L^p−MAXIMAL REGULARITY ON BANACH SPACES WITH A SCHAUDER BASIS

N. J. KALTON AND G. LANCIEN

Abstract. We investigate the problem of L^p-maximal regularity on Banach spaces having a Schauder basis. Our results improve those of a recent paper. We also address the question of L^r-regularity in L^s spaces.

1. Introduction

We will only recall the basic facts and definitions on maximal regularity. For further information, we refer the reader to [2], [4], [8] or [7].

We consider the following Cauchy problem:

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \le t < T \\ u(0) = 0 \end{cases}$$

where $T \in (0, +\infty)$, $-B$ is the infinitesimal generator of a bounded analytic semigroup on a complex Banach space X and u and f are X-valued functions on $[0, T)$. Suppose $1 < p < \infty$. B is said to satisfy L^p*−maximal regularity* if whenever $f \in L^p([0, T); X)$ then the solution

$$u(t) = \int_0^t e^{-(t-s)B} f(s)\, ds$$

satisfies $u' \in L^p([0, T); X)$. It is known that B has L^p-maximal regularity for some $1 < p < \infty$ if and only if it has L^p-maximal regularity for every $1 < p < \infty$ [3], [4], [14]. We thus say simply that B satisfies *maximal regularity (MR).*

As in [7], we define:

Definition 1.1. A complex Banach space X has the *maximal regularity property* (MRP) if B satisfies (MR) whenever $-B$ is the generator of a bounded analytic semigroup.

Let us recall that De Simon [3] proved that any Hilbert space has (MRP), and that the question whether L^q for $1 < q \neq 2 < \infty$ has (MRP) remained open until recently. Indeed, in [7] it is shown that a Banach space with an

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unconditional basis (or more generally a separable Banach lattice) has (MRP) if and only if it is isomorphic to a Hilbert space.

In this paper we attempt to work without these unconditionality assumptions and study the (MRP) on Banach spaces with a finite-dimensional Schauder decomposition. In particular, we show that a UMD Banach space with an (FDD) and satisfying (MRP) must be isomorphic to an ℓ_2 sum of finite dimensional spaces.

In the last question we consider the question of whether the solution u of our Cauchy problem satisfies $u' \in L^2([0,T;L^r)$ if $f \in L^2([0,T);L^s)$.

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2. Notation and background

We will follow the notation of [7]. Let us now introduce more precisely a few notions.

If F is a subset of the Banach space X, we denote by $[F]$ the closed linear span of F. We denote by $(\varepsilon_k)_{k=0}^\infty$ the standard sequence of Rademacher functions on $[0,1]$ and by $(h_k)_{k=0}^\infty$ the standard Haar functions on $[0,1]$ (for convenience we index from 0).

Let $1 \leq p < \infty$. A Banach space X has *type* p if there is a constant $C > 0$ such that for every finite sequence $(x_k)_{k=1}^K$ in X:

$$(\int_0^1 \|\sum_{k=1}^K \varepsilon_k(t)x_k\|^2\,dt)^{1/2} \leq C(\sum_{k=1}^K \|x_k\|^p)^{1/p}.$$

Notice that every Banach space is of type 1. A Banach space X is called (UMD) if martingale difference sequences in $L_2([0,1];X)$ are unconditional i.e. there is a constant K so that for every martingale difference sequence $(f_n)_{n=1}^N$ we have

$$\|\sum_{k=1}^N \delta_k f_k\|_{L_2(X)} \leq K\|\sum_{k=1}^N f_k\|_{L_2(X)}$$

if $\sup_{k\leq N}|\delta_k| \leq 1$.

Let $(E_n)_{n\geq 1}$ be a sequence of closed subspaces of X. Assume that $(E_n)_{n\geq 1}$ is a Schauder decomposition of X and let $(P_n)_{n\geq 1}$ be the associated sequence of projections from X onto E_n. For convenience we will also denote this Schauder decomposition by $(E_n,P_n)_{n\geq 1}$. The decomposition constant is defined by $\sup_n\|\sum_{K=1}^n P_k\|$; this is necessarily finite. If each (E_n) is finite-dimensional we refer to (E_n) as an (FDD) (finite-dimensional decomposition); an unconditional (FDD) is abbreviated to (UFDD).

If $(E_n)_{n\geq 1}$ is a Schauder decomposition of X and $(u_n)_{n=1}^N$ is a finite or infinite sequence (i.e. $N \leq \infty$) of the form $u_n = \sum_{k=r_{n-1}+1}^{r_n} x_k$ where $x_k \in E_k$ and $1 = r_0 < r_1 < .. < r_n < ..$, then $(u_n)_{n\geq 1}$ is called a *block basic sequence* of the decomposition (E_n).

