REMARKS ON THE UNIQUENESS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH QUADRATIC GROWTH CONDITIONS

DAVID ARCOYA, COLETTE DE COSTER, LOUIS JEANJEAN,
AND KAZUNAGA TANAKA

ABSTRACT. In this note we present some uniqueness and comparison results for a class of problem of the form

$$(0.1) \qquad -Lu = H(x, u, \nabla u) + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain, L is a general elliptic second order linear operator with bounded coefficients and H is allowed to have a critical growth in the gradient. In some cases our assumptions prove to be sharp.

1. INTRODUCTION

For a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) and a function $h \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we consider the problem

$$(1.1) \qquad -Lu = H(x, u, \nabla u) + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where L is a general elliptic second order linear operator and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfy the assumptions:

(L) There exists a family of functions $(a^{ij})_{1 \leq i,j \leq N}$ with $a^{ij} \in L^\infty(\Omega) \cap W_{loc}^{1,\infty}(\Omega)$ such that

$$Lu = \sum_{i,j} \frac{\partial}{\partial x_j} \left(a^{ij}(x) \frac{\partial u}{\partial x_i} \right)$$

and, there exists $\eta > 0$ such that, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$,

$$\sum_{i,j} a^{i,j}(x) \xi_i \xi_j \geq \eta |\xi|^2.$$

(H1) There exists a continuous function $C_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $b_1 \in L^p(\Omega)$ such that, for a.e. $x \in \Omega$, all $u \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$,

$$|H(x, u, \xi)| \leq C_1(|u|)(|\xi|^2 + b_1(x)).$$

(H2) There exists a function $b_2 \in L_{loc}^N(\Omega)$ and a continuous function $C_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that, for a.e. $x \in \Omega$, all $u_1, u_2 \in \mathbb{R}$ with $u_1 \geq u_2$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$,

$$H(x, u_1, \xi_1) - H(x, u_2, \xi_2) \leq C_2(|u_1|, |u_2|)(|\xi_1| + |\xi_2| + b_2(x))|\xi_1 - \xi_2|.$$

As we shall see in the proof of Corollary 2.1, a sufficient condition for (H2) is that for a.e. $x \in \Omega$, $H(x, \cdot, \cdot) \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^N)$ with

$$(1.2) \qquad \frac{\partial H}{\partial u}(x, u, \xi) \leq 0, \text{ a.e. } x \in \Omega, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N,$$

and that there exists a function $b_3 \in L_{loc}^N(\Omega)$ and a continuous nondecreasing function $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$(1.3) \qquad \left| \frac{\partial H}{\partial \xi}(x, u, \xi) \right| \leq C(|u|)(|\xi| + b_3(x)), \text{ a.e. } x \in \Omega, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N.$$

Uniqueness of solution for problem (1.1) (with $Lu = \Delta u$) has been first studied in the work [4] and after improved in [3] by requiring weaker regularity conditions on the data. The reader can also see additional uniqueness results in [5] for subcritical nonlinear term H (with respect to ξ), i.e, when its growth is less than a power $|\xi|^q$ with $q < 2$, and in the work [2] for the case that H has a singularity at $u = 0$.

Specifically, in [3] the uniqueness of solution for every h is proved when it is assumed condition (1.3) and the following strengthening of (1.2):

$$\frac{\partial H}{\partial u}(x, u, \xi) \leq -d_0 < 0, \text{ a.e. } x \in \Omega, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N.$$

However, in the case that it is only assumed the general hypothesis (1.2) (together with (1.3)), the authors require to the function h to be sufficiently small in an appropriate sense. Furthermore, adapting the arguments of [3], the case where (1.2)-(1.3) hold and h has sign can also be covered. Nevertheless, the treatment of the general case (1.2)-(1.3) with no assumptions on h seems out of reach with the approach of [3, 4].

The special case of (1.1) given by

$$(1.4) \qquad -\Delta u = d(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

is studied in [1] by an alternative approach. Indeed, if $d, h \in L^p(\Omega)$ for some $p > \frac{N}{2}, \mu \in L^\infty(\Omega)$, then it is proved that (1.4) has at most one solution as soon as $d \leq 0$. Actually this condition is also necessary since [1, Theorem 1.3] proves that (1.4) may have two solutions if $d \gneqq 0$. See also, in that direction, Theorem 2 in [13] or Theorem 1,(iv) in [16]. We also mention that a general condition which guarantees the existence of one solution to (1.4) is derived in [1]. This condition always hold when $d < 0$ but also widely when $d \leq 0$. For example, we have existence of one solution whenever μ and h have opposite sign.

