On some Barenblatt's problems

Guy Vallet

University of Pau (France)

with Adimurthi (Bangalore - India),
Seam Ngonn (Phnom Penh - Cambodia)
Etienne Emmrich (Bielefeld)
Caroline Bauzet and Jacques Giacomoni (Pau)

A (pseudo)parabolic problem

We are interested in :

the Barenblatt's problem

¿ $\exists u \in W^{1,p,2}(0,T;W_0^{1,p}(\Omega),L^2(\Omega))$? such that $u(0,.) = u_0$ in Ω and

$$f(\partial_t u) - \Delta_p u = g \quad \text{in } Q.$$

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Question of Ph. Bénilan to Ki Sik HA in 1975

If A is m-accretif in L^∞ and $\beta \in \mathbb{R}^2$ maximal monotone,
is βA m-accretif in L^∞?

$$g \in \partial_t u + \beta A u.$$

A (pseudo)parabolic problem

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$$¿\ \exists u \in W^{1,p,2}(0,T;W_0^{1,p}(\Omega),L^2(\Omega))\ ? \quad \text{such that} \quad u(0,.) = u_0 \text{ in } \Omega \quad \text{and}$$

$$f(\partial_t u) - \Delta_p u = g \quad \text{in } Q.$$

the Barenblatt's pseudoparabolic problem

$$¿\ \exists u \in W^{1,p,2}(0,T;W_0^{1,p}(\Omega),H_0^1(\Omega))\ ? \quad \text{such that} \quad u(0,.) = u_0 \text{ in } \Omega \quad \text{and}$$

$$f(\partial_t u) - \Delta_p u - \epsilon \Delta \partial_t u = g \quad \text{in } Q.$$

$\Omega \subset \mathbb{R}^d$ Lipschitz bounded, $T > 0$ and $Q :=]0,T[\times \Omega$.

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Application and/or related to

1. Fluid Flow in Fissured Porous Media
   Denote by: [G. I. Barenblatt *et al.*] ('60)
   p_1 the pressure in the fissures,
   p_2 the pressure in the porous blocks.
   Then,

mass conservation
in $-$ flow rate of fissures
Darcy$'$s law

$$\Longrightarrow \quad \begin{aligned} &\operatorname{div}[k_1 \nabla p_1] + a(p_2 - p_1) = 0 \\ &\partial_t p_2 + b(p_2 - p_1) = 0. \end{aligned}$$

i.e. Sobolev equation

$$\partial_t p_1 - \eta \partial_t \Delta p_1 = k \Delta p_1 .$$

Application and/or related to

2. HEAT DIFFUSION IN A TWO-PHASE SUBSTANCE
Denote by: [P. Colli, F. Luterotti, G. Schimperna, U. Stefanelli]('02)
χ volume fraction of one of the phases,
θ the absolute temperature.

balance and constitutive laws
Def. of free energy
pseudo − potential of dissipation

$$\Longrightarrow$$

$$c_s\partial_t\theta + \frac{L}{\theta_c}\theta\partial_t\chi - k\Delta\theta = \mu(\partial_t\chi)^2 + \xi\partial_t\chi + \delta|\nabla\partial_t\chi|^2,$$

$$\mu\partial_t\chi + \xi - \delta\Delta\partial_t\chi - \nu\Delta\chi + \eta = \frac{L}{\theta_c}(\theta - \theta_c) + A,$$

where $\eta \in \beta(\chi)$ and $\xi \in \alpha(\partial_t\chi)$.

Application and/or related to

3. ELASTO-PLASTIC POROUS MEDIA [G. I. Barenblatt *et al.*]('82)

$$\partial_t u + \gamma |\partial_t u| = \Delta u.$$

4. REACTION-DIFFUSION OF TWO SUBSTANCES [M. Ptashnyk]('04)

$$b(t,x,\partial_t u) - \mathrm{div}[a(t,x,\nabla \partial_t u)] - \mathrm{div}[d(t,x)\nabla u] = f(t,x,u).$$

5. ABSTRACT OPERATORS [P. Colli]('92), [T. Roubicek]('05)

$$f \in \partial\psi(\partial_t u) + Au.$$

6. CAPILLARY HYSTERESIS [G. Schimperna *et al.*]('07)

$$f \in \alpha(u_t) - \mathrm{div}[b(\nabla u)] + h(u).$$

7. SEDIMENTARY BASINS [Seam Ngonn *et al.*]('11)

$$f\,(t,x,\partial_t u) - \mathrm{div}[a(x,u,\partial_t u)\nabla u] - \mathrm{div}[b(x,u,\partial_t u)\nabla \partial u_t] = g.$$

