SCALAR CONSERVATION LAWS WITH FRACTIONAL STOCHASTIC FORCING: EXISTENCE, UNIQUENESS AND INVARIANT MEASURE

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OUTLINE

Our aim is to study the following equation

$$\partial_t u(t,x,\omega) + \partial_x \Psi\big(u(t,x,\omega)\big) = \partial_x \dot{F}(t,x,\omega)\ , \tag{1}$$

with

- $x \in \mathbb{R}$,
- $t_0 \geq 0$,
- $u(t,x,\cdot)$ is a random variable with values in $\mathbb{R}$,
- F is a random force.

A deterministic initial data $u(t_0,\cdot) = u_0(\cdot) \in L^\infty(\mathbb{R})$ is given.

Burgers' case :
If the flux function Ψ is the half of the square function and $F = 0$, then we obtain the Burgers' equation

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\partial_x \left(u(t,x)\right)^2 = 0 \ , \\ u(0,x) = u_0(x) \ . \end{cases}$$

Characteristic method :
A characteristic curve ξ is a curve such that $u(t,\xi(t))$ is constant. If u is regular then a characteristic satisfies

$$\begin{cases} \xi'(t) & = u(t,\xi(t))\, ; \\ \xi(0) & = x_0. \end{cases} \qquad \Longrightarrow \qquad \xi(t) = x_0 + u_0(x_0)t.$$

If there exists $x_0 < x_0'$ such that $u_0(x_0) > u_0(x_0')$ then the characteristics intersect each other and at the intersection point (t,x) one should have

$$u(t,x) = u_0(x_0) = u_0(x_0').$$

So we need weak solution.

Weak solution :

$$\int_0^\infty \int_{\mathbb{R}} \frac{\partial\varphi(t,x)}{\partial t} u(t,x)dxdt + \int_{t_0}^\infty \int_{\mathbb{R}} \frac{1}{2}\frac{\partial\varphi(t,x)}{\partial x}\left(u(t,x)\right)^2 dxdt + \int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx = 0\ .$$

For the Riemann problem, $u_0 = -\mathbf{1}_{\mathbb{R}-} + \mathbf{1}_{\mathbb{R}+}$ we have two weak solutions :

1. $u(t,x) = \begin{cases} -1 & \text{if } x+t \le 0\,; \\ +1 & \text{if } x-t \ge 0\,; \\ x/t & \text{otherwise.} \end{cases}$
2. $\tilde{u}(t,x) = u_0(x)$.

In the deterministic case :

- the weak solution is not unique,
- in order to discriminate the "physical" solution one have to introduce the notion of entropy solution
- the selected solution has nice qualitative behavior :
  1. discontinuities that are related with creation of shocks
  2. description of the behavior in term of characteristics
  3. ...

◇ There is a wide literature on deterministic conservation laws :
Dafermos, Evans, Hörmander, Serre, Oleĭnik, Kruzkhov...

◇ In the stochastic case only few works in this area :

- Kim (Indiana 03)
- Vallet-Wittbold (IDAQPRT 09)
- Feng-Nualart (JFA 08)
- Debussche-Vovelle (JFA 10)
- E-Khanin-Mazel-Sinai (Ann. Math. 00)

Outline

This article deals with the Burgers' case (that is $\Psi(u) = u^2/2$) :

$$\partial_t u(t,x,\omega) + \partial_x \big(u(t,x,\omega)\big)^2 = \partial_x \dot{F}(t,x,\omega) \ ,$$

with a stochastic forcing given by

$$F(t,x,\omega) = \sum_{k=1}^{\infty} F_k(x)\dot{B}_k(t)$$

where $(B_k)_{k\geq 1}$ are independent standard Wiener processes on the real line $\mathbb{R}$.

They prove :

- the existence and uniqueness of the solution via a parabolic perturbation method,
- the existence and uniqueness of an invariant measure via a fundamental property of the Brownian paths.