The aim of this paper is to show that the approach of [1] can be generalized to treat, under the assumptions (L)-(H1)-(H2), equation (1.1) and thus to cover additional situations where the approach of [3, 4] is not applicable. As a counterpart of our approach we need to assume that the boundary of Ω is sufficiently smooth, namely that Ω satisfies the following condition (A) of [14, p.6].

Definition 1.1. Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that Ω satisfies condition (A) provided there exist $r_0, \theta_0 > 0$ such that if $x \in \partial\Omega$ and $0 < r < r_0$, then

$$\text{meas}\,\Omega_r \leq (1 - \theta_0)\text{meas}\,B_r(x),$$

for every component Ω_r of $\Omega \cap B_r(x)$, where $B_r(x)$ denotes the ball of radius r centered at the point x.

For the reader convenience, we shall prove in the Appendix (see Lemma A.1), that condition (A) holds true if $\partial\Omega$ is Lipschitz.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that (L)-(H1)-(H2) hold and that Ω satisfies condition (A). Then (1.1) has at most one solution.*

The proof of Theorem 1.1 is divided into two main steps. First we show that any solution of (1.1) belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,N}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. This result is obtained combining classical regularity arguments from [11, 14] which allow to conclude that it belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,q}_{loc}(\Omega)$ for some $\alpha \in (0,1)$ and some $q > 2$. Here the fact that our solutions are bounded seems essential. Then a bootstrap argument of [6, 10] (see also [9]) comes into play. The key ingredient of this bootstrap is an interpolation result due to Miranda [15] which says that any element of $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{2,\frac{q}{2}}_{loc}(\Omega)$ belongs to $W^{1,t}_{loc}(\Omega)$ for a $t > q$. Having obtained the announced required regularity the second step consists in establishing a comparison principle. Roughly speaking, we adapt an argument from [7], based in turn on an original idea from [8] to show in Lemma 2.2 that if $u_1, u_2 \in H^1_0(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are respectively a lower solution and an upper solution of (1.1), then $u_1 \leq u_2$ in Ω. The uniqueness then follows from this comparison principle.

As we already mentioned Theorem 1.1 is sharp for equation (1.4). More globally the condition (H2), which in essence express the fact that $u \to H(x, u, \xi)$ is a non decreasing function and $\xi \to H(x, u, \xi)$ is locally Lipschitz, appears to us as an almost necessary condition to guarantee the uniqueness.

Throughout the rest of the note we assume that $N \geq 3$. The easier case $N = 2$ is left to the reader.

2. Uniqueness results

First we show that, when condition (A) holds, any solution of (1.1) belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,q}_{loc}(\Omega)$ for some $\alpha \in (0,1)$ and $q > 2$. This follows directly from the following two classical regularity results.

Proposition 2.1. *Assume that Ω satisfies the regularity condition (A) and that condition (L) holds. Let u be a solution of*

$$-Lu + a(x, u, \nabla u) = 0, \quad u \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

If there exists a constant $\mu > 0$ and a function $b_1 \in L^p(\Omega)$ with $p > \frac{N}{2}$ such that

$$|a(x, u, \xi)| \le \mu \left[|\xi|^2 + b_1(x) \right], \quad a.e. \ x \in \Omega, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N,$$

then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

Proof. This result is a special case of [14, Theorem IX-2.2 p.441]. □

Proposition 2.2. *Assume that L satisfies condition (L). Let u be a solution of*

$$-Lu + a(x, u, \nabla u) = 0, \quad u \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

If there exists a $\mu > 0$ and a function $b_1 \in L^s(\Omega)$ for some $s > 1$ such that

$$|a(x, u, \xi)| \le \mu \left[|\xi|^2 + b_1(x) \right], \quad a.e. \ x \in \Omega, \ \forall u \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^N,$$

then there exists an exponent $q > 2$ such that $u \in W^{1,q}_{loc}(\Omega)$.

Proof. This result is a special case of [11, Proposition 2.1, p.145]. □

Clearly Propositions 2.1 and 2.2 apply to the solutions of (1.1).

The information that an arbitrary solution of (1.1) belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,q}_{loc}(\Omega)$ for some $\alpha \in (0,1)$ and $q > 2$ is the starting point of a bootstrap argument which relies on the following interpolation result due to C. Miranda [15].