The Barenblatt-Sobolev problem

(with: Adimurthi (Bangalore) and Seam Ngonn (Phnom Penh))

$$f(\partial_t u) - \Delta u - \epsilon \Delta \partial_t u = g \quad \text{in } Q, \qquad \epsilon > 0$$

Hypothesis:

f Lipschitz-continuous, $f(0) = 0$, $f = f_1 - f_2$.

$$(f_1(x) = \int_0^x f'^{+}(s)ds, f_2 = \int_0^x f'^{-}(s)ds).$$

$u_0 \in H_0^1(\Omega)$ and $g \in L^2\left(0, T, H^{-1}(\Omega)\right)$.

Definition (of a solution)

$u \in W^{1,2}(0, T; H_0^1(\Omega), H_0^1(\Omega)), \quad u(0, \cdot) = u_0.,$

$$t \in]0, T[\text{ a.e. }, \quad \forall v \in H_0^1(\Omega),$$

$$\int_\Omega \left\{ f\left(\partial_t u\right) v + \nabla u . \nabla v + \epsilon \nabla \partial_t u . \nabla v \right\} \text{dx} \; =< g, v > .$$

$$f_1(\partial_t u) - \Delta u - \epsilon \Delta \partial_t u = g + f_2(\partial_t u), \quad u(0,.) = u_0.$$

Proposition

λ_1: first eigenvalue of $-\Delta$ in H_0^1, $\quad \epsilon_0 = \frac{||f_2'||_\infty}{\lambda_1}$.

Then,

1. $\epsilon > \epsilon_0 \geq 0$: *there exists a unique solution in $H^1(0,T,H_0^1(\Omega))$.*
2. $\epsilon = \epsilon_0 > 0$: *in $H^1(0,T,H_0^1(\Omega))$, if a solution exists, it is unique.*
   If supp(f_2') is compact, there exists a solution.
3. $0 < \epsilon < \epsilon_0$: *in $H^1(0,T,H_0^1(\Omega))$, solutions are not unique in general.*

Implicit time-discreatization

Consider $N > 0$, $h = \frac{T}{N}$ and $t_k = kh$.

Proposition

$$\exists (u^k)_k \subset H_0^1(\Omega), \quad u^0 = u_0 \quad and \quad \forall v \in H_0^1(\Omega),$$

$$\int_\Omega \left\{ f\left(\frac{u^{k+1} - u^k}{h}\right) v + \nabla u^{k+1} \nabla v + \epsilon \nabla \frac{u^{k+1} - u^k}{h} \nabla v \right\} dx \;=< g^{k+1}, v >,$$

$$where \quad g^{k+1}(x) = \frac{1}{h} \int_{kh}^{(k+1)h} g(t,x)\, dt.$$

Proof

Fixed point - Topological degree - Variational methods - perturbation of Δ ...

$$u^h = \sum_{k=0}^{N-1} u^{k+1} \mathbf{1}_{[t_k, t_{k+1}[}, \qquad \widetilde{u}^h = \sum_{k=0}^{N-1} \left[\frac{u^{k+1} - u^k}{h} (t - t_k) + u^k \right] \mathbf{1}_{[t_k, t_{k+1}[}.$$

Lemma

If $h << 1$.

1. $(\widetilde{u}^h)$ *is bounded in* $H^1(0, T; H_0^1(\Omega))$;
2. $\exists C > 0, \forall t \in [0, T[, \left\| \widetilde{u}^h(t) - u^h(t) \right\|_{H_0^1(\Omega)} \leq C\sqrt{h}$;
3. t *a.e. in* $]0, T[, \exists C(t), \quad ||\partial_t \widetilde{u}^h(t)||_{H_0^1(\Omega)} \leq C(t)$.