The Brownian noise is arbitrary small on an infinite number of arbitrary long time intervals. In other words for all $\varepsilon > 0$, $T > 0$, for almost-all ω, there exists a sequence of random time $(t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall \ n \ , \sup_{t_n - T \leq s \leq t_n} \sum_{k\geq 1} \Big\{ \|F_k\|_{C_b^2(\mathbb{R})} |B_k(s) - B_k(t_n)| \Big\} \ \leq \varepsilon \ .$$

OUTLINE

HYPOTHESES

- The stochastic term : for any t, x, $F(t,x) = \sum_{k=1}^{\infty} F_k(x) B_k(t)$ where :
  - (A) there exists $\lambda > 0$ such that the sequence of processes $\left((B_k(t))_{t\in(-\infty,\infty)}\right)_{k\geq 1}$ satisfies $B_k(\cdot) \in C^\lambda(a,b)$ for any $k \geq 1$, $-\infty < a < b < +\infty$.
  - (B) the sequence $(F_k)_{k\geq 1}$ is such that for any k, the function F_k belongs to $C_b^3(\mathbb{R})$ satisfies $\|F_k\|_{C_b^3(\mathbb{R})} \leq Ck^{-\frac{2+\lambda}{\lambda}}$.
- The flux :
  - (A) Ψ is uniformly convex : there exists $\theta > 0$ such that $\Psi''(v) \geq \theta$ for all $v \in \mathbb{R}$,
  - (B) super-linear growth condition : there exists $k_2 > k_1 > 0$ and two constants l_1, l_2 such that $l_1|v|^{k_1} \leq \frac{\Psi(v)}{|v|} \leq l_2|v|^{k_2}$,
  - (C) there exists L such that $|\Psi'(v) - \Psi'(v')| \leq L|v - v'|$,
  - (D) there exists a positive function $R \mapsto C(R)$ such that $|\Psi^*(v) - \Psi^*(v')| \leq C(R)|v - v'|$ [1] whenever $\max(|v|, |v'|) \leq R$.

1. For a function f from $\mathbb{R} \to \mathbb{R}$, we denote f^* its Legendre transform defined as $f^*(q) = \sup_{p\in\mathbb{R}} \left(pq - f(p)\right)$ for $q \in \mathbb{R}$

OUTLINE

WEAK SOLUTION

DEFINITION

A random field u defined on $[t_0, +\infty) \times \mathbb{R} \times \Omega$ with real values is a weak solution of (1) with initial condition $u(t_0, \cdot) = u_0(\cdot) \in L^\infty(\mathbb{R})$ if :

(I) For all $t > t_0$ and $x \in \mathbb{R}$, $u(t, x, \cdot)$ is measurable with respect to $\mathcal{F}_{t_0,t} = \sigma\{B_k(s), t_0 \leq s \leq t, k \geq 1\}$.

(II) Almost surely, $u(\cdot, \cdot, \omega) \in L^1_{\text{loc}}([t_0, \infty) \times \mathbb{R})$ and $u(t, \cdot, \omega) \in L^\infty(\mathbb{R})$ for any $t \geq t_0$

(III) For all test function $\varphi \in C^2_c(\mathbb{R} \times \mathbb{R})$ (the set of twice differentiable functions with compact support) the following equality holds almost-surely

$$\int_{t_0}^{\infty} \int_{\mathbb{R}} \frac{\partial \varphi(t,x)}{\partial t} u(t,x) dxdt + \int_{t_0}^{\infty} \int_{\mathbb{R}} \frac{\partial \varphi(t,x)}{\partial x} \Psi\big(u(t,x)\big) dxdt = - \int_{\mathbb{R}} u_0(x) \varphi(t_0, x) dx$$
$$- \int_{\mathbb{R}} \sum_{k=1}^{\infty} \left\{ F_k(x) \int_{t_0}^{\infty} \frac{\partial^2 \varphi(t,x)}{\partial t \partial x} (B_k(t) - B_k(t_0)) dt \right\} dx \,. \qquad (2)$$

OUTLINE

WEAK-ENTROPY SOLUTION

DEFINITION

We say that a random field u which is a already a weak solution of Equation (1) is an entropy-weak solution if there exists $C > 0$ such that for almost-all $\omega \in \Omega$,

$$u(t, x+z, \omega) - u(t, x, \omega) \leq C\big(1 + \tfrac{1}{t-t_0}\big)\, z \tag{3}$$

for all $(t, x) \in (t_0, \infty) \times \mathbb{R}$ and $z > 0$.