We denote by $\omega^{<\omega}$ the set of all finite sequences of positive integers, including the empty sequence denoted $\emptyset$. For $a = (a_1, .., a_n) \in \omega^{<\omega}$, $|a| = n$ is the *length* of a ($|\emptyset| = 0$). For $a = (a_1, .., a_k)$ (respectively $a = \emptyset$), we denote $(a, n) = (a_1, .., a_k, n)$ (respectively $(a, n) = (n)$). A subset β of $\omega^{<\omega}$ is a *branch* of $\omega^{<\omega}$ if there exists $(\sigma_n)_{n=1}^\infty \subset \mathbb{N}$ such that $\beta = \{(\sigma_1, .., \sigma_n);\ n \geq 1\}$. In this paper, for a Banach space X, we call a *tree* in X any family $(y_a)_{a\in\omega^{<\omega}} \subset X$. A tree $(y_a)_{a\in\omega^{<\omega}}$ is *weakly null* if for any $a \in \omega^{<\omega}$, $(y_{(a,n)})_{n\geq 1}$ is a weakly null sequence.

Let $(y_a)_{a\in\omega^{<\omega}}$ be a tree in the Banach space X. Let $T \subset \omega^{<\omega}$, $(y_a)_{a\in T}$ is a *full subtree* of $(y_a)_{a\in\omega^{<\omega}}$ if $\emptyset \in T$ and for all $a \in T$, there are infinitely many $n \in \mathbb{N}$ such that $(a, n) \in T$. Notice that if $(y_a)_{a\in T}$ is a full subtree of a weakly null tree $(y_a)_{a\in\omega^{<\omega}}$, then it can be reindexed as a weakly null tree $(z_a)_{a\in\omega^{<\omega}}$

We now state a result of [7] that will be an essential tool for this paper:

Theorem 2.1. *Let $(E_n, P_n)_{n\geq 1}$ be a Schauder decomposition of the Banach space X. Let $Z_n = P_n^* X^*$ and $Z = [\cup_{n=1}^\infty Z_n]$. Assume X has (MRP). Then there is a constant $C > 0$ so that whenever $(u_n)_{n=1}^N$ are such that $u_n \in [E_{2n-1}, E_{2n}]$ and $(u_n^*)_{n=1}^N$ are such that $u_n^* \in [Z_{2n-1}, Z_{2n}]$ then*

$$\left(\int_0^{2\pi} \|\sum_{n=1}^N P_{2n} u_n e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2} \leq C \left(\int_0^{2\pi} \|\sum_{n=1}^N u_n e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2}$$

and

$$\left(\int_0^{2\pi} \|\sum_{n=1}^N P_{2n}^* u_n^* e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2} \leq C \left(\int_0^{2\pi} \|\sum_{n=1}^N u_n^* e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2}.$$

We observe that, by a well-known result of Pisier [12] these inequalities can be replaced by equivalent inequalities (with a modified constant) using ε_k in place of $e^{i2^k t}$:

$$\|\sum_{n=1}^N P_{2n} u_n \varepsilon_n\|_{L_2(X)} \leq C \|\sum_{n=1}^N u_n \varepsilon_n\|_{L_2(X)} \tag{2.1}$$

and

$$\|\sum_{n=1}^N P_{2n} u_n^* \varepsilon_n\|_{L_2(X)} \leq C \|\sum_{n=1}^N u_n^* \varepsilon_n\|_{L_2(X^*)}. \tag{2.2}$$

We refer the reader to [15] for further recent developments in this area.

3. The main results

We begin with a general result on spaces with a Schauder decomposition:

Theorem 3.1. *Let X be a Banach space of type $p > 1$ and with a Schauder decomposition $(E_n)_{n=1}^{\infty}$. If X has (MRP), then there is a constant $C > 0$ so that for any block basic sequence $(u_k)_{k=1}^{N}$ with respect to the decomposition (E_n):*

$$(3.3) \qquad \frac{1}{C}\sum_{k=1}^{N}\|u_k\|^2 \leq \int_0^1 \|\sum_{k=1}^{N}\varepsilon_k(t)u_k\|^2\,dt \leq C\sum_{k=1}^{K}\|u_k\|^2.$$

Proof. If the result is false we can clearly inductively construct an infinite normalized block basic sequence $(u_n)_{n=1}^{\infty}$ so that there is no constant C so that for all finitely nonzero sequences $(a_k)_{k=1}^{\infty}$ we have:

$$(3.4) \qquad \frac{1}{C}\sum_{k=1}^{N}|a_k|^2 \leq \int_0^1 \|\sum_{k=1}^{N}a_k\varepsilon_k(t)u_k\|^2\,dt \leq C\sum_{k=1}^{N}|a_k|^2$$

It therefore suffices to show that (3.4) holds for every normalized block basic sequence $(u_n)_{n=1}^{\infty}$. We can clearly then suppose $u_n \in E_n$.