Test-function $v = \frac{u^{k+1} - u^k}{h}$.

$$\int_\Omega \left\{ f\left(\partial_t \widetilde{u}^h\right) v + \nabla u^h \nabla v + \epsilon \nabla \partial_t \widetilde{u}^h \nabla v \right\} \mathrm{dx} \; =< g^h, v > .$$

Up to a sub-sequence denoted in the same way,

$$\exists u \in H^1\left(0,T;H_0^1(\Omega)\right), \chi \in L^2\left(Q\right) \quad : \quad \begin{cases} \widetilde{u}^h \rightharpoonup u & \text{in } H^1\left(0,T;H_0^1(\Omega)\right), \\ f(\partial_t \widetilde{u}^h) \rightharpoonup \chi & \text{in } L^2\left(Q\right), \\ u^h \rightharpoonup u & \text{in } L^2\left(0,T;H_0^1(\Omega)\right). \end{cases}$$

$$\int_\Omega \left\{ \chi v + \nabla u \nabla v + \epsilon \nabla \partial_t u \nabla v \right\} \mathrm{dx} \; =< g, v > .$$

Problem

$\chi = f(\partial_t u) \quad (f = f_1 - f_2).$

One needs a.e. convergence for $\partial_t \widetilde{u}^h$, *i.e.* information on $\partial_{tt}^2 \widetilde{u}^h$?

Passing to the limits for a fixed t

$$\int_\Omega \left\{ f\left(\partial_t \widetilde{u}^h(t)\right) v + \nabla u^h(t) \nabla v + \epsilon \nabla \partial_t \widetilde{u}^h(t) \nabla v \right\} \mathrm{dx} \; = < g^h(t), v > .$$

$$\widetilde{u}^h \rightharpoonup u \quad \text{in } H^1\left(0, T; H_0^1(\Omega)\right),$$

$\sup_t \left\| \widetilde{u}^h(t) - u^h(t) \right\|_{H_0^1(\Omega)} \leq C\sqrt{h}$

$$u^h(t) \rightharpoonup u(t) \quad \text{in } H_0^1(\Omega),$$
$$g^h(t) \rightarrow g(t) \quad \text{in } L^2(\Omega).$$

$$\int_\Omega \left\{ f\left(\partial_t \widetilde{u}^h(t)\right) v + \nabla u^h(t) \nabla v + \epsilon \nabla \partial_t \widetilde{u}^h(t) \nabla v \right\} \mathrm{dx} \; = < g^h(t), v > .$$

$$\widetilde{u}^h \rightharpoonup u \quad \text{in } H^1\left(0, T; H_0^1(\Omega)\right),$$

$\sup_t \left\| \widetilde{u}^h(t) - u^h(t) \right\|_{H_0^1(\Omega)} \leq C\sqrt{h}$

$$u^h(t) \rightharpoonup u(t) \quad \text{in } H_0^1(\Omega),$$
$$g^h(t) \to g(t) \quad \text{in } L^2(\Omega).$$

t a.e., $\exists C(t), \quad ||\partial_t \widetilde{u}^h(t)||_{H_0^1(\Omega)} \leq C(t)$

$$\exists \xi(t) \in H_0^1(\Omega), \left(\partial_t \widetilde{u}^{h_t}(t)\right) \subset \left(\partial_t \widetilde{u}^h(t)\right),$$
$$\partial_t \widetilde{u}^{h_t}(t) \rightharpoonup \xi(t) \;\text{ in } H_0^1(\Omega).$$

$$\int_\Omega \left\{ f\left(\partial_t \widetilde{u}^h(t)\right) v + \nabla u^h(t)\nabla v + \epsilon \nabla \partial_t \widetilde{u}^h(t)\nabla v \right\} \mathrm{dx} \;=< g^h(t), v > .$$

$$\widetilde{u}^h \rightharpoonup u \quad \text{in } H^1\left(0, T; H_0^1(\Omega)\right),$$

$\sup_t \left\|\widetilde{u}^h(t) - u^h(t)\right\|_{H_0^1(\Omega)} \leq C\sqrt{h}$

$u^h(t) \rightharpoonup u(t) \quad \text{in } H_0^1(\Omega),$

$g^h(t) \to g(t) \quad \text{in } L^2(\Omega).$

t a.e., $\exists C(t), \quad ||\partial_t \widetilde{u}^h(t)||_{H_0^1(\Omega)} \leq C(t)$

$\exists \xi(t) \in H_0^1(\Omega), \left(\partial_t \widetilde{u}^{h_t}(t)\right) \subset \left(\partial_t \widetilde{u}^h(t)\right),$

$\partial_t \widetilde{u}^{h_t}(t) \rightharpoonup \xi(t) \;\text{ in } H_0^1(\Omega).$

$$\int_\Omega \left\{ f\left(\xi(t)\right) v + \nabla u(t)\nabla v + \epsilon \nabla \xi(t)\nabla v \right\} \mathrm{dx} \;=< g(t), v > .$$

The solution to such a problem is unique. Thus,

1. $\partial_t \widetilde{u}^h(t) \rightharpoonup \xi(t)$.
2. ξ is a weakly measurable function.
3. ξ is a measurable function (Th. Pettis).
4. $\partial_t \widetilde{u}^h$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and 1.
   $$\Longrightarrow \partial_t \widetilde{u}^h \rightharpoonup \xi \text{ in } L^2(0,T;H_0^1(\Omega)).$$
5. $\xi = \partial_t u$ and a solution exists.

