An amazing L^2-estimate in reaction-diffusion systems: four applications

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The L^2-estimate

- Let W be a (regular) solution of

$$\begin{cases} \partial_t W - \Delta(A\,W) = 0 \ \ on\ Q_T = (0,T)\times\Omega, \\ W(0) = W_0 \geq 0, \quad W \geq 0 \\ \partial_\nu W = 0\ on\ \Sigma_T = (0,T)\times\partial\Omega, \end{cases}$$

where $A = A(t,x) \in L^\infty(Q_T)$ satisfies

$$0 < \underline{d} \leq A \leq \overline{d} < +\infty, \quad \underline{d}, \overline{d} \in (0,\infty).$$

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- Then, for some $C = C(\underline{d}, \overline{d}, T)$

$$\|W\|_{L^2(Q_T)} \leq C\,\|W_0\|_{L^2(\Omega)}.$$

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- $$= \int_{Q_T} \partial_t \frac{1}{2}\left|\nabla \int_0^t Z(s)ds\right|^2 = \int_{\Omega} \frac{1}{2}\left|\nabla \int_0^T Z(s)ds\right|^2 \geq 0.$$

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- $$\underline{d}\|W\|_{L^2(Q_T)} \le \overline{d}\sqrt{T}\|W_0\|_{L^2(\Omega)}.$$
- The same is true when
$$\partial_t W - \Delta(A\,W) \le 0, \quad W(0) \le W_0, \quad W \ge 0.$$

A first application: *global existence of weak solutions in mass-controlled and quadratic reaction-diffusion systems*

$$\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_T \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

$d_i > 0,\ f_i : [0, \infty)^m \to \boldsymbol{R}$ of class C^1 where

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- or more generally **(M')**
  $\forall r \in [0, \infty[^m,\ \sum_{1 \leq i \leq m} a_i f_i(r) \leq C[1 + \sum_{1 \leq i \leq m} r_i]$
  for some $a_i > 0$

- $$(E)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- **(P) Preservation of Positivity:** quasi-positivity of f
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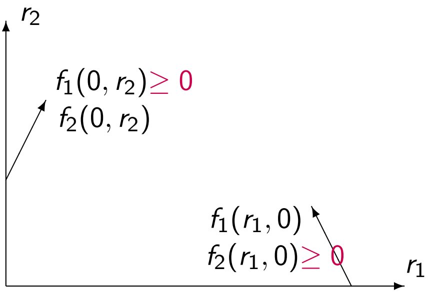

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- **(M):** $\sum_{1 \leq i \leq m} f_i(r_1, ..., r_m) \leq 0 \Rightarrow$ **'Control of the Total Mass'**:

$$\forall t \geq 0, \quad \int_\Omega \sum_{1 \leq i \leq r} u_i(t, x) dx \leq \int_\Omega \sum_{1 \leq i \leq r} u_i^0(x) dx.$$

Add up, integrate on Ω, use $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$:

$$\int_\Omega \partial_t [\sum u_i(t)] dx = \int_\Omega \sum_i f_i(u) dx \leq 0.$$

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- $\Rightarrow L^1(\Omega)$- a priori estimates, uniform in time.

And also an a priori L^2-estimate

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- $$\|u_i\|_{L^2(Q_T)} \leq \|W\|_{L^2(Q_T)} \leq C \|W_0\|_{L^2(\Omega)}.$$

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- **General result (03'): (P)+(M)** + a priori $L^1(Q_T)$-bound on the $f_i(u)$ imply global existence of weak solutions.
- **Or,** use a recent improvement (J.A. Canizo, L. Desvillettes, K. Fellner, F. Otto): There exists $\epsilon(N) > 0$ such that

$$\|W\|_{L^{2+\epsilon}(Q_T)} \leq C\|W_0\|_{L^{2+\epsilon}(\Omega)}.$$

A second application of the L^2-estimate

- $$U_1 + U_2 \underset{k_1}{\overset{1}{\rightleftharpoons}} C \underset{1}{\overset{k_2}{\rightleftharpoons}} U_3 + U_4$$

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- The intermediate C is highly reactive, so that we may assume that $k_1, k_2 \to +\infty$.
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  What is the limit kinetics when space diffusion occurs?
- Mass Action law + Fick's diffusion law lead to the following system for the concentration $u_i(t,x)$ of U_i and $c(t,x)$ of C:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + k_1 c \\ \partial_t u_2 - d_2 \Delta u_2 = -u_1 u_2 + k_1 c \\ \partial_t c - d_c \Delta c = u_1 u_2 - (k_1 + k_2) c + u_3 u_4 \\ \partial_t u_3 - d_3 \Delta u_3 = -u_3 u_4 + k_2 c \\ \partial_t u_4 - d_4 \Delta u_4 = -u_3 u_4 + k_2 c, \end{array}\right\} \textit{on } Q_T \\ + \textit{initial and boundary conditions} \end{array}\right.$$

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- What happens when $k_1 + k_2 \to \infty$?