Proposition 2.3. *Let $\omega \subset \mathbb{R}^N$ be a bounded domain satisfying the cone property. Assume that $0 \le \alpha < 1$, $p \ge 1$ and let*

$$t = \frac{p(2 - \alpha) - \alpha}{1 - \alpha}.$$

Then any element of $\mathcal{C}^{0,\alpha}(\omega) \cap W^{2,p}(\omega)$ belongs to $W^{1,t}(\omega)$.

Proof. This result is [15, Teorema IV]. □

Gathering Propositions 2.1, 2.2 and 2.3 we obtain

Lemma 2.1. *Assume that conditions (L) and (H1) hold and that Ω satisfies condition (A). Then any solution of (1.1) belongs to $\mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,N}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

Proof. Let u be an arbitrary solution of (1.1). As u is bounded, we are in position to apply Propositions 2.1 and 2.2 with $\mu = \max_{[-\|u\|_\infty, \|u\|_\infty]} C_1(|u|)$. Then Propositions 2.1 and 2.2 implies that $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \cap W^{1,q}_{loc}(\Omega)$ for some $\alpha \in (0,1)$ and $q > 2$. If $q \geq N$ then the proof is done, while if $q < N$ we follow a bootstrap argument of [6, 10], see also [9]. Since $u \in W^{1,q}_{loc}(\Omega)$, the function u is a weak solution of

$$(2.1) \qquad\qquad\qquad -Lu = \xi(x),$$

with $\xi(x) = H(x,u,\nabla u) + h(x) \in L^{\frac{q}{2}}_{loc}(\Omega)$, by $(H1)$. By a standard L^p-regularity argument, see for example [17, Theorem 3.8], we deduce that $u \in W^{2,\frac{q}{2}}_{loc}(\Omega)$. Now using Proposition 2.3 which is valid on any regular domain $\omega \subset \Omega$ it follows that

$$u \in W^{1,t_1}_{loc}(\Omega) \quad \text{where} \quad t_1 = \frac{\frac{q}{2}(2-\alpha) - \alpha}{1-\alpha} > q.$$

If $t_1 \geq N$ we are again done. Otherwise from (2.1) and the cited classical regularity argument, $u \in W^{2,\frac{t_1}{2}}_{loc}(\Omega)$. Denoting

$$(2.2) \qquad\qquad t_n = \frac{\frac{t_{n-1}}{2}(2-\alpha) - \alpha}{1-\alpha} > t_{n-1} > q > 2$$

by a bootstrap argument we get $u \in W^{2,\frac{t_n}{2}}_{loc}(\Omega)$ for all $n \in \mathbb{N}$ as long as $t_{n-1} < N$. We now claim that the increasing sequence $\{t_n\}$ exceeds the value N. Indeed, arguing by contradiction, if $t_n < N$ for every $n \in \mathbb{N}$, then the limit l of $\{t_n\}$ has to be $l = 2$. This contradicts that $t_n > q > 2$. At this point the proof of the lemma is completed. $\qquad\square$

The motivation to observe that any solution of (1.1) has an additional regularity appears in the next comparison principle in $H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Recall that u_1 is a *lower solution of* (1.1) if $u_1^+ \in H^1_0(\Omega)$ and, for all $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$ a.e. $x \in \Omega$, we have

$$\sum_{i,j=1}^N \int_\Omega a^{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \leq \int_\Omega H(x, u_1, \nabla u_1)\, \varphi + \int_\Omega h\, \varphi.$$

In the same way, u_2 is an *upper solution of* (1.1) if $u_2^- \in H^1_0(\Omega)$ and, for all $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$ a.e. $x \in \Omega$, we have

$$\sum_{i,j=1}^N \int_\Omega a^{ij} \frac{\partial u_2}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \geq \int_\Omega H(x, u_2, \nabla u_2)\, \varphi + \int_\Omega h\, \varphi.$$

Lemma 2.2. *Assume that the hypotheses (L) and (H2) hold. Then if $u_1, u_2 \in H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are respectively a lower solution and an upper solution of (1.1), then $u_1 \leq u_2$ in Ω.*

Proof. Here we adapt an argument from [7], based in turn on an original idea from [8]. Consider the function $v = u_1 - u_2$, which satisfies

(2.3)
$$\begin{aligned} -Lv &\leq H(x, u_1, \nabla u_1) - H(x, u_2, \nabla u_2), && \text{in } \Omega, \\ v &\leq 0, && \text{on } \partial\Omega, \\ v &\in H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega}). \end{aligned}$$