Uniqueness : L^2-method.

Generalization

Seam Ngonn - G. V.

$$\exists u \in H^1(0,T;H_0^1(\Omega)) \quad \text{such that} \quad u(0,.) = u_0 \text{ in } \Omega \quad \text{and}$$

$$f(t,x,\partial_t u) - \operatorname{div}\Big[a(x,u,\partial_t u)\nabla u + b(x,u,\partial_t u)\nabla\partial_t u\Big] = g \quad \text{in } Q.$$

E. Emmrich - G. V.

Let V be a separable reflexive Banach space

$M : V \to V'$, $\quad A : V \times V \to V'$, $\quad f \in L^\infty(0,T,V')$,

$$\exists u \in W^{1,\infty}(0,T,V) \quad \text{such that } u(t=0) = u_0 \in V$$

$$M(\partial_t u) + A(u,\partial_t u) = f \text{ in } V',\ t \in (0,T)\ a.e.$$

Uniqueness : L^2 − method.

Existence : $V = \{u \in H^1\left(0, T, H_0^1(\Omega)\right),\ u(0,.) = 0\}$

$\|.\|_V : u \mapsto \|u\|_V = ||\nabla \partial_t u||_{L^2(]0,T[\times\Omega)}$

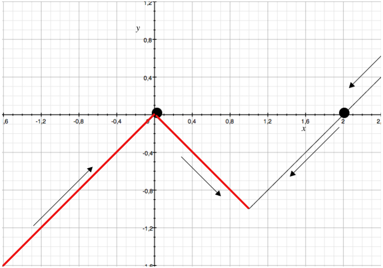
$\Psi : V \to V,\ S \mapsto \Psi(S) = u_S - u_0 :$

$$f_1(\partial_t u_S) - \Delta u_S - \epsilon_0 \Delta \partial_t u_S = g + f_2(\partial_t S)\,.$$

- $\exists C = C(g, supp(f_2'), \epsilon_0, u_0),\ \Psi(V) \subset \overline{B}_V(0, C)$.
- Ψ is non expansive.

Fixed point theorem of Browder.

Set :
$u(t,x) = \alpha(t)\varphi_1(x)$
and
φ_1 eigenfunction.



$$\varphi_1(x)\left[(f + \lambda_1 Id)(\dot{\alpha}(t)) + \alpha(t)\lambda_1\right] = 0$$

i.e.

$$g(\dot{\alpha}(t)) + \alpha(t) = 0$$

Existence of multiple solution of the ODE governing α.

The p-Laplace-Barenblatt equation

(With J. Giacomoni (Pau))

$$(P):\begin{cases} f(.,\partial_t u) - \Delta_p u - \epsilon\Delta\partial_t u = g & g\in L^2(Q),\\ u=0 & on\ \Sigma =]0,T[\times\Gamma,\\ u(0,.) = u_0 \in W_0^{1,p}(\Omega), & \end{cases}$$

where:

1. $\Delta_p u = Div\left(|\nabla u|^{p-2}\nabla u\right), \qquad \frac{2d}{d+2} < p.$
2. $|f(x,t)| \leq c_1(x) + c_2|t|, \quad \exists C < \epsilon\lambda_1,\ f(x,t)sgn(t) \geq -C|t| - c_1(x).$
3. $u \mapsto f(u) - \epsilon\Delta u$ is monotone.