- $$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + k_1 c & =: f_1 \\ \partial_t u_2 - d_2 \Delta u_2 = -u_1 u_2 + k_1 c & =: f_2 \\ \partial_t c - d_c \Delta c = u_1 u_2 - (k_1 + k_2) c + u_3 u_4 & =: f_c \\ \partial_t u_3 - d_3 \Delta u_3 = -u_3 u_4 + k_2 c & =: f_3 \\ \partial_t u_4 - d_4 \Delta u_4 = -u_3 u_4 + k_2 c & =: f_4, \end{cases}$$

- Positivity is preserved and

$$f_1 + f_2 + 2 f_c + f_3 + f_4 = 0.$$

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- $\Rightarrow$

$$\int_\Omega u_1(t) + u_2(t) + 2c(t) + u_3(t) + u_4(t) = \int_\Omega u_1^0 + u_2^0 + 2c^0 + u_3^0 + u_4^0.$$

$$\forall T > 0,\ \sum_i \|u_i\|_{L^2(Q_T)} + \|c\|_{L^2(Q_T)} < \ C(\text{ independent of } k_1, k_2).$$

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- Global existence of bounded classical solutions holds (not trivial).

$k_1 + k_2 \to +\infty$?

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- Quasi-steady state approximation:
  "$\partial_t c - d_c \Delta c = 0$" as "$k_1 + k_2 = +\infty$"
  or $\lim[(k_1 + k_2)c - u_1 u_2 - u_3 u_4] = 0$
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- $\partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + \lim \frac{k_1}{k_1+k_2}(u_1 u_2 + u_3 u_4)$
  or
  $\partial_t u_1 - d_1 \Delta u_1 = -\alpha u_1 u_2 + (1-\alpha) u_3 u_4$
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- The chemical reaction

$$U_1 + U_2 \underset{k_1}{\overset{1}{\rightleftharpoons}} C \underset{1}{\overset{k_2}{\rightleftharpoons}} U_3 + U_4$$

"tends" to the limit chemical reaction:

$$U_1 + U_2 \underset{1-\alpha}{\overset{\alpha}{\rightleftharpoons}} U_3 + U_4$$

The limit system (D. Bothe-MP)

- **Theorem.** The solution $(u_1^k, u_2^k, c^k, u_3^k, u_4^k)$, $k = (k_1, k_2)$ of the previous system converges as $k_1 + k_2 \to +\infty$ in $L^2(Q_T)^5$ for all $T > 0$ to $(u_1, u_2, 0, u_3, u_4)$ solution of

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Boundary layer at $t = 0$: the new initial values are
$u_1^0 + \alpha c^0,\ u_2^0 + \alpha c^0,\ u_3^0 + (1-\alpha) c^0,\ u_4^0 + (1-\alpha) c^0.$

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- Global existence of classical solutions is open for (S_{lim}) when the d_i are different. Known for $N = 1, 2$. Estimate of the Hausdorff dimension of the (possible) blow up set for $N \geq 3$ (Th. Goudon, A. Vasseur). See also L. Desvillettes, K. Fellner, M.P., J. Vovelle, +M. Bisi, F. Conforto, L. Desvillettes.

A third application of the L^2-estimate

- (D. Bothe, MP, G. Rolland, '11)

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k[u_1 u_2 - u_3] \\ \partial_t u_2 - d_2 \Delta u_2 = -k[u_1 u_2 - u_3] \\ \partial_t u_3 - d_3 \Delta u_3 = \ \ k[u_1 u_2 - u_3] \\ U_1 + U_2 \ \underset{k}{\overset{k}{\rightleftharpoons}} \ U_3 \end{cases}$$

What is the limit kinetics when $k \to +\infty$?

