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A SURVEY ON THE SZLENK INDEX AND SOME OF ITS APPLICATIONS

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Abstract. We describe how the Szlenk index has been used in various areas of the geometry of Banach spaces. We cover the following domains of application of this notion: non existence of universal spaces, linear classification of $C(K)$ spaces, descriptive set theory, renorming problems and non linear classification of Banach spaces.

1. Introduction

In this survey paper, we have selected some of the fields, where the Szlenk index and its variants, have been fruitfully used in the geometry of Banach spaces.

In section 2, we define the different slicing indices, including the Szlenk index, that will be studied in this paper. Then, we gather some elementary or technical facts in section 3.

The Szlenk index of a Banach space X, denoted $\mathrm{Sz}(X)$, is an ordinal number, which is invariant under linear isomorphisms. As it is suggested by its name, it was first introduced by W. Szlenk [62] in order to prove that there is no separable reflexive Banach space universal for the class of all separable reflexive Banach spaces. This aspect is briefly treated in section 4. Very recent developments are also indicated.

Another striking fact is that the isomorphic classification of a separable $C(K)$ space is perfectly determined by the value of its Szlenk index. This is a consequence of some classical work by C. Bessaga and A. Pełczyński [8], D.E. Alspach and Y. Benyamini [1] and C. Samuel [59]. We give in section 5 a recent short proof ([33]) of the computation of $\mathrm{Sz}(C(\alpha))$ when α is a countable ordinal.

It follows rather simply from Baire's Great Theorem that a separable Banach space has a separable dual if and only if its Szlenk index is countable. This can be made more precise by using the Kunen-Martin Theorem and other tools from descriptive set theory. The interactions between the geometry of Banach spaces and descriptive set theory have been widely investigated and for a complete reference, we suggest the survey paper by S.A. Argyros, G. Godefroy and H.P. Rosenthal [3]. In section 6, we try to give a simple example of this

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type of results by showing, using the work of B. Bossard ([9],[10]), how one can justify the following statement: the set of all separable Banach spaces with a separable dual is coanalytic non Borel and the Szlenk index is a coanalytic rank for this set.

These tools from descriptive set theory, together with a geometric construction using a dentability index allow us to prove in section 7 a first renorming result. More precisely, we show that a Banach space with countable Szlenk index admits an equivalent norm, whose dual norm is locally uniformly rotund. Then we explain how this dentability index provides us with a simple geometric formula for building an equivalent uniformly convex norm with power type modulus on super-reflexive Banach spaces.

Section 8 is devoted to another renorming question. We consider the problem of building an equivalent norm with a dual weak* uniformly Kadec-Klee norm on a Banach space with finite Szlenk index. This was solved positively by H. Knaust, E. Odell and T. Schlumprecht ([44]), who along their way gave a detailed study of the linear structure of these spaces. We also present a different approach ([30]) yielding some precise quantitative estimates that will be crucial in the last section.

Finally, in section 9, we show that the Szlenk index, when finite, is, in a very precise quantitative way, an invariant under Lipschitz homeomorphisms or uniform homeomorphisms. Then we derive some important results on the non linear classification of certain classical Banach spaces, such as c_0, subspaces of c_0, subspaces of ℓ_p $(p > 2)$ or quotients of ℓ_p $(p > 2)$.

We also refer the reader to a more synthetic survey on this subject that has already been written by G. Godefroy ([28]).

2. Notation

We first give the definition of the Szlenk index and the Szlenk derivation. Suppose X is a real Banach space and K is a weak*-compact subset of X^*. For $\varepsilon > 0$ we let $\mathcal{V}$ be the set of all relatively weak*-open subsets V of K such that the norm diameter of V is less than ε and $s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$. Then we define inductively $s_\varepsilon^\alpha K$ for any ordinal α by $s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K)$ and $s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K$ if α is a limit ordinal.

We denote by B_{X^*} the closed unit ball of X^*. We then define $\mathrm{Sz}(X, \varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we write $\mathrm{Sz}(X, \varepsilon) = \infty$. The *Szlenk index* is defined by $\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X, \varepsilon)$. Let us point out that this definition differs slightly from the original definition of W. Szlenk in [62]. However, both definitions coincide if X is a separable Banach space which does not contain any isomorphic copy of $\ell_1(\mathbb{N})$ (see [46] for instance).

We also introduce an alternative *convex Szlenk index*. If K is a weak*-compact and convex subset of X^*, we may define $c_\varepsilon K = \overline{\mathrm{co}}^*(s_\varepsilon K)$ (namely, the weak*-closed convex hull of $s_\varepsilon K$). Then $\mathrm{Cz}(X,\varepsilon)$ and $\mathrm{Cz}(X)$ are defined as before, using the derivation c_ε instead. Let us notice that a geometrical way to define this derivation is to say that it removes the weak*-slices that can be covered by a union (and therefore by a finite union) of weak*-open sets of diameter less than ε.

Now, if K is a weak*-compact and convex subset of X^*, we call *weak*-slice* of K any non empty set of the form $S = \{x^* \in K,\ x^*(x) > t\}$, where $x \in X$ and $t \in \mathbb{R}$. Then we denote by $\mathcal{S}$ the set of all weak*-slices of K of norm diameter less than ε and $d_\varepsilon K = K \setminus \cup\{S : S \in \mathcal{S}\}$. From this derivation, we define similarly the *weak*-dentability index* of X that we denote $\mathrm{Dz}(X,\varepsilon)$ and $\mathrm{Dz}(X)$.

We shall also briefly use the following derivation on subsets of X itself. Let C be a closed convex subset of X. A *slice* of C is a set of the form $T = \{x \in C,\ x^*(x) > t\}$, where $x^* \in X^*$ and $t \in \mathbb{R}$. We denote by $\mathcal{T}$ the set of all slices of C of norm diameter less than ε and $D_\varepsilon C = C \setminus \cup\{T : S \in \mathcal{T}\}$. From this derivation, we define $\mathrm{D}(X,\varepsilon)$ to be the least ordinal α so that $D_\varepsilon^\alpha B_X = \emptyset$, if such an ordinal exists, and $\mathrm{D}(X,\varepsilon) = \infty$ otherwise. The *dentability index* of X is as usual $\mathrm{D}(X) = \sup_{\varepsilon>0} \mathrm{D}(X,\varepsilon)$.

3. A few basic properties

In the following proposition, we gather without proof some elementary facts.

Proposition 3.1. *Let X be a Banach space.*
(i) $Sz(X) \leq Cz(X) \leq Dz(X)$.
(ii) The indices Sz, Cz and Dz are invariant under linear isomorphisms.
(iii) For each of our three definitions, the indices of subspaces or quotients of X are bounded above by the index of X.
(iv) X is finite dimensional if and only if $Sz(X)=Cz(X) = 1$.
(v) If $X \neq \{0\}$, then for any $\varepsilon > 0$, $Dz(X,\varepsilon) \geq \varepsilon^{-1}$ and therefore $Dz(X) \geq \omega$, where ω is the first infinite ordinal.
(vi) Since weak*-compactness implies that $Dz(X,\varepsilon)$ is never a limit ordinal, we have that $Dz(X) \leq \omega$ if and only if $Dz(X,\varepsilon) < \omega$ for any $\varepsilon > 0$. The same is true for $Sz(X)$ or $Cz(X)$.
(vii) If X is separable, K a weak*-compact subset of X^* and $\varepsilon > 0$, we denote

$$l_\varepsilon K = \{x^* \in K\ \exists (x_n^*)_{n\geq 1} \subset K \text{ st } \forall n\ \|x^* - x_n^*\| \geq \varepsilon \text{ and } x_n^* \xrightarrow{w^*} x^*\}.$$

Then the index associated with this derivation is equal to $Sz(X)$.

Remark 3.2. Despite the lack of compactness, we also have that $D(X,\varepsilon)$ is never a limit ordinal. It relies on the fact that $D_\varepsilon^\alpha B_X$ is convex symmetric and therefore contains 0, whenever it is not empty.

Our next proposition, is a rather technical fact, that was first mentioned in a paper of A. Sersouri [61] on similar indices.

Proposition 3.3. *Let X be a Banach space. Then, either $Sz(X) = \infty$ or there exists an ordinal α such that $Sz(X) = \omega^\alpha$.*
The same is true for the other indices.

Proof. We only give the argument for the Szlenk index and use the following fact: for any Banach space X and any ordinal α

$$\frac{1}{2}s_\varepsilon^\alpha B_{X^*} + \frac{1}{2}B_{X^*} \subset s_{\varepsilon/2}^\alpha B_{X^*}. \tag{3.1}$$

The proof is a straightforward transfinite induction.
Then let us show that $\mathrm{Sz}(X) > \omega^\alpha$ implies that $\mathrm{Sz}(X) \geq \omega^{\alpha+1}$. This will clearly yield the conclusion.
So assume that $\mathrm{Sz}(X) > \omega^\alpha$. We can find $\varepsilon > 0$ such that $s_\varepsilon^{\omega^\alpha} B_{X^*} \neq \emptyset$. Then, by (3.1), $0 \in s_{\varepsilon/2}^{\omega^\alpha} B_{X^*}$ and $\frac{1}{2}B_{X^*} \subset s_{\varepsilon/4}^{\omega^\alpha} B_{X^*}$. So

$$0 \in s_{\varepsilon/4}^{\omega^\alpha}(\frac{1}{2}B_{X^*}) \subset s_{\varepsilon/4}^{\omega^\alpha.2} B_{X^*}.$$

Proceeding inductively, we can show that for any integer n, $0 \in s_{\varepsilon/2^{n+1}}^{\omega^\alpha.2^n} B_{X^*}$. Thus $Sz(X) \geq \omega^{\alpha+1}$. □

Another simple but useful fact is the following.