- The above entropy condition is the historical *"condition E"*.
- This condition will ensure us the uniqueness of bounded weak solution.
- From (3), for $t > t_0$ the function $x \mapsto u(t, x) - Cx$ is nonincreasing, and consequently has left and right hand limits at each point.
- Thus also $x \mapsto u(t, x)$ has left and right hand limits at each point, with $u(t, x-) \geq u(t, x+)$ (classical form of the entropy condition).

OUTLINE

THEOREM

Let $u_0 \in L^\infty(\mathbb{R})$. There exists a unique entropy-weak solution to the stochastic scalar conservation law (1) *such that $u(t_0, x) = u_0(x)$. For $t \geq t_0$, this solution is given by the following Lax-Oleĭnik type formula :*

$$u(t,x,\omega) = \frac{\partial}{\partial_x}\left(\inf_{\substack{\xi \,\in\, H^1(t_0,\,t) \\ \xi(t) \,=\, x}} \left\{ \mathcal{A}_{t_0,t} + \int_0^{\xi(t_0)} u_0(z)dz \right\} \right), \tag{4}$$

with

$$\begin{aligned} \mathcal{A}_{t_0,t}(\xi) = \int_{t_0}^{t} \Big\{ \Psi^*(\dot{\xi}(s)) - \textstyle\sum_{k\geq 1} \big(B_k(s) - B_k(t_0)\big) F_k'(\xi(s))\dot{\xi}(s) \Big\} ds \\ + \textstyle\sum_{k\geq 1} \big(B_k(t) - B_k(t_0)\big) F_k(\xi(t)) \, . \end{aligned} \tag{5}$$

Outline

THE "STOCHASTIC" INTEGRAL

- The trajectories $\omega \to B_k(t)(\omega)$ are $\lambda-$Hölder continuous ;
- If the curve ξ is regular ($C^1(\tau,t)$) then the stochastic term of the action exists as a Riemann-Stieltjes integral

$$\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s) = -\int_\tau^t \sum_{k\geq 1}\big(B_k(s) - B_k(\tau)\big)F_k'(\xi(s))\dot{\xi}(s)ds + \sum_{k\geq 1}\big(B_k(t) - B_k(\tau)\big)F_k(\xi(t)) \quad (6)$$

- If $\xi(t)$ is fixed to be x, then the second term in the above equality is independent on ξ, hence the action is redefined for $\xi \in C^1(\tau,t)$ as

$$\mathcal{A}_{\tau,t}(\xi) = \int_\tau^t \Psi^*(\dot{\xi}(s))ds + \int_\tau^t \sum_{k\geq 1} F_k(\xi(s))dB_k(s) \ .$$

Link with HJB equation

Formally let φ a test function in $C_c^2(\mathbb{R}\times\mathbb{R})$, thanks to an integration by parts one rewrites (2) :

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \frac{\partial\varphi(t,x)}{\partial t} u(t,x)dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \frac{\partial\varphi(t,x)}{\partial x}\Psi\big(u(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx - \int_{\mathbb{R}}\sum_{k=1}^{\infty}\left\{F_k(x)\int_{t_0}^{\infty}\frac{\partial^2\varphi(t,x)}{\partial t\partial x}(B_k(t)-B_k(t_0))dt\right\}dx$$

as

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)u(t,x)dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(u(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx + \int_{\mathbb{R}}\int_{t_0}^{\infty}\partial_t\varphi(t,x)v(t,x)dtdx \qquad (7)$$

with

$$v(t,x) = \sum_{k=1}^{\infty} F_k'(x)(B_k(t)-B_k(t_0)) \,. \qquad (8)$$

LINK WITH HJB EQUATION

Consequently,

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)\big[u(t,x)-v(t,x)\big]dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(u(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx$$

with $w = u + v$ we obtain

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_t\varphi(t,x)w(t,x)dxdt + \int_{t_0}^{\infty}\int_{\mathbb{R}} \partial_x\varphi(t,x)\Psi\big(w(t,x)+v(t,x)\big)dxdt = -\int_{\mathbb{R}} u_0(x)\varphi(t_0,x)dx\ .$$