We next use a theorem of Figiel and Tomczak-Jaegermann [5] combined with [13] (see also [10] p.112) that, since X has nontrivial type for every $n \in \mathbb{N}$ there exists $\varphi(n) \in \mathbb{N}$ so that any subspace F of X with dimension $\varphi(n)$ has a subspace H of dimension n which is 2-complemented in X and 2-isomorphic to ℓ_2^n.

Assume (3.4) is false. Then we can inductively find a sequence $(a_n)_{n\geq 1}$ and an increasing sequence $(r_n)_{n\geq 0}$ with $r_0 = 0$ so that $r_{2n} > r_{2n-1}+\varphi(r_{2n-1}-r_{2n-2})$ for $n \geq 1$,

$$\sum_{r_{2n}+1}^{r_{2n+1}}|a_k|^2 = 1$$

and either

$$\int_0^1 \|\sum_{k=r_{2n}+1}^{r_{2n+1}}a_k\varepsilon_k(t)u_k\|^2\,dt > 2^n$$

or

$$\int_0^1 \|\sum_{k=r_{2n}+1}^{r_{2n+1}}a_k\varepsilon_k(t)u_k\|^2\,dt < 2^{-n}.$$

In order to create new Schauder decompositions of X, we will need the following elementary lemma, that we state without a proof:

Lemma 3.2. *Let $(E_n)_{n\geq 1}$ be a Schauder decomposition of a Banach space X. Assume that each E_n has a finite Schauder decomposition $(F_{n,k})_{k=1}^{m_n}$ with a uniform bound on the decomposition constant. Then $(F_{1,1},\cdots,F_{1,m_1},F_{2,1},\cdots,F_{2,m_2},\cdots)$ is also a Schauder decomposition of X.*

We denote the induced decomposition by $\sum_{n=1}^{\infty}\oplus(\sum_{k=1}^{m_n}\oplus F_{n,k})$.

Now by assumption $E_{r_{2n-1}+1}+\cdots+E_{r_{2n}}$ which has dimension at least $\varphi(r_{2n}-r_{2n-1})$ contains a subspace H_n which is 2-Hilbertian and 2-complemented in X. Let G_n be the complement of H_n in $E_{r_{2n-1}+1}+\cdots+E_{r_{2n}}$ by the projection of norm 2. At the same time $[u_k]$ is 1-complemented (by the Hahn-Banach theorem) in E_k for $r_{2n-1}+1\leq k\leq r_{2n}$ and let F_k be its associated complement. We thus have a new Schauder decomposition:

$$(F_1,[u_1],F_2,[u_2],\cdots,F_{r_1},[u_{r_1}],H_1,G_1,F_{r_2+1},[u_{r_2+1}],\cdots,[u_{r_3}],H_2,G_2,\cdots).$$

If we write $D_n=F_{r_{2n-2}+1}+\cdots+F_{r_{2n-1}}+G_n$ then we have a Schauder decomposition

$$\sum_{n=1}^{\infty}\oplus(D_n\oplus H_n\oplus\sum_{k=r_{2n-2}+1}^{r_{2n}}\oplus[u_k]).$$

Next select a normalized basis $(v_k)_{k=r_{2n-2}+1}^{r_{2n-1}}$ of H_k which is 2-equivalent to the canoncal basis of $\ell_2^{r_{2n}-r_{2n-1}}$. It is easy to see that we can obtain a new Schauder decomposition by interlacing the (v_k) with the (u_k) i.e.:

$$\sum_{n=1}^{\infty}(D_n\oplus[u_{r_{2n-2}+1}]\oplus[v_{r_{2n-2}+1}]\oplus\cdots\oplus[u_{r_{2n-1}}]\oplus[v_{r_{2n-1}}]). \tag{3.5}$$

Now again using Lemma 3.2 we can form two further decompostions:

$$\sum_{n=1}^{\infty}(D_n\oplus[u_{r_{2n-2}+1}+v_{r_{2n-2}+1}]\oplus[v_{r_{2n-2}+1}]\oplus\cdots\oplus[u_{r_{2n-1}}+v_{r_{2n-1}}]\oplus[v_{r_{2n-1}}]), \tag{3.6}$$

and

$$\sum_{n=1}^{\infty}(D_n\oplus[u_{r_{2n-2}+1}+v_{r_{2n-2}+1}]\oplus[u_{r_{2n-2}+1}]\oplus\cdots\oplus[u_{r_{2n-1}}+v_{r_{2n-1}}]\oplus[u_{r_{2n-1}}]). \tag{3.7}$$