Proposition

$\exists u \in W^{1,p,2}(0,T;W_0^{1,p}(\Omega),H_0^1(\Omega)),\ u(0)=u_0,\ t\ a.e.,\ \forall v\in W_0^{1,max(2,p)}(\Omega),$

$$\int_\Omega [f(.,\partial_t u)v\,dx + |\nabla u|^{p-2}\nabla u\nabla v - \epsilon\nabla\partial_t u\nabla v]\,dx = \int_\Omega gv\,dx\ .$$

Implicit time-discreatization

$X = W_0^{1,max(2,p)}(\Omega); \quad N > 0, h = \frac{T}{N}.$

Proposition

$\exists (u^k)_k \subset X, \quad u^0 = u_0 \quad and \quad \forall v \in X,$

$$\int_\Omega \left\{ f\left(\frac{u^{k+1} - u^k}{h}\right) v + |\nabla u^{k+1}|^{p-2} \nabla u^{k+1} \nabla v + \epsilon \nabla \frac{u^{k+1} - u^k}{h} \nabla v \right\} dx$$

$$= \int_\Omega g^{k+1} v \, dx \, ,$$

Proof

$$J : X \to \mathbb{R},$$

$$v \mapsto \Delta t \int_\Omega \int_0^{\frac{v-u^0}{\Delta t}} f(.,s) \mathrm{d}s \mathrm{d}x + \frac{1}{p} \int_\Omega |\nabla v|^p \mathrm{d}x + \Delta t \frac{\epsilon}{2} \int_\Omega |\nabla \frac{v - u^0}{\Delta t}|^2 \mathrm{d}x.$$

$\exists u \in W^{1,\infty,2}(0,T,W_0^{1,p}(\Omega),H_0^1(\Omega)) \quad \exists \chi \in L^2(0,T,L^2(\Omega)),$

- $\partial_t\widetilde{u}^{\Delta t} \rightharpoonup \partial_t u \quad in\ L^2(0,T,H_0^1(\Omega)),$
- $\widetilde{u}^{\Delta t},\ u^{\Delta t} \rightharpoonup u \quad in\ L^\infty(0,T,W_0^{1,p}(\Omega)) - *,$
- $\widetilde{u}^{\Delta t} -\ u^{\Delta t} \to 0 \quad in\ L^p(0,T,X),$
- $f(.,\partial_t\widetilde{u}^{\Delta t}) \rightharpoonup \chi \quad in\ L^2(0,T,L^2(\Omega)).$

$$\chi - ¿\ \Delta_p u? - \epsilon\Delta\partial_t u = g$$

$$¿\ \chi = f(.,\partial_t u)?$$

$$\int_Q g^{\Delta t}[\widetilde{u}^{\Delta t} - u]dx + C\Delta t = \int_Q f(.,\partial_t\widetilde{u}^{\Delta t})[\widetilde{u}^{\Delta t} - u]dx$$
$$- < \Delta_p\widetilde{u}^{\Delta t}, \widetilde{u}^{\Delta t} - u >$$
$$-\epsilon < \Delta\partial_t[\widetilde{u}^{\Delta t} - u], \widetilde{u}^{\Delta t} - u >$$

$p > \frac{2d}{d+2}$ and Simon's compactness, Δ linear ...

$$\Rightarrow \quad \limsup_{\Delta t} \int_0^T < -\Delta_p\widetilde{u}^{\Delta t}, \widetilde{u}^{\Delta t} - u > dt \leq 0$$

$$(S^+\text{-prop}) \Rightarrow \widetilde{u}^{\Delta t} \to u \text{ in } L^p(0,T,W^{1,p}(\Omega)).$$

$$\chi - \Delta_p u - \epsilon\Delta\partial_t u = g$$

$$\chi - \Delta_p u - \epsilon\Delta\partial_t u = g$$

and
$$\int_\Omega \chi\partial_t u dx + \epsilon\|\nabla\partial_t u\|_2^2 + \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p = \int_\Omega g\partial_t u dx$$

$f(.,\partial_t\widetilde{u}^{\Delta t}) - \Delta_p u^{\Delta t} - \epsilon\Delta\partial_t\widetilde{u}^{\Delta t} = g^{\Delta t}$

$$\chi - \Delta_p u - \epsilon\Delta\partial_t u = g$$

and

$$\int_\Omega \chi\partial_t u dx + \epsilon\|\nabla\partial_t u\|_2^2 + \frac{1}{p}\frac{d}{dt}\|\nabla u\|_p^p = \int_\Omega g\partial_t u dx$$

$$\int_\Omega f(.,\partial_t\widetilde{u}^{\Delta t})\partial_t\widetilde{u}^{\Delta t}dx + \epsilon\|\nabla\partial_t\widetilde{u}^{\Delta t}\|_2^2 + \frac{1}{p}\frac{d}{dt}\|\nabla\widetilde{u}^{\Delta t}\|_p^p \leq \int_\Omega g^{\Delta t}\partial_t\widetilde{u}^{\Delta t}dx$$