Hence w is a solution of the stochastic scalar conservation law

$$\partial_t w + \text{div}_x \Psi(w+v) = 0$$

and if we integrate with respect to the space variable x this equation, we derive the HJB equation

$$\partial_t W + \Psi(\partial_x W + v) = 0\ .$$

where W is such that $\partial_x W = w$.
This HJB is related to an optimization problem with an action involving the Legendre transform of $p \mapsto \Psi(p+v)$. Thanks to the behavior under translation of the Legendre transformation, one have $\big(\Psi(\cdot + v)\big)^*(q) = \Psi^*(q) - vq$ and we obtain the action in (5).

OUTLINE

PROPERTIES OF THE MINIMIZERS

To simplify $\Psi(v) = \Psi^*(v) = v^2/2$ and the noise is $F(t,x) = F(x)B(t)$.

If γ is a minimizer of $\mathcal{A}$ on $[t_1,t_2]$, that is

$$\mathcal{A}_{t_1,t_2}(\gamma) = \inf_{\xi \in \mathcal{H}^{t_1,t_2}_{x_1,x_2}} \left\{ \int_{t_1}^{t_2} \left(\frac{1}{2}(\dot{\xi}(s))^2 - (B(s) - B(t_1))F'(\xi(s))\dot{\xi}(s) \right) ds \right.$$
$$\left. + (B(t_2) - B(t_1))F(\xi(t_2)) \right\}$$

then we have the following properties :

- regularity : $\dot{\gamma} \in C^1(t_1,t_2)$;
- Euler-Lagrange equations : for $t_1 \leq r \leq s \leq t_2$

$$\dot{\gamma}(s) - \dot{\gamma}(r) = \int_r^s F'(\gamma(\tau))dB(\tau) \,. \tag{9}$$

- Bound on velocities : for $t_2 - t_1 \geq 1$, then there exists a constant c such that

$$\|\dot{\gamma}\|_{t_1,t_2,\infty} \leq c \, \|F\|_{C^2} \left\{ \sup_{t_1 \leq r \leq r' \leq t_2} |B(r) - B(r')| \right\} \tag{10}$$

- $u(t, x-) = \sup_\gamma \dot{\gamma}(t)$
- $u(t, x+) = \inf_\gamma \dot{\gamma}(t)$
- The bound (10) on velocities implies

$$\left|\mathcal{A}_{t_1,t_2}(\gamma)\right| \leq c \, \|F\|_{C^2} \left\{ \sup_{t_1 \leq r \leq r' \leq t_2} |B(r) - B(r')| \right\} \tag{11}$$

Outline

ONE-SIDED MINIMIZERS

DEFINITION

Let $t \in \mathbb{R}$. A piecewise C^1 curve $\xi :]-\infty, t] \to \mathbb{R}$ is a one-sided minimizer if

(I) for any $\gamma \in H^1(-\infty, t)$ such that $\gamma(t) = \xi(t)$ and $\gamma = \xi$ on $]-\infty, \tau]$ for some $\tau < t$, it holds that $\mathcal{A}_{s,t}(\xi) \leq \mathcal{A}_{s,t}(\gamma)$ for any $s \leq \tau$;

(II) for any $s \leq t$, $|\xi(s) - \xi(t)| \leq 1$.

PROPOSITION

For every $x \in \mathbb{R}$ and $t \in \mathbb{R}$, there exists a one-sided minimizer γ such that $\gamma(t) = x$.