Proposition 3.4. *Let X be a Banach space and α an ordinal. Assume that*

$$\forall \varepsilon > 0 \ \ \exists \delta(\varepsilon) > 0 \ \ s_\varepsilon^\alpha(B_{X^*}) \subset (1-\delta(\varepsilon))B_{X^*}.$$

Then

$$Sz(X) \leq \alpha.\omega$$

A similar statement is true for the other indices.

Proof. Let $\varepsilon > 0$. An easy homogeneity argument shows that for any $n \in \mathbb{N}$:

$$s_\varepsilon^{\alpha.n}(B_{X^*}) \subset (1-\delta(\varepsilon))^n B_{X^*}.$$

Consequently, there exists an integer N so that $s_\varepsilon^{\alpha.N}(B_{X^*}) \subset \frac{\varepsilon}{3}B_{X^*}$ and therefore $s_\varepsilon^{\alpha.N+1}(B_{X^*}) = \emptyset$. This finishes the proof. □

We shall also need the following property of the Szlenk index.

Proposition 3.5. *Let X be a Banach space.*

a) The fonction $Sz(X,.)$ is submultiplicative. More precisely:

$$\forall \varepsilon > 0 \ \forall \varepsilon' > 0 \ \ Sz(X, \varepsilon\varepsilon') \leq Sz(X,\varepsilon).Sz(X,\varepsilon').$$

b) If $Sz(X) = \omega$, then

$$\exists C > 0 \ \exists p \in [1, +\infty) \ \ \forall \varepsilon > 0 \ Sz(X, \varepsilon) \leq C\varepsilon^p.$$

Proof. Let $\varepsilon > 0$ and $\varepsilon' > 0$. It is enough to show (omitting the obvious case when $\mathrm{Sz}(X,\varepsilon') = \infty$) that for any ordinal α,

$$s_{\varepsilon\varepsilon'}^{\alpha.\mathrm{Sz}(X,\varepsilon')}(B_{X^*}) \subset s_\varepsilon^\alpha(B_{X^*}).$$

This will be achieved with a transfinite induction on α. The statement is clearly true for $\alpha = 0$ and passes easily to limit ordinals. So, let us assume it is true for some ordinal α. Let now $x^* \in B_{X^*} \setminus s_\varepsilon^{\alpha+1}(B_{X^*})$. We need to show that $x^* \notin s_{\varepsilon\varepsilon'}^{(\alpha+1).\mathrm{Sz}(X,\varepsilon')}(B_{X^*})$, so we may assume that $x^* \in s_\varepsilon^\alpha(B_{X^*})$. Then, there is a weak*-open set V containing x^* and such that $\mathrm{diam}(V \cap s_\varepsilon^\alpha(B_{X^*})) < \varepsilon$ and therefore $\mathrm{diam}(V \cap s_{\varepsilon\varepsilon'}^{\alpha.\mathrm{Sz}(X,\varepsilon')}(B_{X^*})) < \varepsilon$. But, we have by homogeneity that every set C with diameter less than ε satisfies $s_{\varepsilon\varepsilon'}^{\mathrm{Sz}(X,\varepsilon')}(C) = \emptyset$. So $x^* \notin s_{\varepsilon\varepsilon'}^{(\alpha+1).\mathrm{Sz}(X,\varepsilon')}(B_{X^*})$, and the proof of a) is finished.

The statement b) is classical for $\mathbb{N}$-valued submultiplicative functions. □

Remark 3.6. We do not know if the functions $\mathrm{Cz}(X,.)$ and $\mathrm{Dz}(X,.)$ are submultiplicative, but we will see in sections 7.2 and 8 that they satisfy the property stated in b). One of the ingredients for that, will be the next simple comparison lemma, that we state and prove now.

Lemma 3.7. *Let X be a Banach space and $L^2(X)$ be the space of all square Bochner-integrable functions with respect to the Lebesgue measure on $[0,1]$. Then, for any $\varepsilon > 0$:*

$$Dz(X, 2\varepsilon) \le Sz(L^2(X), \varepsilon).$$

Proof. For simplicity, we denote $K = B_{X^*}$ and $L = B_{(L^2(X))^*}$. We recall that $L^2(X^*)$ is canonically identified with a subspace of $(L^2(X))^*$. Then, it is enough to show that for any $\varepsilon > 0$ and any ordinal α:

$$\forall k \in \mathbb{N} \ \ \forall x_1^*, .., x_k^* \in d_{2\varepsilon}^\alpha(K), \ \ \sum_{i=1}^k x_i^* \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \in s_\varepsilon^\alpha(L).$$

For a given $\varepsilon > 0$, this will be proved by transfinite induction. The statement is clear for $\alpha = 0$ and passes trivially to limit ordinals. So assume it is true for an ordinal α and let $x_1^*, .., x_k^* \in d_{2\varepsilon}^{\alpha+1}(K)$. It follows easily from the Hahn-Banach Theorem, that for all $i \in \{1, .., k\}$, x_i^* belongs to the weak*-closed convex hull of $d_{2\varepsilon}^\alpha(K) \setminus (x_i^* + \varepsilon B_{X^*})$. By induction hypothesis, $\phi = \sum_{i=1}^k x_i^* \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \in s_\varepsilon^\alpha(L)$. Let V be a weak*-open subset of $(L^2(X))^*$ containing ϕ. Then, there exist $n \in \mathbb{N}$ and $(x_{i,j}^*)_{1\le i\le k,\ 1\le j\le n} \subset d_{2\varepsilon}^\alpha(K)$ so that:

$$\forall i \le k \ \forall j \le n \ \ \|x_{i,j}^* - x_i^*\| \ge \varepsilon \ \text{ and } \ f = \sum_{i=1}^k \Big(\frac{1}{n} \sum_{j=1}^n x_{i,j}^*\Big) \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \in V.$$

We now consider the function ψ of period 1 and such that, for all $i \leq k$ and all $j \leq n$, $\psi = x^*_{i,j}$ on $[\frac{i-1}{k} + \frac{j-1}{kn}, \frac{i-1}{k} + \frac{j}{kn}[$. Next we set, for $l \in \mathbb{N}$ and $t \in [0,1]$, $\psi_l(t) = \psi(lt)$. Then the sequence $(\psi_l)_l$ converges to f for the weak*-topology of $(L^2(X))^*$. So, for l large enough, $\psi_l \in V$. On the other hand, $\psi_l \in s^\alpha_\varepsilon(L)$ (by induction hypothesis) and $\|\psi_l - \phi\| \geq \varepsilon$. This finishes our proof.

□

Not surprisingly, these indices are related to Baire's fundamental characterization of pointwise limits of sequences of continuous functions. Indeed we have:

Theorem 3.8. *Let X be a separable Banach space. The following assertions are equivalent.*
(i) X^* is separable.
(ii) $Dz(X) < \omega_1$, where ω_1 is the first uncountable ordinal.
(iii) $Cz(X) < \omega_1$.
(iv) $Sz(X) < \omega_1$.
(v) The identity map from (B_{X^*}, w^*) into $(X^*, \| \ \|)$ is of first Baire class.

Proof. Notice first that the separability of X implies that (B_{X^*}, w^*) is a complete metric space.

(i) ⇒ (ii). A fundamental result of I. Namioka and R. Phelps ([53]) asserts that if X^* is separable, then any non empty bounded subset of X^* has non empty weak*-slices of arbitrarily small diameter. So, for any $\varepsilon > 0$, $(d^\alpha_\varepsilon B_{X^*})_\alpha$ is a strictly decreasing family of weak*-closed subsets of B_{X^*}. Since (B_{X^*}, w^*) is separable, we get that for any $\varepsilon > 0$, $\mathrm{Dz}(X, \varepsilon) < \omega_1$. Thus $\mathrm{Dz}(X) = \sup_{n \in \mathbb{N}} \mathrm{Dz}(X, 1/n) < \omega_1$.

(ii) ⇒ (iii) and *(iii) ⇒ (iv)* are obvious.

(iv) ⇒ (v). Let $\phi = Id : (B_{X^*}, w^*) \to (X^*, \| \ \|)$. If $\mathrm{Sz}(X) < \omega_1$, then it follows from Baire's lemma that for any non empty weak*-closed subset F of B_{X^*}, $\phi_{|F}$ has a point of continuity. Indeed, otherwise, there would exist F weak*-closed subset of B_{X^*} and $n \in \mathbb{N}$ such that the set F_n of all x^* in F so that the oscillation of $\phi_{|F}$ at x^* is at least $1/n$ has a non empty interior in F. Thus, for any ordinal α and any $\varepsilon < 1/n$, the interior of F_n is included in $s^\alpha_\varepsilon(F)$ and therefore in $s^\alpha_\varepsilon(B_{X^*})$. Hence we can apply Baire's great theorem (see [4] for a historical reference) to conclude that ϕ is the pointwise limit of a sequence of weak* to norm continuous functions.

(v) ⇒ (i). Let (f_n) be a sequence of continuous functions from (B_{X^*}, w^*) into $(X^*, \| \ \|)$ so that for any $x^* \in B_{X^*}$, $\|f_n(x^*) - x^*\| \to 0$. Since B_{X^*} is weak*-separable, we have that for any n, $f_n(B_{X^*})$ is norm separable. Thus B_{X^*}, which is included in the norm closure of $\cup_n f_n(B_{X^*})$, is norm separable.

□

Remark 3.9. The oscillation index of a Baire class 1 function, among other indices, is defined and studied in detail by A. Kechris and A. Louveau in [43]. Then, the Szlenk index is simply the oscillation index of the identity map from (B_{X^*}, w^*) into $(X^*, \| \|)$.