and

$$\limsup_{\Delta t}\int_Q f(.,\partial_t\widetilde{u}^{\Delta t})\partial_t\widetilde{u}^{\Delta t}dx + \epsilon\|\nabla\partial_t\widetilde{u}^{\Delta t}\|_2^2 \leq \int_\Omega \chi\partial_t u dx + \epsilon\|\nabla\partial_t u\|_2^2$$

$$f(.,\partial_t u) - \Delta_p u - \epsilon\Delta\partial_t u = g.$$

The p(x)-Laplace-Barenblatt equation

(With J. Giacomoni)

$$(P):\begin{cases} f(.,\partial_t u)-\Delta_{\overrightarrow{p}}u=g & in\ Q=]0,T[\times\Omega,\\ u=0 & on\ \Sigma=]0,T[\times\Gamma,\\ u(0,.)=u_0 & in\ \Omega,\end{cases}$$

where:

1. $\Delta_{\overrightarrow{p}}u=\sum_{i=1}^{d}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\right)$
2. $\overrightarrow{p}=(p_i)_{i=1,..,d}:\Omega\mapsto]p^-,p^+[\subset]\frac{2d}{d+2},\infty[$ log-continuous
3. f is non-decreasing w. r. t. λ and
$$\exists b_1\in L^1(\Omega),\ b_2\geq 0,\qquad \lambda f(.,\lambda)\geq b_2|\lambda|^2-b_1$$
4. $g\in L^2(Q)$ and $u_0\in W_0^{1,\overrightarrow{p}}(\Omega)$

$$L^{p_i(.)} = \left\{ u : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega |u(x)|^{p_i(x)} \text{ dx } < \infty \right\}.$$

$$W_0^{1,\overrightarrow{p}}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) \,/\, i = 1,..,d \ \partial_{x_i} u \in L^{p_i(x)}(\Omega) \right\}.$$

Endowed with the Luxembourg norm :

$$\|u\|_{L^{p_i(.)}} = \inf \left\{ \lambda > 0 / \ \rho_{p_i(.)}(\frac{u}{\lambda}) \leq 1 \right\} \text{ where } \rho_{p_i(.)}(f) = \int_\Omega |f|^{p_i(x)}\, dx,$$

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = \sum_{i=1}^{d} \|\partial_{x_i} u\|_{L^{p_i(.)}} \quad or \quad \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = [\sum_{i=1}^{d} \|\partial_{x_i} u\|^r_{L^{p_i(.)}}]^{\frac{1}{r}}.$$

Proposition

$\exists u \in H^1(Q) \cap L^\infty(0,T; W_0^{1,\overrightarrow{p}}(\Omega))$, $u(0) = u_0$, t a.e., $\forall v \in W_0^{1,\overrightarrow{p}}(\Omega)$,

$$\int_\Omega \left[f(.,\partial_t u) v \, dx + \sum_{i=1}^{d} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v \right] dx = \int_\Omega g v \, dx$$

proof

Time-discretization: $f(.,\partial_t \tilde{u}^h) - \Delta_{\overrightarrow{p}} u^h = g^h$.

$$\min_{u \in W_0^{1,\overrightarrow{p}}(\Omega)} E(u) = \Delta t \int_\Omega F(\cdot, \frac{u-u_0}{\Delta t})\,\mathrm{d}x + \sum_{i=1}^{d} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\,\mathrm{d}x - \int_\Omega g u\,\mathrm{d}x,$$

where $F(.,\lambda) = \int_0^\lambda f(.,\sigma)d\sigma$.

a priori estimates, compactness (J. Simon), S^+ prop. of $\Delta_{\overrightarrow{p}}$

$\Rightarrow$ convergence of u^h in $L^2(0,T;W_0^{1,\overrightarrow{p}}(\Omega))$;

monotonocity argument $(f) \Rightarrow$ convergence in $L^2(Q)$.

Theorem

If $p(x) \in [p^-, p^+] \subset]1, +\infty[$, $L^{p(x)}$ *is uniformly convex.*

Corollary

$$(f_n), f \subset L^{p(x)}(\Omega); \qquad f_n \rightharpoonup f \text{ in } L^{p(x)}(\Omega),$$
$$\text{or} \quad f_n \to f \text{ a.e. in } \Omega.$$

Then,

$$\rho_p(f_n) \to \rho_p(f) \; \Rightarrow \; f_n \to f \text{ in } L^{p(x)}(\Omega).$$

$$\rho_p(f) = \int_\Omega |f|^{p_i(x)} dx.$$

Lemma (Morawetz)

$p \in [p_*, p^*] \subset]1, \infty[$.