PROPOSITION

Intersection of one sided minimizers :
For almost-all ω, for any distinct one-sided minimizers γ_1 and γ_2 on $]-\infty, t_1]$ and $]-\infty, t_2]$, if γ_1 and γ_2 intersect at time t in a point x, then $t_1 = t_2 = t$ and $\gamma_1(t_1) = \gamma_2(t_2) = x$.

The above result is true if we assume that the noise has the following "silence property".

"SILENCE PROPERTY"

Our noise must satisfy that it is arbitrary small on an infinite number of arbitrary long time intervals.

$\forall \varepsilon > 0$, $\forall T > 0$, for almost-all ω, $\exists\ (t_n(\omega))_{n\geq 1}$, such that $t_n(\omega) \to -\infty$ and

$$\forall\ n\ ,\quad \sup_{t_n - T \leq s \leq t_n} |B(s) - B(t_n)|\ \leq \varepsilon\ .$$

In the Brownian case, we consider the events

$$A_n = \left\{ \sup_{-nT - T \leq s \leq -nT} |B(s) - B(-nT)|\ \leq \varepsilon \right\}\ .$$

The events A_n are independent and $\mathbb{P}(A_n) > 0$ for any n. By the Borel-Cantelli lemma,

$$\mathbb{P}\Big(\omega \in A_n \text{ for infinitely many } n\Big) = 1\ .$$

Outline

Definition

A two-sided fractional Brownian motions (fBm in short) with Hurst parameter $H \in (0,1)$ is a Gaussian process $(B(t))_{t\in\mathbb{R}}$ with $B(0) = 0$ and $\mathbb{E}(|B(t) - B(s)|^2) = |t - s|^{2H}$.

Moving average representation :

$$B(t) = \int_{\mathbb{R}} f_t(s)dW_s \quad \text{with}$$
$$f_t(s) = c_H \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) .$$

Proposition

A two-sided fractional Brownian motion satisfies the following property : $\forall \varepsilon > 0, \forall T > 0$, *for almost-all* ω, $\exists\ (t_n(\omega))_{n\geq 1}$, *such that* $t_n(\omega) \to -\infty$ *and*

$$\forall\ n\ , \sup_{t_n - T \leq s \leq t_n} |B(s) - B(t_n)| \ \leq \varepsilon\ .$$

OUTLINE

PROOF

Let $\varepsilon > 0$ and $T > 0$ be fixed. Let $(t_n)_{n\geq 1}$ be a decreasing sequence of negative real numbers such that

$$\begin{cases} \lim_{n\to\infty} t_n = -\infty \ ; \\ t_{n+1} < t_n - T \text{ and} \\ \sum_{n\geq 1}(t_n - t_{n+1})^{H-1} < \infty. \end{cases}$$

We denote $\mathcal{F}_{t_n} = \sigma\{B(r); -\infty < r \leq t_n\}$ and

$$A_n(\varepsilon) = \left\{ \sup_{t_n - T \leq t,s \leq t_n} |B(t) - B(s)| \leq \varepsilon \right\} .$$

The proof is based on the following reversed Borel-Cantelli's lemma :

LEMMA

Let $(\mathcal{F}_n)_{n\geq 1}$ be a decreasing sequence of $\sigma-$fields and $(A_n)_{n\geq 1}$ a sequence of events such that $A_n \in \mathcal{F}_n$. Then the events

$$\left\{ \sum_{k\geq 1} \mathbf{1}_{A_k} < \infty \right\} \quad \textit{and} \quad \left\{ \sum_{k\geq 1} \mathbb{E}\left(\mathbf{1}_{A_k} | \mathcal{F}_{k+1}\right) < \infty \right\}$$

are almost-surely equal.