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- Estimates independent of k:

$$\sup_t \|u_i(t)\|_{L^1(\Omega)} \le C, \ \forall T > 0, \|u_i\|_{L^2(Q_T)} \le C.$$

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- A main difficulty: no a priori $L^1(Q_T)$-estimate on $k(u_1^k u_2^k - u_3^k)$ seems to be true ! (except for $d_1 = d_2 = d_3$).

The entropy inequality

- $$\left\{ \begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = -k[u_1 u_2 - u_3] \\ \partial_t u_2 - d_2 \Delta u_2 = -k[u_1 u_2 - u_3] \\ \partial_t u_3 - d_3 \Delta u_3 = k[u_1 u_2 - u_3] \end{array} \right.$$

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- $$\partial_t \theta_i = \log u_i \, \partial_t u_i \ ; \ \Delta \theta_i = \log u_i \ \Delta u_i + \frac{|\nabla u_i|^2}{u_i},$$

  $$\partial_t \theta_1 - d_1 \Delta \theta_1 + \frac{d_1 |\nabla u_1|^2}{u_1} = -k[u_1 u_2 - u_3] \log u_1,$$

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- $$\sum_i (\partial_t - d_i \Delta)\, \theta_i + \frac{d_i |\nabla u_i|^2}{u_i} = -k[u_1 u_2 - u_3][\log(u_1 u_2) - \log u_3] \leq 0.$$

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- Integrating leads to the bound

  $$\int_{Q_T} \sum_i \frac{d_i |\nabla u_i|^2}{u_i} + k[u_1 u_2 - u_3][\log \frac{u_1 u_2}{u_3}] \leq C \; (\textit{independent of } k) \; .$$

Passing to the limit as $k \to \infty$

- Recall the estimates

$$\sup_t \|u_i(t)\|_{L^1(\Omega)} \leq C, \ \forall T > 0, \|u_i\|_{L^2(Q_T)} \leq C.$$

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- Next, we use for $i = 1, 2$ the identity

$$\partial_t (u_i + u_3) - \Delta(d_i u_i + d_3 u_3) = 0$$

to show that $\partial_t \sqrt{u_i + u_3} \in L^2\left(0, T; H^{-1}(\Omega)\right) + L^1(Q_T)$
By Aubin-Simon type of compactness and using again the above equation, we deduce that $u_i + u_3$ are relatively compact in $L^2(Q_T)$.

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- We use the pointwise entropy inequality to prove that all three functions converge a.e.. Whence their convergence in $L^2(Q_T)$.

A general convergence result

(D. Bothe, M.P., G. Rolland)

$$\begin{cases} \partial_t u_1^k - d_1 \Delta u_1^k = -k[u_1^k u_2^k - u_3^k] \\ \partial_t u_2^k - d_2 \Delta u_2^k = -k[u_1^k u_2^k - u_3^k] \\ \partial_t u_3^k - d_3 \Delta u_3^k = k[u_1^k u_2^k - u_3^k] \end{cases}$$

Theorem. Up to a subsequence, the u_i^k converge in $L^2(Q_T), \forall T > 0$ to a weak global nonnegative solution of

$$(Lim) \begin{cases} \left.\begin{array}{l} \partial_t(u_1 + u_3) - \Delta(d_1 u_1 + d_3 u_3) = 0 \\ \partial_t(u_2 + u_3) - \Delta(d_2 u_2 + d_3 u_3) = 0 \end{array}\right\} + \textit{boundary cond.} \\ u_1 u_2 = u_3. \\ (u_1 + u_3)(0) = u_1^0 + u_3^0, \; (u_2 + u_3)(0) = u_2^0 + u_3^0, \end{cases}$$

About the problem (Lim)

$$(Lim)\begin{cases} \left.\begin{array}{c}\partial_t(u_1+u_3)-\Delta(d_1u_1+d_3u_3)=0\\ \partial_t(u_2+u_3)-\Delta(d_2u_2+d_3u_3)=0\end{array}\right\} + \textit{ boundary cond.}\\ u_1u_2=u_3.\\ (u_1+u_3)(0)=u_1^0+u_3^0,\ (u_2+u_3)(0)=u_2^0+u_3^0,\end{cases}$$

If we set, $w_1 := u_1 + u_3, w_2 = u_2 + u_3$, then it is equivalent to the 2×2 cross-diffusion system

$$(Lim')\begin{cases} \left.\begin{array}{c}\partial_t w_1-\Delta\psi_1(w_1,w_2)=0\\ \partial_t w_2-\Delta\psi_2(w_1,w_2)=0\end{array}\right\} + \textit{ boundary cond.}\\ w_1(0)=u_1^0+u_3^0,\ w_2(0)=u_2^0+u_3^0,\end{cases}$$

where $\psi = (\psi_1, \psi_2) : [0, \infty[^2 \to R^2$ is C^∞ and the Jacobian matrix $D\psi(w_1, w_2)$ satisfies the spectral conditions for this problem to have unique local classical solution (see H. Amann's theory).