Then there exist constants $M > 1$, *independent of* p *and* $C(p)$ *defined as follows*

$$C(p) = \begin{cases} \frac{M}{2^p}\left(\frac{1}{1-2^{1-p}}\right)^{\frac{p}{2}} \text{ if } p < 2 \\ \frac{1}{2} \text{ if } p \geq 2 \end{cases}$$

such that $\forall\, a,\, b \in \mathbb{R}$,

$$\left|\frac{a-b}{2}\right|^p \leq C(p)\left(|a|^p + |b|^p\right)^{1-s}\left(|a|^p + |b|^p - 2\left|\frac{a+b}{2}\right|^p\right)^s$$

where $s = \min\left(1, \frac{p}{2}\right)$.

Monotonicity argument

Lemma

Consider $T > 0$, $Q =]0, T[\times\Omega$, $u_0 \in W_0^{1,\overrightarrow{p}}(\Omega)$, $g \in L^2(Q)$.

If $u \in H^1(Q) \cap C_w([0,T], W_0^{1,\overrightarrow{p}}(\Omega))$ *is the solution of*

$$\partial_t u - \Delta_{\overrightarrow{p}} u = g \text{ in } Q, \quad \text{with } u(0,.) = u_0.$$

Then, $u \in C([0,T], W_0^{1,\overrightarrow{p}}(\Omega))$ *and* $\forall t \in [0,T]$,

$$\int_{]0,t[\times\Omega} |\partial_t u|^2 \,\mathrm{d}x\mathrm{d}\sigma + \sum_{i=1}^{d} \int_\Omega \frac{1}{p_i(x)} |\nabla u(t)|^{p_i(x)}\, dx$$

$$= \sum_{i=1}^{d} \int_\Omega \frac{1}{p_i(x)} |\nabla u_0|^{p_i(x)}\, dx + \int_{]0,t[\times\Omega} g\partial_t u \,\mathrm{d}x\mathrm{d}\sigma.$$

The stochastic Barenblatt equation

(With C. Bauzet (Pau))

$$(P) : \begin{cases} f(\partial_t u) - \Delta u = 0 & \text{in } Q =]0, T[\times \Omega, \\ u = 0 & \text{on } \Sigma =]0, T[\times \Gamma, \\ u(0, .) = u_0 & \text{in } \Omega, \end{cases}$$

The stochastic Barenblatt equation

$$(P) : \begin{cases} \partial_t u - f^{-1}(\Delta u) = 0 & \textit{in } Q =]0,T[\times\Omega, \\ u = 0 & \textit{on } \Sigma =]0,T[\times\Gamma, \\ u(0,.) = u_0 & \textit{in } \Omega, \end{cases}$$

The stochastic Barenblatt equation

$$(P):\begin{cases} du - f^{-1}(\Delta u)dt = hdw & in\ Q =]0,T[\times\Omega, \\ u = 0 & on\ \Sigma =]0,T[\times\Gamma, \\ u(0,.) = u_0 & in\ \Omega, \end{cases}$$

where:

1. $(\Theta, \mathcal{F}, P)$ is a probability space, $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ a standard 1-D Brownien motion associated to $(\mathcal{F}_t)$, s.t. $w_0 = 0$.
2. $u_0 \in H_0^1(\Omega)$, $h \in \mathcal{N}_W^2(0,T,H_0^1(\Omega))$ $= L^2(]0,T[\times\Theta, H_0^1(\Omega))$-predictable.
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i.e.

$$f\left(\partial_t[u - \int_0^t hdw]\right) - \Delta u = 0 \quad in\ Q =]0,T[\times\Omega,$$

Proposition

$\forall h \in \mathcal{N}_W^2(0, T, H_0^1(\Omega) \cap H^2(\Omega)),\ \exists! u \in \mathcal{N}_W^2(0, T, H_0^1(\Omega))$ *solution of (P).*

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Set $U = u - \int_0^t h dw$,

$$f\left(\partial_t U\right)-\Delta U=\int_0^t \Delta h\, dw = g$$

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$\forall h \in \mathcal{N}_W^2(0,T,H_0^1(\Omega)\cap H^2(\Omega))$, $\exists! u \in \mathcal{N}_W^2(0,T,H_0^1(\Omega))$ *solution of (P).*