For every $c \in \mathbb{R}$, let us consider the set $\Omega_c = \{x \in \Omega : |v(x)| = c\}$ and

$$J = \{c \in \mathbb{R} : \operatorname{meas} \Omega_c > 0\}.$$

As $|\Omega|$ is finite, J is at most countable and, since for all $c \in \mathbb{R}$, $\nabla v = 0$ a.e. on Ω_c, we also have

(2.4)
$$\nabla v = 0 \text{ a.e. in } \bigcup_{c \in J} \Omega_c.$$

Define $Z = \Omega \setminus \bigcup_{c \in J} \Omega_c$ and, for all $k \geq 0$, choose $\varphi = (v - k)^+ \in H^1_0(\Omega) \cap L^\infty(\Omega)$ as test function in (2.3), to deduce by condition (L) that

$$\eta \left\| \nabla (v - k)^+ \right\|^2_{L^2(\Omega)} \leq \int_\Omega \left(H(x, u_1, \nabla u_1) - H(x, u_2, \nabla u_2) \right) (v - k)^+ \, dx.$$

Let $A_k = \{x \in \Omega : v(x) \geq k\}$. By (2.4), (H2) and $\|u_1\|_\infty \leq R$, $\|u_2\|_\infty \leq R$ for some $R > 0$ we obtain a constant $M > 0$ such that

$$\begin{aligned} \eta \left\| \nabla (v - k)^+ \right\|^2_{L^2(\Omega)} &\leq \int_{A_k} \left(H(x, u_1, \nabla u_1) - H(x, u_2, \nabla u_2) \right) (v - k)^+ \, dx \\ &\leq M \int_{A_k \cap Z} (|\nabla u_1| + |\nabla u_2| + b_2) |\nabla v| (v - k)^+ \, dx \\ &= M \int_{A_k \cap Z} (|\nabla u_1| + |\nabla u_2| + b_2) |\nabla (v - k)^+| (v - k)^+ \, dx. \end{aligned}$$

Since $v \in \mathcal{C}(\overline{\Omega})$ and $v \leq 0$ on $\partial\Omega$, we have that $(v - k)^+$ has a compact support in Ω, for all $k > 0$, and hence $(|\nabla u_1| + |\nabla u_2| + b(x)) \in L^N(A_k \cap Z)$. This implies that

(2.5)
$$\begin{aligned} \eta \left\| \nabla (v - k)^+ \right\|^2_{L^2(\Omega)} &\leq M \| |\nabla u_1| + |\nabla u_2| + b \|_{L^N(A_k \cap Z)} \| \nabla (v - k)^+ \|_{L^2(\Omega)} \| (v - k)^+ \|_{L^{2^*}(\Omega)} \\ &\leq \mathcal{S}_N^{-1} M \| |\nabla u_1| + |\nabla u_2| + b \|_{L^N(A_k \cap Z)} \| \nabla (v - k)^+ \|^2_{L^2(\Omega)}, \end{aligned}$$

where $2^* = 2N/(N-2)$ and $\mathcal{S}_N$ denotes the Sobolev constant.

We want to prove that $v \leq 0$. Indeed, assume by contradiction that $v^+ \not\equiv 0$ and consider the non-increasing function F defined on $]0, \|v^+\|_\infty]$ by

$$F(k) = \mathcal{S}_N^{-1} M \| |\nabla u_1| + |\nabla u_2| + b \|_{L^N(A_k \cap Z)}, \quad \forall 0 < k < \|v^+\|_\infty$$

and $F(\|v^+\|_\infty) = 0$. By definition of Z we have that F is continuous and we can choose $0 < k_0 < \|v^+\|_\infty$ such that $F(k_0) < \eta$. By (2.5), $\eta \| \nabla (v - k_0)^+ \|^2_{L^2(\Omega)} \leq F(k_0) \| \nabla (v - k_0)^+ \|^2_{L^2(\Omega)}$, which implies that $\| \nabla (v - k_0)^+ \|_{L^2(\Omega)} = 0$, i.e. $v \leq k_0 <$

$\|v^+\|_\infty$, a contradiction proving that necessarily $v^+ = 0$ and hence $u_1 \le u_2$. This concludes the proof. $\qquad\square$

Proof of Theorem 1.1. Let u_1 and u_2 be two solutions of (1.1). By Lemma 2.1 we know that u_1 and u_2 belong to $\mathcal{C}(\overline{\Omega}) \cap W^{1,N}_{loc}(\Omega)$. Thus it follows from Lemma 2.2 that $u_1 = u_2$. $\qquad\square$

Remark 2.1. In the proof of Theorem 1.1 the requirement that Ω satisfies condition (A) is used to show that any solution of (1.1) belongs to $\mathcal{C}(\overline{\Omega})$. In turn this property is used only in Lemma 2.2 to guarantee that for any solution u of (1.1) the set $A_k = \{x \in \Omega : u(x) \ge k\}$ is compact for any $k > 0$. It is an open question if the conclusion of Theorem 1.1 holds true without assumption (A).