Let us now recall that a Banach space X is called an *Asplund space* if every convex continuous function defined on a convex open subset U of X is Fréchet differentiable on a dense $\mathcal{G}_\delta$ subset of U. The following statement summarizes the classical theory of Asplund spaces (see the book of R. Deville, G. Godefroy and V. Zizler [17] for a complete reference).

Theorem 3.10. *Let X be a Banach space. The following assertions are equivalent:*
(i) X is an Asplund space.
(ii) Every bounded non empty subset of X^* has non empty weak*-slices of arbitrarily small diameter.
(iii) X^* has the Radon-Nikodým property.
(iv) Every separable subspace of X has a separable dual (in particular, a separable Banach space is an Asplund space if and only if its dual is separable).
(v) $Dz(X) < \infty$.
(vi) $Cz(X) < \infty$.
(vii) $Sz(X) < \infty$.

We will now show that if the indices $\mathrm{Sz}(X)$, $\mathrm{Dz}(X)$ and $\mathrm{Cz}(X)$ of a Banach space X are countable, then they are determined by the separable subspaces of X ([48]). More precisely, we have:

Theorem 3.11. *Let X be a Banach space and let $\alpha < \omega_1$. If $Dz(X) > \alpha$, then there is a closed separable subspace Y of X such that $Dz(Y) > \alpha$.*
The same is true for the other indices.

Proof. We will only indicate the proof for the dentability index Dz. For this, it is enough to show that there is a family of separable subspaces of X, $(X_\alpha)_{\alpha<\omega_1}$ such that for any countable ordinal α and any $\gamma \leq \alpha$: if $x^* \in d_\varepsilon^\gamma B_{X^*}$ then $x^*_{|X_\alpha}$, the restriction of x^* to X_α, belongs to $d_\varepsilon^\gamma B_{X^*_\alpha}$. Let us denote this statement by $(H_{\alpha,\gamma})$.

We will first build (X_α) by transfinite induction. We pick $x \neq 0$ in X and set $X_0 = \mathbb{R}x$.
If α is a limit ordinal, we define X_α to be the closed linear span of $\cup_{\beta<\alpha} X_\beta$.
If $\alpha = \beta + 1$, we call $V_0 = X_\beta$. Let D_0 be a countable dense subset of V_0 and $\mathcal{S}_0$ be the collection of all half spaces $S = \{x^* \in X^* : \ x^*(z) > q\}$ with $z \in D_0$ and $q \in \mathbb{Q}$. If $S \in \mathcal{S}_0$ intersects $d_\varepsilon^{\gamma+1} B_{X^*}$ for some $\gamma \leq \beta$, then the diameter of $S \cap d_\varepsilon^\gamma B_{X^*}$ is greater than ε and therefore we can find u^*, v^* in $S \cap d_\varepsilon^\gamma B_{X^*}$ and $x = x(\gamma, S)$ in B_X such that $(u^* - v^*)(x) > \varepsilon$. Let us now denote by V_1

the closed linear span of $X_\beta \cup \{x(\gamma, S),\ \gamma \le \beta,\ S \in \mathcal{S}_0\}$ and pick a countable dense subset D_1 of V_1. Then $V_2, D_2,\ V_3, D_3$,.. are constructed similarly and X_α is the closed linear span of $\bigcup_{n=0}^{\infty} V_n$.

We will now prove by induction on α, that $(H_{\alpha,\gamma})$ is true for any $\gamma \le \alpha$. If $\alpha = 0$ or if α is a limit ordinal, the conclusion is clear. So assume that $\alpha = \beta + 1$ and that for any $\gamma \le \beta$, $(H_{\beta,\gamma})$ is true. Clearly, we only need to prove $(H_{\alpha,\alpha})$. So let $x^* \in d_\varepsilon^{\beta+1} B_{X^*}$ and let S be a slice of $d_\varepsilon^\beta B_{X^*}$ containing x^*. We may assume that $S = \{x^* \in X^* :\ x^*(z) > q\}$, with z in some D_n and q in $\mathbb{Q}$ (following the notation used above in the construction of $X_{\beta+1}$). Let u^* and v^* in $S \cap d_\varepsilon^\beta B_{X^*}$ such that $(u^* - v^*)(x(\beta, S)) > \varepsilon$. By induction hypothesis $u^*_{|X_\beta}$ and $v^*_{|X_\beta}$ belong to $d_\varepsilon^\gamma B_{X^*_\beta}$. Thus the diameter of $S \cap d_\varepsilon^\beta B_{X^*_\beta}$ and therefore the diameter of $S \cap d_\varepsilon^\beta B_{X^*_\alpha}$ are greater than ε. So $x^*_{|X_\alpha} \in d_\varepsilon^{\gamma+1} B_{X^*_\alpha}$. □

Remark 3.12. It is also proved in [48] that if X is a Banach space with separable dual such that $\mathrm{Sz}(X) > \alpha$ for some countable ordinal α, then there is a quotient $Y = X/Z$ of X, with a shrinking basis and such that $\mathrm{Sz}(Y) > \alpha$.

In a recent work [63], S. Todorcevic proved that under a suitable "Baire-like" axiom, every Banach space of density character $\aleph_1$, has a quotient with a transfinite basis. One can wonder if this deep result leads to a non separable version of the above remark.

4. Universal spaces

Let us first recall that a Banach space X is universal for a class of Banach spaces $\mathcal{F}$ if any Banach space Y in $\mathcal{F}$ is isomorphic to a subspace of X. For instance, it is a well known result, due to S. Mazur, that $C(\Delta)$, the space of continuous functions on the Cantor set, is universal for the class of all separable Banach spaces. In [62], W. Szlenk introduced his index to prove the following fundamental result.

Theorem 4.1. *There is no separable reflexive Banach space universal for all reflexive separable Banach spaces.*

Proof. Since every separable reflexive Banach space has a countable Szlenk index, it is enough to build a family $(X_\alpha)_{\alpha<\omega_1}$ of separable reflexive Banach spaces such that for any countable ordinal α, $\mathrm{Sz}(X_\alpha, 1) > \alpha$. We define X_α inductively as follows: $X_1 = \ell_2$, $X_{\alpha+1} = X_\alpha \oplus_1 \ell_2$ and $X_\alpha = (\sum_{\beta<\alpha} \oplus X_\beta)_{\ell_2}$ if α is a limit ordinal. One can prove, by a straightforward transfinite induction that for any countable ordinal α, X_α is separable and reflexive. Then, another induction shows that for any $\alpha < \omega_1$, $0 \in s_1^\alpha B_{X^*}$. This relies on the following: if $x^* \in s_1^\alpha B_{X^*}$, then for any $n \in \mathbb{N}$ $(x^*, e_n) \in s_1^\alpha (B_{X^* \oplus_\infty \ell_2})$, where (e_n) is the canonical basis of ℓ_2. Since $(0, e_n)$ is weak* null in $X^* \oplus_\infty \ell_2 = (X \oplus_1 \ell_2)^*$, we get that $(x^*, 0) \in s_1^{\alpha+1}(B_{(X \oplus_1 \ell_2)^*})$. □

Note that J. Bourgain [12] significantly improved this result by showing:

Theorem 4.2. *If a Banach space X is universal for all separable reflexive Banach spaces then X contains an isomorphic copy of $C(\Delta)$.*

Remarks 4.3. 1) A recent important result of E. Odell and T. Schlumprecht [55] asserts the existence of a separable reflexive space, which is universal for the class of all separable superreflexive Banach spaces. In fact, they obtain a stronger result, which is that this space is universal for the class of all reflexive Banach spaces X such that $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.
In a work in preparation, E. Odell, T. Sclumprecht and A. Zsak improve this result and show that for any fixed countable ordinal α, there is a reflexive space which is universal for all reflexive Banach spaces X such that $\mathrm{Sz}(X) \leq \alpha$ and $\mathrm{Sz}(X^*) \leq \alpha$ (private communication).

2) However, one can show, with a simple transfinite induction (see [45] for details), that the spaces $(X_\alpha)_{\alpha<\omega_1}$ satisfy the following property: for any $\alpha < \omega_1$, X_α admits a Schauder basis $(e_n^\alpha)_{n=1}^\infty$ so that

$$\forall n \geq 1\ \forall x \in \mathrm{sp}\{e_1^\alpha, .., e_n^\alpha\}\ \forall y \in \mathrm{sp}\{e_{n+1}^\alpha, ..\}\ \ \|x+y\|^2 \geq \|x\|^2 + \|y\|^2.$$

Then, it follows easily that for any $\alpha < \omega_1$, $\mathrm{Sz}(X_\alpha^*) \leq \omega$. Thus, there is no separable reflexive space universal for the class of reflexive spaces X with $\mathrm{Sz}(X) \leq \omega$. As well, there is no separable reflexive space universal for the class of reflexive spaces X with $\mathrm{Sz}(X^*) \leq \omega$.

3) Universality problems in a descriptive set theoretical context are revisited in the important recent work by S. Argyros and P. Dodos ([2]). This point of view is further developed in an even more recent work of P. Dodos and V. Ferenczi ([19]). As a consequence of their study of some strongly bounded classes of Banach spaces (a notion introduced in [2]), they obtain for instance that for every countable ordinal α, there exists a Banach space Y_α with separable dual such that every separable Banach space X with $\mathrm{Sz}(X) \leq \alpha$ embeds into Y_α.

5. The Szlenk index of $C(K)$ spaces

Another classical feature of the Szlenk index is that it perfectly describes the isomorphic classification of $C(K)$ spaces when K is a countable compact space. In this section, we shall detail this property.