A fourth application of the L^2-estimate

- Existence of solutions to the cross-diffusion system where $a_i : (0,\infty)^m \to [\underline{d},\infty)$ continuous (only), $\underline{d} > 0$:

$$\begin{cases} \partial_t u_i - \Delta[a_i(\tilde{u})u_i] = 0, \ i = 1, ..., m \\ \tilde{u}_i - \delta_i \Delta \tilde{u}_i = u_i, \qquad \delta_i > 0 \ (\ \delta_i = 1 \ below), \\ \partial_\nu u_i = \partial_\nu \tilde{u}_i = 0, \ u_i(0) = u_i^0 \geq 0. \end{cases}$$

  Model proposed by M. Bendahmane, Th. Lepoutre, A. Marrocco, B. Perthame (partial results in dimensions N=1,2).

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- **THEOREM.** (Th. Lepoutre, MP, G. Rolland, '11): Existence of global solutions satisfying for all $T > 0, p < \infty$

$$u_i \in L^p(Q_T), \tilde{u}_i \in C^\alpha(Q_T) \cap L^p\left(0, T; W^{2,p}(Q_T)\right),$$

$$u_i(t) - \Delta\left[\int_0^t a_i(\tilde{u})u_i\right] = u_i^0.$$

  If, moreover, a_i is locally Lipschitz continuous, the solution is classical, unique and

$$u_i \in L^\infty(Q_T), \partial_t u_i, \Delta(a_i(\tilde{u})u_i) \in L^p_{loc}\left((0, T]; L^p(\Omega)\right).$$

Ideas of the proof

- We first truncate the nonlinearities $a_i(\cdot)$ and solve the problem. We use the L^2 estimate $+$compactness
  - to solve the linear problem $\partial_t w - \Delta(Aw) = 0, w(0) = w_0$ where $A \in L^\infty(Q_T), 0 < \underline{a} \leq A \leq \overline{a} < \infty$; moreover $A^n \to A$ a.e implies $w^n \to w$ in $L^2(Q_T)$, for all T.
  - to apply a Leray-Schauder fixed-point theorem in X^m where $X = \{v \in L^2(Q_T); \partial_t \tilde{v} \in L^2(Q_T),\ \tilde{v} = (I - \Delta)^{-1} v\}$
  $v \in X^m \to u$ solution of $\partial_t u_i - \Delta\left(a_i(\tilde{v})u_i\right) = 0,\ u_i(0) = u_i^0$.

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- We exploit that $a_i(\tilde{u})$ is continuous in this equation to deduce the L^p regularity.

Proof of uniqueness when a_i locally Lipschitz

- If u, v are two solutions, we have ,

$$\partial_t(u_i - v_i) - \Delta[a_i(\tilde{u})(u_i - v_i) + v_i(a_i(\tilde{u}) - a_i(\tilde{v}))] = 0$$

$$\textit{or} \;\; \partial_t U_i - \Delta[a_i(\tilde{u})U_i + v_i A_i \cdot \widetilde{U}] = 0, \; (I - \delta_i \Delta)\widetilde{U}_i = U_i$$

$$U_i = u_i - v_i, \widetilde{U}_i = \tilde{u}_i - \tilde{v}_i, \; A_i = \int_0^1 Da_i(t\tilde{u} + (1-t)\tilde{v})dt \in L^\infty$$

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- And we prove that the dual problem is solvable

$$\partial_t \varphi_i + a_i(\tilde{u})\Delta\varphi_i + (I - \delta_i\Delta)^{-1}[B_i \, \Delta\varphi] = F_i \in L^2(Q_T)$$

$$\forall i = 1, ...m, \; \varphi_i, \partial_t\varphi_i, \Delta\varphi_i \in L^2(Q_T), \varphi_i(T) = 0.$$

$$B_i = [B_{i1}...B_{im}], \;\; B_{ij} = v_j A_{ji}.$$

These L^2-estimates are essentially the dual of the main one we started with !