As a corollary of Lemma 2.2 and Theorem 1.1 we obtain the following result which, under the condition (A), improves the results in [3, 4] concerning (1.1).

Corollary 2.1. *Assume that Ω satisfies condition (A) and that condition (L) holds. Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying (H1) and such that, for a.e. $x \in \Omega$, $H(x, \cdot, \cdot) \in \mathcal{C}^1(\mathbb{R}^{N+1})$ with*

$$\frac{\partial H}{\partial u}(x, u, \xi) \le 0, \;\; a.e. \; x \in \Omega, \; \forall u \in \mathbb{R}, \; \forall \xi \in \mathbb{R}^N.$$

Assume moreover there exists a function $b_3 \in L^N_{loc}(\Omega)$ and a continuous nondecreasing function $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left| \frac{\partial H}{\partial \xi}(x, u, \xi) \right| \le C(|u|)(|\xi| + b_3(x)), \;\; a.e. \; x \in \Omega, \; \forall u \in \mathbb{R}, \; \forall \xi \in \mathbb{R}^N.$$

If $u_1, u_2 \in H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are respectively a lower solution and an upper solution of (1.1), then $u_1 \le u_2$ in Ω. In particular, (1.1) has at most one solution.

Proof. We just have to prove that (H2) holds. Let $x \in \Omega$, $u_1, u_2 \in \mathbb{R}$ with $u_1 \ge u_2$ and $\xi_1, \xi_2 \in \mathbb{R}^N$. Define the function $F(t) = H(x, tu_1 + (1-t)u_2, t\xi_1 + (1-t)\xi_2)$,

for every $t \in [0,1]$. Observe that $F \in \mathcal{C}^1(\mathbb{R})$. Moreover we have

$$
\begin{aligned}
H(x, u_1, \xi_1) - H(x, u_2, \xi_2) &= F(1) - F(0) \\
&= \int_0^1 \frac{d}{dt} F(t)\, dt \\
&= \int_0^1 \frac{\partial H}{\partial u}(x, tu_1 + (1-t)u_2, t\xi_1 + (1-t)\xi_2)\, dt\, (u_1 - u_2) \\
&\quad + \int_0^1 \frac{\partial H}{\partial \xi}(x, tu_1 + (1-t)u_2, t\xi_1 + (1-t)\xi_2) \cdot (\xi_1 - \xi_2)\, dt \\
&\leq \int_0^1 \left| \frac{\partial H}{\partial \xi}(x, tu_1 + (1-t)u_2, t\xi_1 + (1-t)\xi_2) \right|\, dt\, |\xi_1 - \xi_2| \\
&\leq \int_0^1 C_0(|tu_1 + (1-t)u_2|)\, [|t\xi_1 + (1-t)\xi_2| + b_3(x)]\, dt\, |\xi_1 - \xi_2| \\
&\leq C_0(|u_1| + |u_2|)\, [|\xi_1| + |\xi_2| + b_3(x)]\, |\xi_1 - \xi_2|.
\end{aligned}
$$

This proves that (H2) is valid and we can apply Lemma 2.2 and Theorem 1.1 to conclude. $\qquad \square$

Appendix A. Sufficient conditions for condition (A)

We prove in this section that Ω satisfies condition (A) whenever $\partial\Omega$ is Lipschitz. Recall that, by [12, Theorem 1.2.2.2], $\partial\Omega$ is Lipschitz if and only if the uniform cone condition is satisfied.