Now we can apply Theorem 2.1. If we use decomposition (3.6) we note that $u_k=(u_k+v_k)-v_k$ and so for a suitable C and all n,

$$\|\sum_{k=r_{2n-2}+1}^{r_{2n}}a_k(u_k+v_k)\varepsilon_k\|_{L_2(X)}\leq C\|\sum_{k=r_{2n-2}+1}^{r_{2n-1}}a_ku_k\varepsilon_k\|_{L_2(X)}.$$

However, using decomposition (3.5) there is also a constant C' so that

$$\|\sum_{k=r_{2n-2}+1}^{r_{2n}}a_kv_k\varepsilon_k\|_{L_2(X)}\leq C'\|\sum_{k=r_{2n-2}+1}^{r_{2n}}a_k(u_k+v_k)\varepsilon_k\|_{L_2(X)}.$$

This leads to an estimate:

$$\left(\sum_{k=r_{2n-2}+1}^{r_{2n-1}} |a_k|^2\right)^{\frac{1}{2}} \leq C_1 \| \sum_{k=r_{2n-2}+1}^{r_{2n-1}} a_k u_k \varepsilon_k \|_{L_2(X)}.$$

If we use decomposition (3.7) instead we obtain an estimate:

$$\| \sum_{k=r_{2n-2}+1}^{r_{2n-1}} a_k u_k e^{i2^k t} \|_{L_2(X)} \leq C_2 \left(\sum_{k=r_{2n-2}+1}^{r_{2n-1}} |a_k|^2\right)^{\frac{1}{2}}.$$

Combining gives us (3.4) and completes the proof. □

Let us first use this result to give a mild improvement of a result from [7]:

Theorem 3.3. *Let X be a reflexive space with an (FDD) and with non-trivial type which embeds into a space Y with a (UFDD). If X has (MRP) then X is isomorphic to an ℓ_2-sum of finite-dimensional spaces $(\sum_{n=1}^{\infty} \oplus E_n)_{\ell_2}$.*

Proof. Using Proposition 1.g.4 of [9] (cf. [6]) we can block the given (FDD) to produce an (FDD) (E_n) so that $(E_{2n})_{n=1}^{\infty}$ and $(E_{2n-1})_{n=1}^{\infty}$ are both (UFDD)'s. Let us denote, as in Theorem 2.1, the dual (FDD) of X^* by $(Z_n)_{n=1}^{\infty}$. Now it follows applying Theorem 3.1 to both X and X^* (which also has (MRP)) that there exists a constant C so that if $x_n \in E_n$ and $x_n^* \in Z_n$ are two finitely nonzero sequences

$$\| \sum_{k=1}^{\infty} x_{2k-j} \| \leq C (\sum_{k=1}^{\infty} \|x_{2k-j}\|^2)^{\frac{1}{2}}$$

$$\| \sum_{k=1}^{\infty} x_{2k-j}^* \| \leq C (\sum_{k=1}^{\infty} \|x_{2k-j}^*\|^2)^{\frac{1}{2}}$$

for $j = 0, 1$. Hence

$$\| \sum_{k=1}^{\infty} x_k \| \leq 2C (\sum_{k=1}^{\infty} \|x_k\|^2)^{\frac{1}{2}}$$

$$\| \sum_{k=1}^{\infty} x_k^* \| \leq 2C (\sum_{k=1}^{\infty} \|x_k^*\|^2)^{\frac{1}{2}}.$$

Now for given x_k we may find $y_k^* \in X^*$ with $\|y_k^*\| = \|x_k\|$ and $y_k(x_k^*) = \|x_k^*\|$. Let $x_k^* = P_k^* y_k^*$ (where $P_k : X \to E_k$ is the projection associated with the FDD (E_n)). Then $\|x_k^*\| \leq C_1 \|x_k\|$ where $C_1 = \sup_n \|P_n\| < \infty$. Hence if $(x_k)_{k=1}^{\infty}$ is finitely nonzero, we have

$$\| \sum_{k=1}^{\infty} x_k^* \| \leq 2CC_1 (\sum_{k=1}^{\infty} \|x_k\|^2)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\|x_k\|^2 &= \sum_{k=1}^{\infty} x_k^*(x_k)\\ &= (\sum_{k=1}^{\infty} x_k^*)(\sum_{k=1}^{\infty} x_k)\\ &\leq 2CC_1(\sum_{k=1}^{\infty}\|x_k^*\|^2)^{\frac{1}{2}}\|\sum_{k=1}^{\infty} x_k\|\end{aligned}$$

so that we obtain the lower estimate:

$$(\sum_{k=1}^{\infty}\|x_k\|^2)^{\frac{1}{2}} \leq 2CC_1\|\sum_{k=1}^{\infty} x_k\|.$$

This completes the proof. □

We next give another application to (UMD)-spaces with (MRP).