First, we need to recall some standard facts about ordinals. For that purpose, we follow the notation of [58]. An ordinal α is identified with the set of ordinals β such that $\beta < \alpha$. For an ordinal α, we denote $\alpha+ = \alpha + 1$. We always consider that the sets of ordinals are topological spaces equipped with the order topology. Then, for any ordinal α, $C(\alpha+)$ is the space of all real valued continuous functions on $[0, \alpha]$ equipped with the supremum norm and $C_0(\alpha) = \{f \in C(\alpha+),\ f(\alpha) = 0\}$. Note that for any infinite α, $C(\alpha+)$

is isomorphic to $C_0(\alpha)$. Through the natural isometries, we will identify the dual space of $C(\alpha+)$ to $\ell_1([0,\alpha])$ and the dual space of $C_0(\alpha)$ to $\ell_1([0,\alpha))$. For α and β countable ordinals, we set $e_\alpha(\beta)=1$ if $\alpha=\beta$ and 0 otherwise.

The isomorphic classification of these spaces is described by the following fundamental result due to C. Bessaga and A. Pełczyński [8].

Theorem 5.1. *Let α and β be two ordinals so that $\omega \le \alpha \le \beta < \omega_1$. Then $C(\alpha+)$ is isomorphic to $C(\beta+)$ if and only if $\beta < \alpha^\omega$.*

Then, C. Samuel [59] performed the following computation.

Theorem 5.2. *For any $0 \le \alpha < \omega_1$,*

$$Sz(C(\omega^{\omega^\alpha}+)) = \omega^{\alpha+1}.$$

By a result of A.A. Milutin [52], $C(K)$ and $C(L)$ are isomorphic, whenever K and L are two compact uncountable metrizable spaces. In that case, $C(K)$ and $C(L)$ are not Asplund. So we can state:

Corollary 5.3. *The Szlenk index characterizes the isomorphism class of $C(K)$ spaces, for K compact and metrizable.*

Samuel's proof contains more information but is rather complicated and relies on the following deep result of D. Alspach and Y. Benyamini [1].

Theorem 5.4. *Let X be a $\mathcal{L}^\infty$-space. Then X has a quotient isomorphic to $C(\omega^{\omega^\alpha}+)$ if and only if $Sz(X) > \omega^\alpha$.*

We shall indicate here a direct and elementary proof due to P. Hájek and G. Lancien [33]. H.P. Rosenthal conjectured that there should be such a proof in his survey paper on $C(K)$ spaces [58].

Proof. Showing the inequality $\mathrm{Sz}(C(\omega^{\omega^\alpha}+)) \ge \omega^{\alpha+1}$ is the easy part of the proof. Indeed, using the fact that the set $(e_\gamma)_{\gamma\le\beta}$ is 2-separated for the norm of $\ell_1([0,\beta])$ and w^*-homeomorphic to $[0,\beta]$, we get that for any $\beta<\omega_1$, $\mathrm{Sz}(C(\omega^\beta+),1) > \beta$ (see [58] for details). Then Proposition 3.3 implies that $\mathrm{Sz}(C(\omega^{\omega^\alpha}+)) \ge \omega^{\alpha+1}$.

So we now concentrate on the converse inequality. For a fixed $0 \le \alpha < \omega_1$, we denote $Z = \ell_1([0,\omega^{\omega^\alpha}))$ equipped with the weak*-topology induced by $C_0(\omega^{\omega^\alpha})$. Then, for all $\gamma < \omega^{\omega^\alpha}$, we set $Z_\gamma = \ell_1([0,\gamma])$ equipped with the weak*-topology induced by $C(\gamma+)$ and P_γ the canonical projection from Z onto Z_γ. The following lemma is the crucial step of our argument (in this statement, the Szlenk derived sets are meant with the weak*-topologies described above for Z and Z_γ).

Lemma 5.5. *Let $\alpha<\omega_1$, $\gamma<\omega^{\omega^\alpha}$, $\beta<\omega_1$ and $\varepsilon>0$.*

If $z \in s^\beta_{3\varepsilon}(B_Z)$ and $\|P_\gamma z\| > 1-\varepsilon$, then $P_\gamma z \in s^\beta_\varepsilon(B_{Z_\gamma})$.

Proof. Given $\alpha < \omega_1$, $\gamma < \omega^{\omega^\alpha}$, and $\varepsilon > 0$, we denote, for $\beta < \omega_1$, by (H_β) the implication to be proved. This will be done by transfinite induction on β. (H_0) is trivially true and (H_β) passes easily to limit ordinals. So assume (H_β) is true and let us prove $(H_{\beta+1})$. Let $z \in B_Z$ such that $\|P_\gamma z\| > 1-\varepsilon$ and $P_\gamma z \notin s_\varepsilon^{\beta+1}(B_{Z_\gamma})$. We need to show that $z \notin s_{3\varepsilon}^{\beta+1}(B_Z)$, so we may assume that $z \in s_{3\varepsilon}^{\beta}(B_Z)$ and therefore that $P_\gamma z \in s_\varepsilon^\beta(B_{Z_\gamma})$. Then, there is a weak*-open subset V of Z_γ containing $P_\gamma z$ such that $d =$diam$(V \cap s_\varepsilon^\beta(B_{Z_\gamma})) < \varepsilon$. Using the Hahn-Banach separation theorem, we may choose V so that $V \cap (1-\varepsilon)B_{Z_\gamma} = \emptyset$. We may also assume that

$$V = \bigcap_{i=1}^n \{x \in Z_\gamma,\ f_i(x) > \alpha_i\},$$

where $\alpha_i \in \mathbb{R}$ and $f_i \in C(\gamma+)$. We now define functions $g_i \in C_0(\omega^{\omega^\alpha})$ by $g_i = f_i$ on $[0,\gamma]$ and $g_i = 0$ on $(\gamma, \omega^{\omega^\alpha})$. Then we consider the weak*-open subset of Z:

$$U = \bigcap_{i=1}^n \{y \in Z,\ g_i(y) > \alpha_i\}.$$

It is clear that $z \in U \cap s_{3\varepsilon}^\beta(B_Z)$. For any $y \in U \cap s_{3\varepsilon}^\beta(B_Z)$, $P_\gamma y \in V$, so $\|P_\gamma y\| > 1-\varepsilon$ and by the induction hypothesis $P_\gamma y \in V \cap s_\varepsilon^\beta(B_{Z_\gamma})$. Therefore for all $y, y' \in U \cap s_{3\varepsilon}^\beta(B_Z)$, $\|P_\gamma y - P_\gamma y'\| \le d < \varepsilon$. Since moreover $\|P_\gamma y\| > 1-\varepsilon$ and $\|P_\gamma y'\| > 1-\varepsilon$, we have that $\|y - y'\| \le d + 2\varepsilon < 3\varepsilon$. We have shown that diam$(U \cap s_{3\varepsilon}^\beta(B_Z)) < 3\varepsilon$ and therefore $z \notin s_{3\varepsilon}^{\beta+1}(B_Z)$. This finishes our induction.

□

In order to conclude the proof of Theorem 5.2, it is enough to show that

$$\forall 0 \le \alpha < \omega_1\ \ \forall \gamma < \omega^{\omega^\alpha}\ \ \forall \varepsilon > 0\ \ \ s_\varepsilon^{\omega^\alpha}(B_{Z_\gamma}) = \emptyset. \tag{5.2}$$

This will be done by transfinite induction on α. If $\alpha = 0$, then for any $\gamma < \omega$, Z_γ is finite dimensional and therefore $s_\varepsilon(B_{Z_\gamma}) = \emptyset$. So the statement is true for $\alpha = 0$. It also passes easily to limit ordinals. So assume now that it is true for α. Then Lemma 5.5 implies that

$$\forall \varepsilon > 0\ \ s_\varepsilon^{\omega^\alpha}(B_Z) \subset (1 - \frac{\varepsilon}{3})B_Z, \tag{5.3}$$

where $Z = \ell_1([0, \omega^{\omega^\alpha}))$ is equipped with the weak*-topology induced by $C_0(\omega^{\omega^\alpha})$. It now follows from (5.3) and Proposition 3.4 that

$$\forall \varepsilon > 0\ \ s_\varepsilon^{\omega^{\alpha+1}}(B_Z) = \emptyset \tag{5.4}$$

Now, Theorem 5.1 implies that for any $\omega^{\omega^\alpha} \le \gamma < \omega^{\omega^{\alpha+1}}$, $C(\gamma+)$ is isomorphic to $C(\omega^{\omega^\alpha}+)$ and therefore to $C_0(\omega^{\omega^\alpha})$. So $s_\varepsilon^{\omega^{\alpha+1}}(B_{Z_\gamma}) = \emptyset$, for any $\varepsilon > 0$ and any $\gamma < \omega^{\omega^{\alpha+1}}$. This finishes our induction (Note that we only used the "if" part of Theorem 5.1, which is the easy one). □

Remark 5.6. It is also shown in [33], that

$$\forall \alpha \in [\omega_1, \omega_1.\omega), \quad \mathrm{Sz}(C(\alpha+)) = \omega_1.\omega.$$

However, Z. Semadeni proved in [60] that when $\omega_1 \leq \alpha < \beta < \omega_1.\omega$, $C(\alpha+)$ and $C(\beta+)$ are isomorphic if and only if $\omega_1.n \leq \alpha < \beta < \omega_1.(n+1)$, for some integer n. So, unlike in the separable case, the Szlenk index does not distinguish the isomorphic classes for the non separable $C(\alpha+)$ spaces.

6. The Szlenk index as a coanalytic rank

Our first task will be to prove the following improvement of Theorem 3.8, which is based on ideas of B. Bossard (see [9] or [10]).