$h, \hat{h} \in \mathcal{N}_w^2(H_0^1(\Omega)\cap H^2(\Omega))$, $u_0, \hat{u}_0 \in H_0^1(\Omega)$,

$$\begin{aligned} & E\int_{]0,t[\times\Omega} (\partial_t(U-\hat{U}))^2 \mathrm{d}x\mathrm{d}s + \frac{1}{2}E||\nabla(u-\hat{u})(t)||^2 \\ \leq \quad & \frac{1}{2}E||\nabla(u_0-\hat{u}_0)||^2 + E\int_{]0,t[\times\Omega} |\nabla(h-\hat{h})|^2 \mathrm{d}x\mathrm{d}s. \end{aligned}$$

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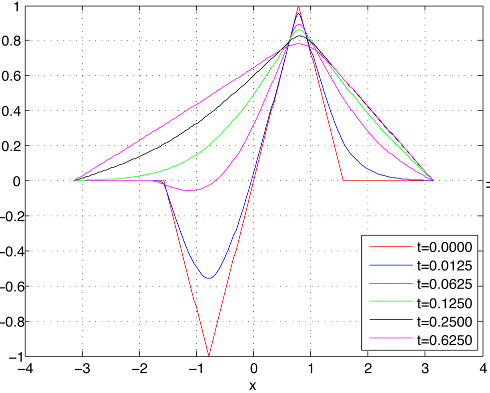
$H : H_0^1(\Omega) \to H_0^1(\Omega)$ Lipschitz:

There exists a unique solution of

$$f\left(\partial_t[u - \int_0^t H(u) dw]\right) - \Delta u = 0.$$

$f(\partial_t u) - \Delta u - \epsilon \Delta \partial_t u = 0$ **where** $f(s) = \frac{s^+}{10} - 10 s^-$



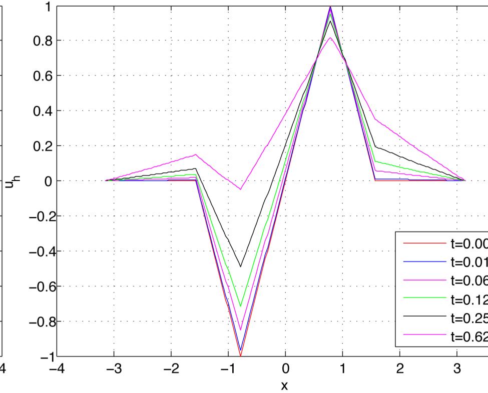


Figure: $\epsilon = 0$ and $\epsilon = 0.5$

$f(\partial_t u) - \Delta u - \epsilon \Delta \partial_t u = 0$ **where** $f(s) = \dfrac{s^+}{10} - 10 s^-$

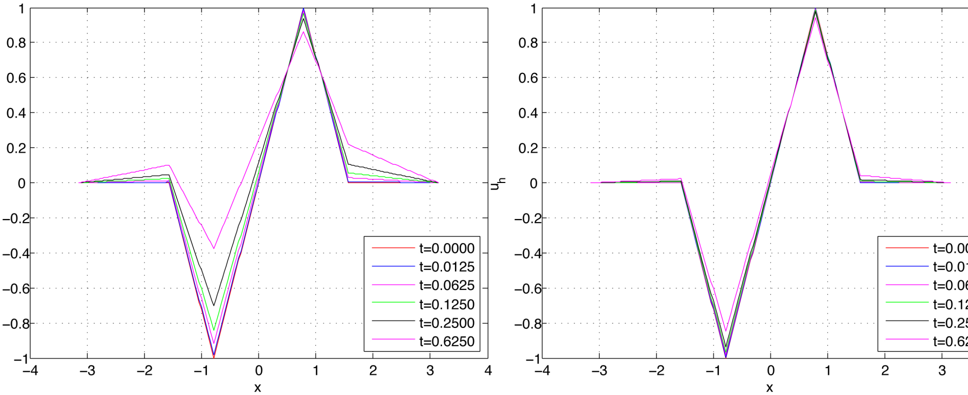


Figure: $\epsilon = 1$ and $\epsilon = 5$

$f(\partial_t u) - \Delta u - \epsilon \Delta \partial_t u = 0$ **where** $f(s) = \dfrac{s^+}{10} - 10 s^-$