{n,p})_{1\leq n,p\leq N}$ in $\mathbb{C}^{N^2}$:

$$\Big\| \sum_{1\leq n,p\leq N} \varphi_m(i2^n)\varphi_m(i2^p)\alpha_{n,p}(e_{2^n} \otimes e_{2^p})x_{n,p} \Big\|_{L^2(\mathbb{T}^2;X)} \leq$$
$$MC_\nu^2 K \sup_{1\leq n,p\leq N} |\alpha_{n,p}| \, \Big\| \sum_{1\leq n,p\leq N} (e_{2^n} \otimes e_{2^p})x_{n,p} \Big\|_{L^2(\mathbb{T}^2;X)}.$$

Finally, for all z in Σ_π, $\lim_{m\to+\infty} \varphi_m(z) = 1$, so

$$\Big\| \sum_{1\leq n,p\leq N} \alpha_{n,p}(e_{2^n} \otimes e_{2^p})x_{n,p} \Big\|_{L^2(\mathbb{T}^2;X)} \leq$$
$$MC_\nu^2 K \sup_{1\leq n,p\leq N} |\alpha_{n,p}| \, \Big\| \sum_{1\leq n,p\leq N} (e_{2^n} \otimes e_{2^p})x_{n,p} \Big\|_{L^2(\mathbb{T}^2;X)},$$

which yields property (α). $\diamond$

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