Theorem 6.1. *There exists a universal function $\psi : [1,\omega_1) \to [1,\omega_1)$ such that if $\alpha < \omega_1$ and X is a Banach space satisfying $Sz(X) \leq \alpha$, then $Dz(X) \leq \psi(\alpha)$.*

Before to proceed with the proof, we need to introduce some notation and background. Let $K = (B_{\ell_\infty}, \sigma(\ell_\infty, \ell_1))$. K is a compact metrizable space. We denote by $\mathcal{F}$ be the set of all closed subsets of K and we equip $\mathcal{F}$ with the usual Hausdorff topology. Then $\mathcal{F}$ is again compact and metrizable. The next lemma is due to B. Bossard ([9]):

Lemma 6.2. *For any $\varepsilon > 0$, s_ε and d_ε are Borel maps on $\mathcal{F}$.*

Proof. We will only detail the argument for d_ε. Then it is enough to prove that for any open subset O of K, $H = \{F \in \mathcal{F},\ d_\varepsilon F \subset O\}$ is a Borel subset of $\mathcal{F}$.
For $x \in \ell_1$ and $a \in \mathbb{R}$, we denote $S_{x,a} = \{x^* \in K,\ x^*(x) > a\}$. Let V be the set of all linear combinations with rational coefficients of the elements of the canonical basis of ℓ_1. Then, we clearly have:

$$H = \{F \in \mathcal{F} :\ \forall x^* \in F\backslash O\ \exists (x,a) \in V \times \mathbb{Q},\ x^* \in S_{x,a} \text{ and } \mathrm{diam}(S_{x,a} \cap F) < \varepsilon\}.$$

Let $\mathcal{P}$ be the set of all finite subsets of $V \times \mathbb{Q}$. Then, by compactness, we get that

$$H = \bigcup_{I \in \mathcal{P}} \Big(\{F \in \mathcal{F},\ F \subset \bigcup_{(x,a)\in I} S_{x,a} \cup O\} \ \cap \bigcap_{(x,a)\in I} H_{x,a} \Big),$$

where $H_{x,a} = \{F \in \mathcal{F},\ \mathrm{diam}(F \cap S_{x,a}) < \varepsilon\}$. Then it easy to verify that $H_{x,a}$ is a closed subset of $\mathcal{F}$. Finally, since $\mathcal{P}$ is countable, we obtain that H is a $\mathcal{G}_{\delta\sigma}$ subset of $\mathcal{F}$. □

Proof of Theorem 6.1. In view of Theorem 3.11, we may clearly assume that X is separable. Then X is isometric to a quotient of ℓ_1 and we can identify

B_{X^*} with an element of $\mathcal{F}$. So, it is enough to build ψ such that if $\alpha < \omega_1$, $F \in \mathcal{F}$ and $s_\varepsilon^\alpha F = \emptyset$ for any $\varepsilon > 0$, then $d_\varepsilon^{\psi(\alpha)} F = \emptyset$ for any $\varepsilon > 0$.
Let us denote

$$\mathcal{B}_\alpha = \{F \in \mathcal{F},\ \forall \varepsilon > 0\ \ s_\varepsilon^\alpha F = \emptyset\}.$$

By Lemma 6.2, $\mathcal{B}_\alpha$ is a Borel subset of $\mathcal{F}$. On the other hand, any F in $\mathcal{B}_\alpha$ is norm separable and therefore weak*-dentable (see [18]), so

$$\forall \varepsilon > 0\ \ \mathcal{B}_\alpha \subset \{F \in \mathcal{F},\ d_\varepsilon^{\omega_1} F = \emptyset\} = C_\varepsilon.$$

The crucial point is now to use the results of C. Dellacherie [14] on the applications of the Kunen-Martin Theorem to the study of analytic derivations. It follows from this work that for any $\varepsilon > 0$ and any Borel subset B of C_ε, there exists $\beta < \omega_1$ such that $B \subset \{F \in \mathcal{F},\ d_\varepsilon^\beta F = \emptyset\}$. So,

$$\forall \alpha < \omega_1\ \forall n \in \mathbb{N}\ \exists \psi_n(\alpha) < \omega_1,\ \ \mathcal{B}_\alpha \subset \{F \in \mathcal{F},\ d_{1/n}^{\psi_n(\alpha)} F = \emptyset\}.$$

We can now conclude the proof by taking $\psi(\alpha) = \sup_{n \geq 1} \psi_n(\alpha)$. □

Remark 6.3. Let us now denote

$$\psi(\alpha) = \sup_{\mathrm{Sz}(X) \leq \alpha} \mathrm{Dz}(X).$$

It follows from Proposition 3.1 (*iv)* and *v)*) that $\psi(1) = \omega$. It is easy to see that $\psi(\omega) \geq \omega^2$. Indeed, $\mathrm{Sz}(c_0(\mathbb{N})) = \omega$, while $\mathrm{Dz}(X) \leq \omega$ if and only if X is super-reflexive (as we will see in the next sections).
It is proved in [33] that $\psi(\omega) = \omega^2$, but the values of $\psi(\omega^\alpha)$, for $\alpha \geq 2$ are not known.
Similarly, we can define

$$\phi(\alpha) = \sup_{\mathrm{Sz}(X) \leq \alpha} \mathrm{Cz}(X), \quad \text{and} \quad \theta(\alpha) = \sup_{\mathrm{Cz}(X) \leq \alpha} \mathrm{Dz}(X).$$

As we will show in section 8, $\phi(\omega) = \omega$. But we do not know if the function ϕ is the identity.
Let us also mention, that it follows from recent results of F. García, L. Oncina, J. Orihuela and S. Troyanski (see [25] and [26]), that $\theta(\alpha) \leq \omega^\omega \alpha$.

We are very grateful to the referee for suggesting the following statement:

Proposition 6.4. *There is an uncountable set $S \subset [1, \omega_1)$ such that ψ is the identity on S.*

Proof. Let $A = \psi([1, \omega_1))$. We define $f : A \to [1, \omega_1)$ by

$$\forall \alpha \in A,\ \ f(\alpha) = \inf_{X,\ \mathrm{Dz}(X) \geq \alpha} \mathrm{Sz}(X).$$

Assume that $f(\alpha) < \alpha$, for all $\alpha \in A$. Then, the so-called "pressing down lemma" and the fact that A is uncountable imply the existence of an uncountable subset B of A such that f is constant on B (we use here the simplest

version of this lemma due to J. Nowák [54]). This is clearly in contradiction with Theorem 6.1. So, there is $\alpha \in A$ such that $f(\alpha) = \alpha$. Let $\beta \in [1, \omega_1)$ so that $\psi(\beta) = \alpha$. We now wish to prove that $\beta = \alpha$. So let us assume that $\beta < \alpha$.
If there exists a separable Banach space X such that $\mathrm{Sz}(X) \leq \beta$ and $\mathrm{Dz}(X) = \alpha$, we directly have a contradiction with the equality $f(\alpha) = \alpha$. Otherwise, it follows from Proposition 3.3, that $\alpha = \omega^\lambda$, where λ is a limit ordinal. Besides, with can find a sequence of separable Banach spaces $(X_n)_{n=0}^\infty$ such that $\sup_n \mathrm{Dz}(X_n) = \alpha$ and for all $n \in \mathbb{N}$, $\mathrm{Sz}(X_n) \leq \beta$. Let now X be the ℓ_2-sum of the X_n's. We have that $\mathrm{Dz}(X) \geq \alpha$. On the other hand, one can show, using some techniques in the spirit of Lemma 5.5, that $\mathrm{Sz}(X) \leq \beta.\omega < \alpha$. This is again a contradiction.

This proves the existence of $\alpha \geq 1$, such that $\psi(\alpha) = \alpha$. The same reasoning can be applied to f restricted to any uncountable subset of A. This implies that $S = \{\alpha \in [1, \omega_1), \quad \psi(\alpha) = \alpha\}$ is uncountable.

□

Note. Let us say a few words on the formal setting developed by B. Bossard (see [9] and [10]) in which Theorem 6.1 can be properly interpreted. The space $E = C(\Delta)$ of all continuous functions on the Cantor set Δ is isometrically universal for all separable Banach spaces. So we can consider the set of all closed subspaces of E, denoted by $\mathcal{G}(E)$, as being the "set of all separable Banach spaces". The set $\mathcal{F}$ of all closed subsets of E can be equipped with the Effros Borel-structure, generated by the sets $\{F \in \mathcal{F},\ F \cap O_n \neq \emptyset\}$, where $(O_n)_{n=1}^\infty$ is a basis for the topology of E. The above construction does not depend on the choice of the basis $(O_n)_{n=1}^\infty$ and defines a standard Borel structure. Then one can show that $\mathcal{G}(E)$ is a Borel subset of $\mathcal{F}$ and therefore inherits its Borel structure. Let us also recall that a subset A of a standard Borel space B is analytic if there is a standard Borel space B' such that A is the canonical projection on B of a Borel subset of $B \times B'$. Then a subset C of B is coanalytic in B if $B \setminus C$ is analytic. Finally, if C is coanalytic in B, we say that a map $r : B \to \omega_1$ is a coanalytic rank for C if $C = \{x \in B,\ r(x) < \omega_1\}$ and the sets $\{(x, y) \in B \times B,\ r(x) < r(y)\}$ and $\{(x, y) \in B \times B,\ x \in C \text{ and } r(x) \leq r(y)\}$ are coanalytic in $B \times B$.

The following result is fundamental in the descriptive set theory (see the book of A. Kechris [42] for a complete exposition).

Theorem 6.5. *1) Let C be a coanalytic subset of a standard Borel space. Then C admits a coanalytic rank.*

2) Let C be a coanalytic subset of a standard Borel space B and r be a coanalytic rank for C. Then, for every $\alpha < \omega_1$, $B_\alpha = \{x \in B,\ r(x) \leq \alpha\}$ is a Borel subset of B. Moreover, for every analytic subset A of C, there is $\alpha < \omega_1$ such that $A \subset B_\alpha$.