Theorem 3.4. *Let X be a (UMD) Banach space with an (FDD) satisfying (MRP). Then X is isomorphic to an ℓ_2-sum of finite dimensional spaces, $(\sum_{n=1}^{\infty}\oplus E_n)_{\ell_2}$.*

Proof. Let (E_n) be the given (FDD) of X. We will show first that there is a blocking (F_n) of (E_n) which satisfies an upper 2−estimate i.e. if there is a constant A so that if (x_n) is block basic with respect to (F_n) and finitely non-zero then

$$\|\sum_{n=1}^{\infty} x_n\| \leq A(\sum_{n=1}^{\infty}\|x_n\|^2)^{\frac{1}{2}}. \tag{3.8}$$

Once this is done, the proof can be completed easily. Indeed if (Z_n) is the dual decomposition to (F_n) for X^* then we can apply the fact that X^* also has (MRP) (X is reflexive) to block (Z_n) to obtain a decomposition which also has an upper 2-estimate. Thus we can assume (F_n) and (Z_n) both have an upper 2-estimate and then repeat the argument used in Theorem 3.3 to deduce that $X = (\sum_{n=1}^{\infty}\oplus F_n)_{\ell_2}$.

Since X necessarily has type $p > 1$, we can apply Theorem 3.1 and assume (E_n) obeys (3.3).

We now introduce a particular type of tree in the space $L_2([0,1);X)$. Let $\mathcal{D}_n$ for $n \geq 0$ be the sub-algebra of the Borel sets of $[0,1)$ generated by the dyadic intervals $[(k-1)2^{-n}, k2^{-n})$ for $1 \leq k \leq 2^n$. Let $\mathbb{E}_n$ denote the conditional expectation operator $\mathbb{E}_n f = \mathbb{E}(f|\mathcal{D}_n)$.

We will say that a tree $(f_a)_{a\in\omega^{<\omega}}$ is a *martingale difference tree* or (MDT) if

- each f_a is $\mathcal{D}_{|a|}$− measurable,
- if $|a| > 0$ then $\mathbb{E}_{|a|-1} f_a = 0$,

- there exists N so that if $|a| > N$ then $f_a = 0$.

In such a tree the partial sums along any branch form a dyadic martingale which is eventually constant.

We will prove the following Lemma:

Lemma 3.5. *There is a constant K so that if $(f_a)_{a\in\omega^{<\omega}}$ is a weakly null (MDT), there is a full subtree $(f_a)_{a\in T}$ so that for any branch β we have:*

$$\|\sum_{a\in\beta} f_a\|_{L_2(X)} \leq K(\sum_{a\in\beta}\|f_a\|^2_{L_2(X)})^{\frac{1}{2}}.$$

Proof. For each a we define integers $m_-(a)$ and $m_+(a)$. If $f_a \neq 0$ we set $m_-(a)$ to be the greatest m so that

$$\|\sum_{k=1}^{m} P_m f_a\|_{L_2(X)} \leq 2^{-|a|-1}\|f_a\|_{L_2(X)}$$

and $m_+(a)$ to be the least $m > m_-(a)$ so that

$$\|\sum_{k=m+1}^{\infty} P_k f_a\|_{L_2(X)} \leq 2^{-|a|-1}\|f_a\|_{L_2(X)}.$$

If $f_\emptyset = 0$ we set $m_-(\emptyset) = 0$ and $m_+(\emptyset) = 1$; if $f_a = 0$ where $a \neq \emptyset$ we set $m_-(a)$ to be the last member of a and $m_+(a) = m_-(a) + 1$.

Since (f_a) is weakly null we have $\lim_{n\to\infty} m_-(a,n) = \infty$ for every a. It is then easy to pick a full subtree T so that $m_-(a,n) > m_+(a)$ whenever $a, (a,n) \in T$. Now let $g_a = \sum_{k=m_-(a)+1}^{m_+(a)} f_a$. Then $\|f_a - g_a\|_{L_2(X)} \leq 2^{-|a|}\|f_a\|_{L_2(X)}$.

For any branch β of T, we have that $g_a(t)$ is a block basic sequence with respect to (E_n) for every $0 \leq t < 1$. Hence

$$\left(\int_0^1 \|\sum_{a\in\beta}\epsilon_{|a|}(s)g_a(t)\|^2_X ds\right)^{\frac{1}{2}} \leq C\left(\sum_{a\in\beta}\|g_a(t)\|^2_X\right)^{\frac{1}{2}}.$$

Integrating again we have

$$\left(\int_0^1 \|\sum_{a\in\beta}\epsilon_{|a|}(s)g_a\|^2_{L_2(X)} ds\right)^{\frac{1}{2}} \leq C\left(\sum_{a\in\beta}\|g_a\|^2_{L_2(X)}\right)^{\frac{1}{2}}.$$