For $t \geq t_{n+1}$ we set

$$B^{n+1}(t) = \mathbb{E}(B(t)|\mathcal{F}_{t_{n+1}}) \qquad \text{and} \qquad \overline{B}^{n+1}(t) = B(t) - B^{n+1}(t) .$$

By the gaussian property of the fBm, $\overline{B}^{n+1}(t)$ is independent of $\mathcal{F}_{t_{n+1}}$. We denote

$$\widetilde{A}_n(\varepsilon) = \left\{ \sup_{t_n - T \leq t, s \leq t_n} \left| B^{n+1}(t) - B^{n+1}(s) \right| \leq \varepsilon \right\},$$

$$\overline{A}_n(\varepsilon) = \left\{ \sup_{t_n - T \leq t, s \leq t_n} \left| \overline{B}^{n+1}(t) - \overline{B}^{n+1}(s) \right| \leq \varepsilon \right\}.$$

After elementary calculus we obtain

$$\mathbb{E}\left(\mathbf{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}\right) \geq \mathbb{P}(A_n(\varepsilon/4)) - \mathbb{P}((\widetilde{A}_n(\varepsilon/4))^c) - \mathbf{1}_{(\widetilde{A}_n(\varepsilon/2))^c}. \tag{12}$$

If one has

$$\sum_{n \geq 1} \mathbb{P}((\widetilde{A}_n(\varepsilon))^c) < \infty \qquad \text{and} \qquad \mathbb{P}(A_n(\varepsilon)) \geq \exp\left(\frac{-cT}{\varepsilon^H}\right), \tag{13}$$

then $\sum_{n \geq 1} \mathbb{E}(\mathbf{1}_{A_n(\varepsilon)}|\mathcal{F}_{t_{n+1}}) = \infty$ a.s.

Using the reversed conditional Borel-Cantelli's lemma we deduce that $\sum_{n \geq 1} \mathbf{1}_{A_n(\varepsilon)} = \infty$ a.s., which implies the expected "silence property".

PROOF OF $\sum_{n\geq 1} \mathbb{P}((\widetilde{A}_n(\varepsilon))^c) < \infty$

Garsia-Rodemich-Rumsey inequality : let f be a continuous function, ρ and g two continuous stricly increasing functions on $[0,\infty)$ with $\rho(0) = g(0) = 0$ and $\lim_{x\to\infty} \rho(x) = \infty$,

$$|f(t) - f(s)| \leq 8 \int_0^{t-s} \rho^{-1}\left(\frac{4C_{s,t}}{u^2}\right) dg(u)$$

$$\text{with} \quad C_{s,t} = \int_s^t \int_s^t \rho\left(\frac{|f(t') - f(s')|}{g(|t' - s'|)}\right) ds'\,dt' \;.$$

We apply the above inequality with $\rho(u) = u^4$ and $g(u) = u$:

$$|B^{n+1}(t) - B^{n+1}(s)| \leq \delta_n \times |t-s|^{1/2} \qquad \text{with}$$

$$\delta_n = c\left(\int_{t_n-T}^{t_n}\int_{t_n-T}^{t_n}\left(\frac{|B^{n+1}(t') - B^{n+1}(s')|}{|t'-s'|}\right)^4 ds'\,dt'\right)^{1/4} \;.$$

Proof of $\sum_{n\geq 1} \mathbb{P}((\widetilde{A}_n(\varepsilon))^c) < \infty$

For $t_n - T \leq s \leq t \leq t_n$:

$$B^{n+1}(t) - B^{n+1}(s) = \mathbb{E}\left[\int_{-\infty}^{t_{n+1}} c_H \left\{ (s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}} \right\} dW_r \middle| \mathcal{F}_{t_{n+1}} \right]$$

and for $p \geq 1$ we obtain

$$\begin{aligned} \mathbb{E}\left(|B^{n+1}(t) - B^{n+1}(s)|^{2p} \right) &\leq c \left(\int_{-\infty}^{t_{n+1}} \left| (s-r)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}} \right|^2 dr \right)^p \\ &\leq c \left((t-s)(t_n - t_{n+1})^{H-1} \right)^{2p} . \end{aligned}$$

By the Garsia-Rodemich-Rumsey inequality there exists a random variable δ_n with $\mathbb{E}(|\delta_n|^2) \leq cT(t_n - t_{n+1})^{2(H-1)}$ and we obtain

$$\sup_{t_n - T \leq t, s \leq t_n} \left| B^{n+1}(t) - B^{n+1}(s) \right| \leq c\, T^{1/2}\, \delta_n .$$

We obtain the convergence since $\sum_{n\geq 1} (t_n - t_{n+1})^{H-1} < \infty$.