In this setting, B. Bossard proved the following.

Theorem 6.6. *The set $C = \{X \in \mathcal{G}(E),\ X^* \text{ is separable}\}$ is coanalytic non Borel in $\mathcal{G}(E)$ and the applications Sz and Dz are coanalytic ranks for C.*

For an overview of the applications of the descriptive set theory in Banach space geometry, we refer the reader to the survey paper by S.A. Argyros, G. Godefroy and H.P. Rosenthal [3] and references therein (including in particular the work of B. Bossard).

7. Locally uniformly rotund norms, uniformly rotund norms and slicing indices

7.1. **The Szlenk index and locally uniformly rotund norms.** First, we recall that a norm $\| \ \|$ of a Banach space X is *locally uniformly rotund* (in short LUR) if $\lim \|x - x_n\| = 0$, whenever $\|x\| = \|x_n\| = 1$ for all $n \geq 1$ and $\lim_{n\to\infty} \|x + x_n\| = 2$. Our first proposition, whose proof can be found in [17], describes the well known duality between local uniform rotundity and Fréchet smoothness:

Proposition 7.1. *Let $(X, \| \ \|)$ be a Banach space such that the dual norm of $\| \ \|$ is LUR. Then $\| \ \|$ is Fréchet differentiable on $X \setminus \{0\}$ (in short F-smooth).*

It is known that if a Banach space has an equivalent F-smooth norm then it is an Asplund space (see [17]) but R. Haydon proved that the converse is false [32]. However, it follows from the work of Asplund, Kadets and Klee that this equivalence is true in the separable case. More precisely, we have:

Theorem 7.2. *Let X be a separable Banach space. The following assertions are equivalent:*
(i) X is an Asplund space.
(ii) X admits an equivalent norm whose dual norm is LUR.
(iii) X admits an equivalent F-smooth norm.

Our next result ([48]) can be seen as a non separable extension of this theorem.

Theorem 7.3. *Let X be a Banach space. If $Sz(X) < \omega_1$, then X admits an equivalent norm whose dual norm is LUR.*

Proof. By Theorem 6.1, we have that $\mathrm{Dz}(X) < \omega_1$. For $n \geq 1$ and $\alpha < \mathrm{Dz}(X, 2^{-n})$, we choose $a_{n,\alpha} > 0$ such that

$$\sum_{n=1}^{\infty} \sum_{\alpha < \mathrm{Dz}(X, 2^{-n})} a_{n,\alpha}^2 = 1.$$

Then, we set

$$\forall x^* \in X^* : \ f(x^*) = \|x^*\|^2 + \sum_{n=1}^{\infty} \sum_{\alpha < \mathrm{Dz}(X,2^{-n})} f_{n,\alpha}^2(x^*),$$

where $f_{n,\alpha}(x^*) = a_{n,\alpha}\,\mathrm{dist}(x^*, d_{2^{-n}}^{\alpha} B_{X^*})$.

Lemma 7.4. *Let x^* in X^* and $(x_k^*)_{k=1}^{\infty}$ be a sequence in X^* such that*

$$\forall k \geq 1 \ f(x^*) = f(x_k^*) = 1 \quad \text{and} \quad \lim_{k\to\infty} f(\frac{x^* + x_k^*}{2}) = 1.$$

Then

$$\lim_{k\to\infty} \|x^* - x_k^*\| = 0.$$

Proof. The first step is to show that

$$\forall n \geq 1 \ \forall \alpha < \mathrm{Dz}(X, 2^{-n}), \quad \lim_{k\to\infty} f_{n,\alpha}(x_k^*) = f_{n,\alpha}(x^*). \tag{7.5}$$

By convexity of f, we get that

$$\left(\frac{\|x^*\| + \|x_k^*\|}{2}\right)^2 + \sum_{n=1}^{\infty} \sum_{\alpha < \mathrm{Dz}(X,2^{-n})} \left(\frac{f_{n,\alpha}(x^*) + f_{n,\alpha}(x_k^*)}{2}\right)^2 \to 1.$$

Then, (7.5) follows easily from the uniform convexity of $\ell_2(\mathbb{N})$.

Let us now fix $\varepsilon > 0$ and pick n so that $2^{-n} < \varepsilon/2$. Then, there exists $\alpha < \mathrm{Dz}(X, 2^{-n})$ such that $x^* \in d_{2^{-n}}^{\alpha} B_{X^*} \setminus d_{2^{-n}}^{\alpha+1} B_{X^*}$. Using (7.5) and a simple approximation argument, we may as well assume that $(x_k^*) \subset d_{2^{-n}}^{\alpha} B_{X^*}$. Thus $(x^* + x_k^*)/2$ is also in $d_{2^{-n}}^{\alpha} B_{X^*}$. If we assume that $\|x^* - x_k^*\| \geq \varepsilon$, then any weak*-slice of $d_{2^{-n}}^{\alpha} B_{X^*}$ containing $(x^* + x_k^*)/2$ has diameter at least $\varepsilon/2$ and therefore $(x^* + x_k^*)/2$ belongs to $d_{2^{-n}}^{\alpha+1} B_{X^*}$. This implies that

$$f_{n,\alpha+1}(\frac{x^* + x_k^*}{2}) = 0 \text{ and } \frac{1}{2}(f_{n,\alpha+1}(x^*) + f_{n,\alpha+1}(x_k^*)) \geq \frac{1}{2} f_{n,\alpha+1}(x^*) > 0.$$

Since $\lim_{k\to\infty} f(\frac{x^*+x_k^*}{2}) = 1$ and all $f_{m,\beta}$'s are convex, this is impossible for k large enough. Therefore $\lim_{k\to\infty} \|x^* - x_k^*\| = 0$. □

Clearly, $f(x^*) \geq \|x^*\|^2$. On the other hand, for any $n \in \mathbb{N}$ and any $\alpha < \mathrm{Dz}(X, 2^{-n})$, $0 \in d_{2^{-n}}^{\alpha}(B_{X^*})$ and therefore $\mathrm{dist}(x^*, d_{2^{-n}}^{\alpha} B_{X^*}) \leq \|x^*\|$. Thus $f(x^*) \leq 2\|x^*\|^2$. Moreover, since the sets $d_{2^{-n}}^{\alpha} B_{X^*}$ are weak*-closed, convex and symmetric, f is weak* lower semicontinuous and the Minkowski functional N of $\{x^* \in X^*,\ f(x^*) \leq 1\}$ is the dual norm of an equivalent norm on X. Finally, the local uniform rotundity of N follows easily from Lemma 7.4. □

We will now explain how this theorem applies to the $C(K)$-spaces. If K is a compact space, we denote by K' the set of all accumulation points of K. Then, we define inductively $K^{\alpha+1} = (K^{\alpha})'$ for any ordinal α and $K^{\alpha} = \cap_{\beta<\alpha} K^{\beta}$ if α is a limit ordinal. A compact space K is said to be scattered if $K^{\alpha} = \emptyset$

for some ordinal α. it is known (see [53]) that $C(K)$ is an Asplund space if and only if K is scattered. R. Haydon [32] showed the existence of a compact space K such that K^{ω_1} is a singleton, but so that $C(K)$ does not admit any equivalent Gâteaux-smooth norm. However, we can deduce from Theorem 7.3 the following result of R. Deville [15], which can now be seen as an optimal result about F-smooth renormings of $C(K)$-spaces:

Theorem 7.5. *Let K be a compact space such that $K^{\omega_1} = \emptyset$. Then $C(K)$ has an equivalent norm whose dual norm is LUR.*

In view of Theorem 7.3, we have to show that $\mathrm{Sz}(C(K)) < \omega_1$. More precisely, we will prove:

Proposition 7.6. *Let K be a compact space such that $K^{\omega^\alpha} \neq \emptyset$ and $K^{\omega^{\alpha+1}} = \emptyset$, with $\alpha < \omega_1$. Then $Sz(C(K)) = \omega^{\alpha+1}$.*

Proof. A straightforward transfinite induction shows that if $t \in K^\beta$, then the Dirac mass δ_t belongs to $s_1^\beta B_{C(K)^*}$. Thus $\mathrm{Sz}(C(K)) > \omega^\alpha$ and therefore, by Proposition 3.3, $\mathrm{Sz}(C(K)) \geq \omega^{\alpha+1}$.

We now turn to the converse inequality. Let X be a separable subspace of $C(K)$. For $t \in K$, we denote by $\phi(t)$ the restriction of δ_t to X. Clearly, ϕ is a continuous map from K into B_{X^*} equipped with its weak*-topology. Then $L = \phi(K)$ is metrizable and compact and X embeds isometrically into $C(L)$ in a canonical way. By a simple transfinite induction, we get that for any ordinal β, $L^\beta \subset \phi(K^\beta)$ (see Lemma VI.8.1 in [17] for details). So $L^{\omega^{\alpha+1}} = \emptyset$. Therefore L is countable and it follows from Theorem 5.2 that $\mathrm{Sz}(C(L)) \leq \omega^{\alpha+1}$. Thus $\mathrm{Sz}(X) \leq \omega^{\alpha+1}$. Finally, Theorem 3.11 about separable determination yields the conclusion. □

7.2. **The dentability index, uniformly rotund renormings and supereflexivity.** We recall that, if $(X, \|\ \|)$ is a Banach space, the *modulus of convexity* of the norm $\|\ \|$ is defined by

$$\forall \varepsilon \in (0,2]\ \ \delta_{\|\ \|}(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2},\ \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\}.$$

The norm $\|\ \|$ is *uniformly rotund* (in short UR) if $\delta_{\|\ \|}(\varepsilon) > 0$, for all $\varepsilon > 0$. The norm $\|\ \|$ is said to have a *power type modulus of convexity* if

$$\exists C > 0\ \exists p \geq 2\ \text{ such that }\ \forall \varepsilon \in (0,2]\ \ \delta_{\|\ \|}(\varepsilon) \geq C\varepsilon^p.$$

Without any use of supereflexivity, we shall prove the following (this is taken from [47]).