From this we get

$$\left(\int_0^1 \|\sum_{a\in\beta}\epsilon_{|a|}(s)f_a\|^2_{L_2(X)} ds\right)^{\frac{1}{2}} \leq 2C\left(\sum_{a\in\beta}\|f_a\|^2_{L_2(X)}\right)^{\frac{1}{2}} + \sum_{a\in\beta}2^{-|a|}\|f_a\|.$$

Estimating the last term by the Cauchy-Schwarz inequality and using the fact that X is (UMD) we get the Lemma. $\square$

Now we introduce a functional Φ on X by defining $\Phi(x)$ to be the infimum of all $\lambda > 0$ so that for every weakly null (MDT) $(f_a)_{a\in\omega^{<\omega}}$ with $f_\emptyset = x\chi_{[0,1)}$ we have a full subtree T so that for any branch β

$$\|\sum_{a\in\beta} f_a\|^2_{L_2(X)} \le \lambda + 2K^2 \sum_{\substack{a\in\beta\\ a\neq\emptyset}} \|f_a\|^2_{L_2(X)}. \tag{3.9}$$

Note that that since

$$\|\sum_{a\in\beta} f_a\|^2_{L_2(X)} \le 2(\|x\|^2 + \|\sum_{\substack{a\in\beta\\ a\neq\emptyset}} f_a\|^2_{L_2(X)})$$

we have an estimate $\Phi(x) \le 2\|x\|^2$. By considering the null tree we have $F(x) \ge \|x\|^2$. It is clear that Φ is continuous and 2-homogeneous. Most importantly we observe that Φ is convex; the proof of this is quite elementary and we omit it. It follows that we can define an equivalent norm by $|||x|||^2 = \Phi(x)$ and $\|x\| \le |||x||| \le 2\|x\|$ for $x \in X$.

Next we prove that if $x \in X$ and (y_n) is a weakly null sequence then

$$\limsup_{n\to\infty}(|||x+y_n|||^2 + |||x-y_n|||^2) \le 2|||x|||^2 + 4K^2 \limsup_{n\to\infty} \|y_n\|^2. \tag{3.10}$$

We first note that we can suppose $\lim_{n\to\infty} |||x \pm y_n|||$ and $\lim_{n\to\infty} \|y_n\|^2$ all exist. Now suppose $\epsilon > 0$. Then we can find weakly null (MDT)'s $(f_a^n)_{a\in\omega^{<\omega}}$ with $f_\emptyset^n \equiv x + y_n$ so that for every full subtree T we have a branch β on which:

$$\|\sum_{a\in\beta} f_a^n\|^2_{L_2(X)} + \epsilon > |||x+y_n|||^2 + 2K^2 \sum_{\substack{a\in\beta\\ a\neq\emptyset}} \|f_a^n\|^2_{L_2(X)}. \tag{3.11}$$

In fact by easy induction we can pick a full subtree so that (3.11) holds for *every* branch. Hence we suppose the original tree satisfies (3.11) for every branch.

Similarly we may find weakly null (MDT)'s $(g_a^n)_{a\in\omega^{<\omega}}$ with $g_\emptyset^n \equiv x - y_n$ and for every branch β,

$$\|\sum_{a\in\beta} g_a^n\|^2_{L_2(X)} + \epsilon > |||x-y_n|||^2 + 2K^2 \sum_{\substack{a\in\beta\\ a\neq\emptyset}} \|g_a^n\|^2_{L_2(X)}.$$

We next consider the (MDT) defined by $h_\emptyset \equiv x$,

$$h_{(n)}(t) = \begin{cases} y_n & \text{if } 0 \le t < \frac{1}{2} \\ -y_n & \text{if } \frac{1}{2} \le t < 1 \end{cases}$$

and if $|a| > 1$ then

$$h_{(a,n)}(t) = \begin{cases} f_a^n(2t-1) & \text{if } 0 \le t < \frac{1}{2} \\ g_a^n(2t) & \text{if } \frac{1}{2} \le t < 1. \end{cases}$$

Now for every branch of the (MDT) $(h_a)_{a\in\omega^{<\omega}}$ with initial element $\{n\}$ we have

$$\|\sum_{a\in\beta} h_a\|^2_{L_2(X)} + \epsilon > \frac{1}{2}(|||x+y_n|||^2 + |||x-y_n|||^2) + 2K^2 \sum_{\substack{a\in\beta\\ |a|>1}} \|h_a\|^2_{L_2(X)}.$$

However, from the definition of $\Phi(x) = |||x|||^2$ it follows that there exists n_0 so that if $n \geq n_0$ we can find a branch β whose initial element is n and such that

$$\|\sum_{a\in\beta} h_a\|^2_{L_2(X)} < |||x|||^2 + 2K^2 \sum_{\substack{a\in\beta\\ |a|>0}} \|h_a\|^2_{L_2(X)} + \epsilon.$$