Definition A.1. ([12, Definition 1.2.2.1, p.10]) Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω satisfies the *uniform cone property* if, for every $x \in \partial\Omega$, there exists a neighbourhood V of x in $\mathbb{R}^N$ and new coordinates $\{y_1, \ldots, y_N\}$ such that
(a) V is a hypercube in the new coordinates, i.e.,

$$ V = \{(y_1, \ldots, y_N) \in \mathbb{R}^N \mid -a_i < y_i < a_i,\ 1 \leq i \leq N\}, $$

for some $a_i > 0$, $i = 1, 2, \ldots, N$;
(b) $y - z \in \Omega$ whenever $y \in \overline{\Omega} \cap V$ and $z \in C$, where C is the open cone $\{z = (z', z_n) \mid (\cot\theta)|z'| < z_n < h\}$ for some $\theta \in\,]0, \pi/2]$ and some $h > 0$.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If $\partial\Omega$ is Lipschitz, then Ω satisfies the condition (A).*

Proof. As it has been mentioned we can assume that $\partial\Omega$ satisfies the uniform cone condition. Arguing as in the proof of [12, Theorem 1.2.2.2], we know that $\{x\} - C \subset \Omega$ but we can also observe that $\{x\} + C \subset \Omega^c$, at least if the distance from x to V^c is greater that $h/\cos\theta$; this last condition can always be achieved by choosing a smaller h. Indeed, if $(\{x\}+C)\cap\overline{\Omega}$ is not empty, let y be a point in the intersection. Then $y \in \overline{\Omega} \cap V$ since $|y_n - x_n| < h$ and consequently, $\{y\} - C \subset \Omega$, but this contradicts the fact that $x \in \{y\} - C$.

For every $r > 0$, we have that $\Omega \cap B_r(x) \subset B_r(x) \setminus (B_r(x) \cap (\{x\} + C))$ and hence

$$
\begin{aligned}
\text{meas}\,(\Omega \cap B_r(x)) \;&\leq\; \text{meas}\,B_r(x) - \text{meas}\,(B_r(x) \cap (\{x\} + C)) \\
&=\; \text{meas}\,B_r(x)\left(1 - \frac{\text{meas}\,(B_r(x) \cap (\{x\} + C))}{\text{meas}\,B_r(x)}\right) \\
&=\; \text{meas}\,(B_r(0) \cap C).
\end{aligned}
$$

If $r \leq h$, the cone $C(\theta, r\cos\theta)$ of vertex 0, opening angle θ and height $r\cos\theta$ satisfies

$$
C(\theta, r\cos\alpha) \subset (B_r(0) \cap C).
$$

Hence,

$$
\frac{1}{N+1}\left(\text{meas}_{\mathbb{R}^{N-1}}B_{\mathbb{R}^{N-1}}(r\sin\theta)\right)r\cos\theta \leq \text{meas}\,(B_r(0) \cap C),
$$

and therefore

$$
\begin{aligned}
\text{meas}\,(\Omega \cap B_r(x)) &\leq \text{meas}\,B_r(x)\left(1 - \frac{\text{meas}\,(B_r(0) \cap C)}{\text{meas}\,B_r(0)}\right) \\
&\leq \text{meas}\,B_r(x)\left(1 - \frac{\frac{1}{N+1}\frac{\pi^{(N-1)/2}}{\Gamma(\frac{N-1}{2}+1)}(r\sin\theta)^{N-1}r\cos\theta}{\frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}r^N}\right) \\
&= \text{meas}\,B_r(x)\left(1 - \frac{\Gamma(\frac{N}{2}+1)(\sin\theta)^{N-1}\cos\theta}{(N+1)\pi^{1/2}\,\Gamma(\frac{N-1}{2}+1)}\right),
\end{aligned}
$$

i.e., condition (A) holds with $r_0 = h$ and $\theta_0 = \frac{\Gamma(\frac{N}{2}+1)(\sin\theta)^{N-1}\cos\theta}{(N+1)\pi^{1/2}\,\Gamma(\frac{N-1}{2}+1)}$. $\qquad\square$

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David Arcoya
Departamento de Análisis Matemático, Universidad de Granada,
C/Severo Ochoa, 18071 Granada, Spain
E-mail address: darcoya@ugr.es

Colette De Coster
Université de Valenciennes et du Hainaut Cambrésis
LAMAV, FR CNRS 2956,
Institut des Sciences et Techniques de Valenciennes
F-59313 Valenciennes Cedex 9, France
E-mail address: Colette.DeCoster@univ-valenciennes.fr

Louis Jeanjean
Laboratoire de Mathématiques (UMR 6623)
Université de Franche-Comté
16, Route de Gray 25030 Besançon Cedex, France
E-mail address: louis.jeanjean@univ-fcomte.fr

Kazunaga Tanaka
Department of Mathematics,
School of Science and Engineering
Waseda University
3-4-1 Ohkubo, Shijuku-ku, Tokyo 169-8555, Japan
E-mail address: kazunaga@waseda.jp