Theorem 7.7. *Let X be a Banach space. The following assertions are equivalent:*

(i) X admits an equivalent UR norm.

(ii) $D(X) \leq \omega$.
(iii) X admits an equivalent UR norm with power type modulus of convexity.

Proof. *(iii)* $\Rightarrow$ *(i)* is trivial.
(i) $\Rightarrow$ *(ii)*. Let $|\ |$ be an equivalent UR norm on X, B its unit ball, $\varepsilon > 0$ and $x \in B \setminus (1 - \delta_{|\ |}(\varepsilon))B$. By Hahn-Banach Theorem there is a slice T of B containing x and whose intersection with $(1 - \delta(\varepsilon))B$ is empty. Then it is clear from the definition of $\delta_{|\ |}$, that the diameter of T is at most ε. So we have shown that $D_\varepsilon(B) \subset (1 - \delta_{|\ |}(\varepsilon))B$. The conclusion now follows from Proposition 3.4.
(ii) $\Rightarrow$ *(iii)*. Let $\|\ \|$ be the original norm on X and B_X its unit ball. For $k \in \mathbb{N}$, $\mathrm{D}(X, 2^{-k})$ is by assumption an integer and we denote $N_k = \mathrm{D}(X, 2^{-k}) - 1$. Then we define on X the following convex function:

$$f(x) = \|x\| + \sum_{k=1}^{\infty} \sum_{n=1}^{N_k} \frac{2^{-k}}{N_k} \operatorname{dist}(x, D^n_{2^{-k}}(B_X)).$$

Finally, we define on X an equivalent norm $|\ |$ as the Minkowski functional of the set $\{x \in X,\ f(x) \leq 1\}$. The crucial step of the proof is to show

Lemma 7.8. *Let $x, y \in X$ so that $f(x) = f(y) = 1$ and $\|x - y\| \geq \varepsilon$. Then*

$$f(\frac{x+y}{2}) \leq 1 - \frac{\varepsilon^2}{32\,(D(X, \frac{\varepsilon}{8}))^2}. \tag{7.6}$$

Proof. Let $k \in \mathbb{N}$ so that $\frac{\varepsilon}{8} \leq 2^{-k} < \frac{\varepsilon}{4}$.
Set $\gamma = \frac{\varepsilon}{4N_k}$ and $n = \max\{m \geq 0,\ x \in D^m_{2^{-k}}(B_X)$ and $y \in D^m_{2^{-k}}(B_X)\}$.
Notice that $\|x - y\| \geq \varepsilon$ implies that $n < N_k$.
Assume that

$$f(\frac{x+y}{2}) > 1 - \frac{2^{-k}\gamma}{N_k}.$$

Then, all functions involved in the definition of f being convex, we get that for all $1 \leq l \leq N_k - n$,

$$\frac{1}{2}\big(d(x, D^{n+l}_{2^{-k}}(B_X)) + d(y, D^{n+l}_{2^{-k}}(B_X)\big) - d(\frac{x+y}{2}, D^{n+l}_{2^{-k}}(B_X)) < \gamma. \tag{7.7}$$

Then we show by induction that

$$\forall 1 \leq l \leq N_k - n, \quad \frac{1}{2}\big(d(x, D^{n+l}_{2^{-k}}(B_X)) + d(y, D^{n+l}_{2^{-k}}(B_X)\big) < l\gamma. \tag{7.8}$$

Indeed, since $x, y \in D^n_{2^{-k}}(B_X)$ and $\|x - y\| \geq \varepsilon$, $\frac{x+y}{2} \in D^{n+1}_{2^{-k}}(B_X)$. Then (7.7) implies that (7.8) is true for $l = 1$.
Assume now that (7.8) is true for l. Then, there exist $x', y' \in D^{n+l}_{2^{-k}}(B_X)$ such that $\frac{1}{2}(\|x - x'\| + \|y - y'\|) < l\gamma$. This implies that $\|x' - y'\| > \frac{\varepsilon}{2}$ and therefore

that $\frac{x'+y'}{2} \in D_{2^{-k}}^{n+l+1}(B_X)$. Thus

$$d(\frac{x+y}{2}, D_{2^{-k}}^{n+l+1}(B_X)) \leq \|\frac{x+y}{2} - \frac{x'+y'}{2}\| < l\gamma.$$

This, together with (7.7) finishes our inductive proof of (7.8).
But, if we apply (7.8) for $l = N_k - n$, we obtain that

$$\frac{1}{2}\big(d(x, D_{2^{-k}}^{n+l}(B_X)) + d(y, D_{2^{-k}}^{n+l}(B_X)\big) < \frac{\varepsilon}{4}.$$

This yields again the existence of $x', y' \in D_{2^{-k}}^{N_k}(B_X)$ with $\|x' - y'\| > \frac{\varepsilon}{2}$ and therefore that $D_{2^{-k}}^{N_k+1}(B_X) \neq \emptyset$, which is a contradiction.
Thus

$$f(\frac{x+y}{2}) \leq 1 - \frac{2^{-k}\gamma}{N_k} \leq 1 - \frac{\varepsilon^2}{32\,(\mathrm{D}(X, \frac{\varepsilon}{8}))^2}.$$

□

End of proof of Theorem 7.7. Now an elementary computation leads us to

$$\exists a > 0 \ \ \forall \varepsilon > 0, \ \ \delta_{|\ |}(\varepsilon) \geq a\varepsilon^2 (\mathrm{D}(X, a\varepsilon))^{-2}. \tag{7.9}$$

Let us finally explain why $|\ |$ has a power type modulus of convexity. Since X is reflexive, by Lemma 3.7 we get

$$\mathrm{D}(X, \varepsilon) = \mathrm{Dz}(X^*, \varepsilon) \leq \mathrm{Sz}((L^2(X))^*, \frac{\varepsilon}{2}).$$

Since X has an equivalent UR norm, so does $L^2(X)$. In particular, $\mathrm{Sz}((L^2(X))^*) \leq \mathrm{Dz}((L^2(X))^*) = \mathrm{D}(L^2(X)) \leq \omega$. So Proposition 3.5 yields:

$$\exists C > 0 \ \exists p \in [1, +\infty) \ \ \forall \varepsilon > 0 \ \mathrm{D}(X, \varepsilon) \leq C\varepsilon^p.$$

Then the conclusion follows immediately from (7.9). □

We can now combine Theorem 7.7 with Enflo's renorming Theorem ([24]), which asserts that every supereflexive Banach space admits an equivalent uniformly rotund norm.

Theorem 7.9. *Let X be a Banach space. The following assertions are equivalent:*
(i) X is supereflexive.
(ii) $D(X) \leq \omega$.
(iii) There exist $C > 0$ and $p \in [1, +\infty)$ such that: $\forall \varepsilon > 0 \ \ D(X, \varepsilon) \leq C\varepsilon^p$.
(iv) X admits an equivalent UR norm with power type modulus of convexity.

Proof. We already know, from the previous proof, that *(ii)* implies *(iii)* and *(iii)* implies *(iv)*.
The definition of uniform rotundity is a property of finite sets (pairs) of vectors. Thus it is clear that, if the norm of X is uniformly rotund, so is the norm of any space Y which is finitely representable in X. Therefore *(iv)* implies *(i)*.

We obtain that *(i)* implies *(ii)*, by combining Enflo's renorming Theorem and the easy implication in Theorem 7.7. □

Comment. One can see this as a new proof of Pisier's improvement ([56]) of Enflo's Theorem, knowing Enflo's result. Let us also mention that a direct proof of the fact that *(i)* implies *(ii)*, using an earlier work of R.C. James ([35]), can be found in [27].

However, the main interest of the above construction is probably not to reprove Pisier's Theorem, but rather to give a simple geometric formula for constructing an equivalent UR norm with power type modulus.

8. The Szlenk index and uniformly Kadec-Klee renormings

The purpose of this section is to give a renorming result similar to Theorem 7.7 for the condition "$\mathrm{Sz}(X) \leq \omega$".

If $(X, \| \ \|)$ is a Banach space, we define $\theta_{\| \ \|}(\varepsilon)$ for $\varepsilon > 0$ to be such that $1 - \theta_{\| \ \|}(\varepsilon)$ is the supremum of $\|x^*\|$ over all $x^* \in B_{X^*}$ so that every weak*-neighbourhood of x^* in B_{X^*} has a $\| \ \|$-diameter greater than ε.

The norm of X^* is *weak* uniformly Kadets-Klee* (in short w^*-UKK) if for any $\varepsilon > 0$, $\theta_{\| \ \|}(\varepsilon) > 0$.

Let us also recall the definition of the modulus of asymptotic smoothness of a Banach space X:

$$\forall \tau > 0, \quad \overline{\rho}(\tau) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} (\|x + \tau y\| - 1).$$

This modulus was first introduced by V.D. Milman in [51]. Then a Banach space X is said to be *uniformly asymptotically smooth* if $\lim_{\tau \to 0} \overline{\rho}(\tau)/\tau = 0$.

The following proposition, which can be found in [22] (Proposition 3.6), relates these two notions.