Combining gives the equation (for $n \geq n_0$),

$$\frac{1}{2}(|||x+y_n|||^2 + |||x-y_n|||^2) \leq |||x|||^2 + 2K^2\|y_n\|^2 + 2\epsilon.$$

This proves (3.10). But note that if y_n is weakly null we have $\liminf_{n\to\infty} |||x-y_n||| \geq |||x|||$ and so we deduce:

$$\limsup_{n\to\infty} |||x+y_n|||^2 \leq |||x|||^2 + 4K^2 \limsup \|y_n\|^2.$$

Using this equation it is now easy to block the Schauder decomposition (E_n) to produce a Schauder decomposition (F_n) with the property that for any N if $x \in F_1 + \cdots + F_N$ and $y \in \sum_{k=N+2}^{\infty} F_k$ then

$$|||x+y||| \leq (1+\delta_N)(|||x|||^2 + 4K^2\|y\|^2)^{\frac{1}{2}},$$

where $\delta_N > 0$ are chosen to be decreasing and so that $\prod_{N=1}^{\infty}(1+\delta_N) \leq 2$. Next suppose (x_k) is any finitely non-zero block basic sequence with respect to (F_n). By an easy induction we obtain for $j = 0, 1$:

$$|||\sum_{k=1}^{n} x_{2k-j}||| \leq 4K^2 \prod_{k=1}^{n-1}(1+\delta_{2k-j})(\sum_{k=1}^{n} |||x_{2k-j}|||^2)^{\frac{1}{2}}.$$

Hence

$$\|\sum_{k=1}^{n} x_k\| \leq 32K^2(\sum_{k=1}^{n} \|x_k\|^2)^{\frac{1}{2}}.$$

This establishes (3.8) and as shown earlier this suffices to complete the proof. □

Remark. Recently Odell and Schlumprecht [11] showed that a separable Banach space X can be embedded in an ℓ_p−sum of finite-dimensional spaces for $1 < p < \infty$ if and only if X is reflexive and every normalized weakly null tree has a branch which is equivalent to the usual ℓ_p−basis. This result is closely related to the proof of the previous theorem.

4. On L^r-regularity in L^s spaces

Let $s \in [1, \infty)$. We consider our usual Cauchy problem:

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \leq t < T \\ u(0) = 0 \end{cases}$$

where $T \in (0, +\infty)$, $-B$ is the infinitesimal generator of a bounded analytic semigroup on $L^s = L^s([0,1])$ and $f \in L^2([0,T); L^s)$. Then we ask the following question: for what values of s and r in $[1, \infty)$ does the solution

$$u(t) = \int_0^t e^{-(t-s)B} f(s)\, ds$$

necessarily satisfies $u' \in L^2([0,T); L^r)$? Thus we introduce the following definition:

Definition 4.1. Let r and s in $[1, \infty)$. We say that (r, s) is a *regularity pair* if whenever $-B$ is the infinitesimal generator of a bounded analytic semigroup on $L^s = L^s([0,1])$ and $f \in L^2([0,T); L^s)$, the solution u of

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \leq t < T \\ u(0) = 0 \end{cases}$$

satisfies $u' \in L^2([0,T); L^r)$.

Notice that it follows from previous results ([3], [8] and [7]) that (s, s) is a regularity pair if and only if $s = 2$. This is extended by our next result:

Theorem 4.2. *Let r and s in $[1, \infty)$. Then (r, s) is a regularity pair if and only if $r \leq s = 2$.*

Proof. It follows clearly from the work of De Simon [3], that if $r \leq s = 2$ then (r, s) is a regularity pair.
So let now (r, s) be a regularity pair. Since L^1 does nat have (MRP) ([8]), we have that $s > 1$. Then, solving our Cauchy problem with $B = 0$, we obtain that $r \leq s$. Thus we can limit ourselves to the case $s > 1$ and $1 \leq r \leq s$.
Then by the closed graph Theorem, for any B so that $-B$ is the infinitesimal generator of a bounded analytic semigroup on $L^s = L^s([0,1])$, there is a constant $C > 0$ such that for any $f \in L^2([0,T); L^s)$:

$$\|u'\|_{L^2(L^s)} \leq C \|f\|_{L^2(L^s)}.$$

Using the inclusion $L^s \subset L^r$ for $r \leq s$, we can now state the following analogue of Theorem 2.1:

Proposition 4.3. *Let $(E_n, P_n)_{n \geq 1}$ be a Schauder decomposition of L^s. Assume that (r, s) is a regularity pair. Then there is a constant $C > 0$ so that whenever*

$(u_n)_{n=1}^N$ *are such that* $u_n \in [E_{2n-1}, E_{2n}]$ *then*

$$\|\sum_{n=1}^N P_{2n} u_n \varepsilon_n\|_{L^2(L^r)} \le C \|\sum_{n=1}^N u_n \varepsilon_n\|_{L^2(L^s)}$$