Proof of $\mathbb{P}(A_n(\varepsilon)) \geq \exp\left(\frac{-cT}{\varepsilon^H}\right)$

Talagrand's small ball estimate :

One needs al least $T\varepsilon^{-H}$ balls of radius ε under the Dudley metric

$$d(s,t) = \left(\mathbb{E}|B(t) - B(s)|^2\right)^{1/2}$$

to cover the time interval $[t_n - T, t_n]$.

Then there exists a constant c such that

$$\log \mathbb{P}\left(\sup_{t_n - T \leq t, s \leq t_n} |B(t) - B(s)| \leq \varepsilon\right) \geq -c\,\frac{T}{\varepsilon^H}$$

Construction of the invariant solution

We denote $\mathcal{M}_{t,x}$ the family of all one-sided minimizers with end x at time t and

$$u^\sharp(t,x,\omega) = \inf_{\gamma\in\mathcal{M}_{t,x}} \dot\gamma(t) \,.$$

Remark

If more than one one-sided minimizer comes to x at time t, there corresponds a non-trivial segment $I(x) = [\gamma_1(t-T), \gamma_2(t-T)]$, where $\gamma_1 < \gamma_2$ on $]-\infty, t]$ (because two different one-sided minimizers can not intersect each other more than once).

Then the segments $I(x)$ are mutually disjoint. Consequently, for almost-all ω, the set of $x \in \mathbb{R}$ with more than one one-sided minimizer is coming to x at time t is at most countable.

Proposition

- (I) *almost-surely, $u^\sharp(t,\cdot,\omega) \in L^\infty(\mathbb{R})$ for any t ;*
- (II) *almost-surely, $u^\sharp(t,\cdot,\omega) \in \mathbb{D}$ for any t ;*
- (III) *given t, the mapping $\omega \mapsto u^\sharp(t,\cdot,\omega)$ is measurable from $(\Omega,\mathcal{F})$ to $(\mathbb{D},\mathcal{D})$;*
- (IV) *on any finite time interval $[t_1,t_2]$, for almost-all ω, $(t,x) \mapsto u^\sharp(t,x,\omega)$ is a weak solution of (1) with initial data $u_0(x) = u^\sharp(t_1,x,\omega)$.*

EXISTENCE AND UNIQUENESS OF THE INVARIANT MEASURE

- $\Omega = C_0(\mathbb{R},\mathbb{R})$,
- θ^τ the shift operator on Ω with increment τ : $\theta^\tau(\omega) = \omega(\cdot + \tau) - \omega(\tau)$ for any $\omega \in \Omega$,
- The operator S^τ_ω : for $v \in L^\infty(\mathbb{R})$, $S^\tau_\omega(v)$ is the solution of (1) at time τ, with initial condition v at time $t_0 = 0$ when the realization of the noise is ω.

By construction of $u^\sharp$,

$$S^t_\omega(u^\sharp(0,\cdot,\omega)) = u^\sharp(t,\cdot,\omega).$$

THEOREM

On $(\Omega \times \mathbb{D}\ ;\ \mathcal{F} \otimes \mathcal{D})$, the measure μ defined by

$$\mu(d\omega,dv) = \delta_{u^\sharp(0,\cdot,\omega)}(dv)\ \mathbb{P}(d\omega) \tag{14}$$

is the unique measure that leaves invariant the transformation

$$\begin{aligned} \Omega \times \mathbb{D} &\longrightarrow \Omega \times \mathbb{D} \\ (\omega, v) &\longrightarrow (\theta^t\omega, S^t_\omega(v)) \end{aligned}$$

with given projection $\mathbb{P}$ on $(\Omega,\mathcal{F})$.

Merci de votre attention