Then our first step will be to show that the Haar system (h_k) satisfies some lower-2 estimates in L^s in the following sense:

Lemma 4.4. *If there exists* $r \le s$ *such that* (r, s) *is a regularity pair, then there is a constant* $C > 0$ *such that for any normalized block basic sequence* $(v_1, ..., v_n)$ *of* (h_k) *and for any* $a_1, .., a_n$ *in* $\mathbb{C}$*:*

$$\|\sum_{k=1}^n a_k v_k\|_{L^s}^2 \ge C \sum_{k=1}^n |a_k|^2.$$

Proof. We first observe that if $1 < p < 2$, it follows from the work of J. Bretagnolle, D. Dacunha-Castelle and J.L. Krivine [1] on p-stable random variables that there is a sequence $(e_n)_{n\ge 1}$ in L^1 which is equivalent to the canonical basis of ℓ_p in any L^q for $1 \le q < p$. Thus (e_n) is weakly null in L^s, and by a gliding hump argument, we may assume that (e_n) is actually a block basic sequence with respect with the Haar basis. If $p = 2$, then the Rademacher functions form a block basic sequence in every L^q for $1 \le q < \infty$.

Now assume the lemma is false. We pick a normalized block basic sequence $(v_1, ..., v_{n_1})$ of (h_k) and $a_1, .., a_{n_1}$ in $\mathbb{C}$ so that

$$\|\sum_{k=1}^{n_1} a_k v_k\|_{L^s}^2 \le \sum_{k=1}^{n_1} |a_k|^2 = 1.$$

Then pick $m_1 \in \mathbb{N}$ such that $(v_1, .., v_{n_1}, e_{m_1})$ is a block basic sequence of (h_k). By induction, we pick a normalized block basic sequence $(v_{n_j+1}, ..., v_{n_{j+1}})$ of (h_k), $a_{n_j+1}, .., a_{n_{j+1}}$ in $\mathbb{C}$ and $m_{j+1} \in \mathbb{N}$ so that $(v_1, .., v_{n_1}, e_{m_1}, v_{n_1+1}, .., v_{n_{j+1}}, e_{m_{j+1}})$ is a block basic sequence of (h_k) and

$$\|\sum_{k=n_j+1}^{n_{j+1}} a_k v_k\|_{L^s}^2 \le \frac{1}{2^j} \sum_{k=n_j+1}^{n_{j+1}} |a_k|^2 = \frac{1}{2^j}.$$

So we can find $(I_k)_{k\ge 1}$ and $(J_k)_{k\ge 1}$ two sequences of finite intervals of $\mathbb{N}$ such that $\{I_k, J_k :\ k \ge 1\}$ is a partition of $\mathbb{N}$ and for all $k \ge 1$, $v_k \in [h_j,\ j \in I_k]$ and $e_{m_k} \in [h_j,\ j \in J_k]$. Then set

$$X_k = [h_j :\ j \in I_k \cup J_k].$$

Then (X_k) is an unconditional Schauder decomposition of L^s. Each X_k can be decomposed into $X_k = E_{2k-1} \oplus E_{2k}$, where $E_{2k-1} = [v_k + e_{m_k}]$, $e_{m_k} \in E_{2k}$ and the corresponding projections are uniformly bounded. So, by Lemma 3.2, $(E_k)_{k\ge 1}$ is a Schauder decomposition of L^s. We can now make use of of

Proposition 4.3. If we decompose $a_k v_k = a_k(v_k + e_{m_k}) - a_k e_{m_k}$ in $E_{2k-1} \oplus E_{2k}$, we obtain that there is a constant $C > 0$ such that for all $n \geq 1$:

$$\|\sum_{k=1}^{n} a_k v_k \varepsilon_k\|_{L^2(L^s)} \geq C(\sum_{k=1}^{n} |a_k|^2)^{1/2}.$$

Since (v_k) is an unconditional basic sequence in L^s, there is a constant $K > 0$ so that for all $n \geq 1$:

$$\|\sum_{k=1}^{n} a_k v_k\|_{L^s}^2 \geq K \sum_{k=1}^{n} |a_k|^2,$$

which is in contradiction with our construction. □

We now conclude the proof of Theorem 4.2. The Haar basis of L^s has a block basic sequence equivalent to the standard basis of $\ell_{\max(s,2)}$. Hence Lemma 4.4 shows that $\max(s,2) \leq p$ whenever $s < p < 2$ or $p = 2$. Thus $s = 2$.

□

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Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211
E-mail address: `nigel@math.missouri.edu`

Equipe de Mathématiques - UMR 6623, Université de Franche-Comté, F-25030 Besançon cedex
E-mail address: `glancien@math.univ-fcomte.fr`