Proposition 8.1. *Let X be a separable Banach space. Then, the norm of X^* is weak* uniformly Kadets-Klee if and only if the norm of X is uniformly asymptotically smooth.*

It is clear that $s_\varepsilon(B_{X^*}) \subset (1 - \theta_{\| \ \|}(\varepsilon))B_{X^*}$ and therefore that $c_\varepsilon(B_{X^*}) \subset (1 - \theta_{\| \ \|}(\varepsilon))B_{X^*}$. It follows now from Proposition 3.4, that $\mathrm{Cz}(X) \leq \omega$ and $\mathrm{Sz}(X) \leq \omega$, whenever X admits an equivalent norm whose dual norm is w^*-UKK. It is then natural to ask whether the converse is true. This question has been answered positively by H. Knaust, E. Odell and T. Schlumprecht ([44]) in the separable case. In fact they obtain the following theorem, which contains much more information on the structure of these spaces.

Theorem 8.2. *Let X be a separable Banach space such that $Sz(X) \leq \omega$. Then there exist a Banach space $Z = Y^*$ with a boundedly complete finite dimensional decomposition (H_j) and $p \in [1, +\infty)$ so that X^* embeds isomorphically* *(norm and weak*) into Z and*

$$\|\sum z_j\|^p \geq \sum \|z_j\|^p \quad \text{for all block bases } (z_j) \text{ of } (H_j). \tag{8.10}$$

Remark 8.3. We already mentioned in Remark 6.3 that $\mathrm{Dz}(X) \leq \omega^2$, whenever $\mathrm{Sz}(X) \leq \omega$. Let us just mention that the proof of this result ([33]) is based on the above structural theorem.

We shall concentrate here on our renorming problem. It follows easily from the upper estimate (8.10) that for every $\varepsilon > 0$, $\theta_{\|\ \|}(\varepsilon) \geq \varepsilon^p$. Then one can deduce.

Corollary 8.4. *Let X be a separable Banach space. The following assertions are equivalent.*

(i) X admits an equivalent norm, whose dual norm is w^*-UKK.
(ii) $Cz(X) \leq \omega$.
(iii) $Sz(X) \leq \omega$.
(iv) X admits an equivalent norm, whose dual norm is w^*-UKK with a power type modulus.
(v) There exist $C > 0$ and $p \in [1, +\infty)$ such that: $\forall \varepsilon > 0$ $Cz(X, \varepsilon) \leq C\varepsilon^p$.
(vi) There exist $C > 0$ and $p \in [1, +\infty)$ such that: $\forall \varepsilon > 0$ $Sz(X, \varepsilon) \leq C\varepsilon^p$.

In [30], the relationship between the different exponents involved in the above statement are precisely described. The two main results are the following:

Theorem 8.5. *Let X be a separable Banach space with $Sz(X) \leq \omega$ and suppose $p > 1$ is such that*

$$\sup_{\varepsilon>0} \varepsilon^p Sz(X, \varepsilon) < \infty.$$

Then

$$\sup_{\varepsilon>0} \varepsilon^p Cz(X, \varepsilon) < \infty.$$

Theorem 8.6. *Let X be a separable Banach space with $Sz(X) \leq \omega$. Then there exists an absolute constant $C > 0$ such that for any $\varepsilon > 0$ there is a 2-equivalent norm $|\ |$ on X so that*

$$\forall \varepsilon > 0 \quad \theta_{|\ |}(\varepsilon) \geq (Cz(X, \frac{\varepsilon}{C}))^{-1}.$$

The main applications of this result will be explained in section 9. But as a first consequence we obtain:

Corollary 8.7. *Let X be a separable Banach space with $Sz(X) \leq \omega$ and define the Szlenk power type of X to be*

$$p_X := \inf\{q \geq 1,\ \sup_{\varepsilon>0} \varepsilon^q Sz(X, \varepsilon) < \infty\}.$$

Then

$p_X = \inf\{q \geq 1,$ *there is an equ. norm* $|\ |$ *on* $X,\ \exists c > 0\ \forall \varepsilon > 0,\ \theta_{|\ |}(\varepsilon) \geq c\varepsilon^q\}.$

Proof. It follows rather easily from the proof of Proposition 3.4, that $p_X \leq q$, whenever X admits an equivalent norm $|\ |$ such that $\theta_{|\ |}(\varepsilon) \geq c\varepsilon^q$, for some $c > 0$ and all $\varepsilon > 0$.

So, we need to show that, conversely, for any $q > p_X$, there is an equivalent norm $|\ |$ on X satisfying

$$\exists c > 0 \quad \forall \varepsilon > 0 \quad \theta_{|\ |}(\varepsilon) \geq c\varepsilon^q. \tag{8.11}$$

Fix $p_X < r < q$. By Theorems 8.5 and 8.6, there exists $C > 0$, such that for each $k \in \mathbb{N}$ there is a norm $|\ |_k$ on X which is 2-equivalent to the original norm and so that $\theta_{|\ |_k}(2^{-k}) \geq C2^{-rk}$. Now define the norm $|\ |$ on X^* by

$$|x^*| = \sum_{k=1}^{\infty} 2^{(r-q)k}|x^*|_k.$$

This defines an equivalent dual norm on X^* and an easy computation shows that it satisfies (8.11).

□

We refer the reader to [30] for the detailed proofs of Theorems 8.5 and 8.6. However, we will try to give the general scheme of the argument. Even for that limited purpose, we need to introduce quite a lot of notation.

Let $(\mathcal{F}\mathbb{N}, \preceq)$ be the set of finite sequences of positive integers equipped with its natural partial order. For $a = (a_1, .., a_n) \in \mathcal{F}\mathbb{N}$, we denote $|a| = n$, $a- = (a_1, .., a_{n-1})$ the predecessor of a and $a+ = \{(a_1, .., a_n, k),\ k \in \mathbb{N}\}$ the set of successors of a. A subset S of $\mathcal{F}\mathbb{N}$ is said to be a *full tree* of height N if the following conditions hold

$\emptyset \in S$.

$\forall a \in S \setminus \{\emptyset\},\ a- \in S$.

If $a \in S$ and $|a| = N$ then $a + \cap S = \emptyset$.

If $a \in S$ and $|a| < N$ then $a + \cap S$ is infinite.

If S is a full tree of height N, a *branch* of S is a set of the form $B = \{b,\ b \preceq a\}$, with $a \in S$ and $|a| = N$.

Let now V be a vector space. A *tree map* of height N in V is any map $(x_a)_{a\in S}$ from a tree of height N into V. If τ is a topology on V, we say that a tree map $(x_a)_{a\in S}$ of height N is τ-*null* if for any $a \in S$ with $|a| < N$, $\{x_b\}_{b\in a+}$ is a τ-null sequence.

It is easy to show, in the spirit of (*vii*) in Proposition 3.1, that $\mathrm{Sz}(X, \varepsilon)$ is equivalent to the maximal height of a weak*-null tree map $(x_a^*)_{a\in S}$ in X^* such that for all $a \in S$, $\|x_a^*\| \geq \varepsilon$ and for any branch B of S, $\|\sum_{a\in B} x_a^*\| \leq 1$. However, we shall use a more efficient dual approach. If $\sigma > 0$, define $N =$

$N(\sigma)$ to be the least integer so that there exists a weakly-null tree map $(x_a)_{a\in S}$ in X of height $N+1$ such that for all $a \in S$, $\|x_a\| \leq \sigma$ and for any branch B of S, $\|\sum_{a\in B} x_a\| > 1$. The crucial step is to prove the following

Theorem 8.8. *Let X be a separable Banach space and $\sigma > 0$.*

(i) If $N(\sigma) < \infty$ there is a norm $|\ |$ on X satisfying $\frac{1}{2}\|x\| \leq |x| \leq \|x\|$ and

$$\limsup_{n\to\infty} |x + x_n| \leq 1 + \frac{1}{N(\sigma)},$$

whenever $|x| = 1$, $\lim_{n\to\infty} |x_n| = \frac{\sigma}{2}$ and $x_n \xrightarrow{w} 0$.

(ii) If $N(\sigma) = \infty$, then, for any $\varepsilon > 0$, there is a norm $|\ |$ on X satisfying $\frac{1}{2}\|x\| \leq |x| \leq \|x\|$ and

$$\limsup_{n\to\infty} |x + x_n| \leq 1 + \varepsilon,$$

whenever $|x| = 1$, $\lim_{n\to\infty} |x_n| = \frac{\sigma}{2}$ and $x_n \xrightarrow{w} 0$.

Sketch of proof. We mention first how the norm is constructed in *(i)*. Define $f_0(x) = \|x\|$ and then for $k > 0$ define $f_k(x)$ to be the infimum of all $\lambda > 0$ so that, whenever $(x_a)_{a\in S}$ is a weakly null tree-map of height k with $\|x_a\| \leq \sigma$ for all $a \in S$, there is a full subtree T of height k of S so that $\|x + \sum_{a\in B} x_a\| \leq \lambda$ for every branch B of T. Then we check that (f_k) is an increasing sequence of convex symmetric functions. The key property of these functions is that

$$\limsup_{n\to\infty} f_k(x + x_n) \leq f_{k+1}(x), \quad \text{whenever } \|x_n\| \leq \sigma \text{ and } x_n \xrightarrow{w} 0.$$

Then we conclude the proof by considering

$$g(x) = \frac{1}{N(\sigma)} \sum_{k=0}^{N(\sigma)-1} f_k(x) \ \text{ and } \ |\ | \text{ to be the gauge of } \{x \in X,\ g(x) \leq 2\}.$$

The proof is almost identical for *(ii)*, except that one considers $g_m(x) = \frac{1}{m}\sum_{k=0}^{m-1} f_k(x)$ for arbitrarily large choices of m.

□

Comment. For a Banach space X, having an equivalent norm, whose dual norm is weak*-UKK, can be roughly interpreted as: X^* has an "ℓ_1-like" behavior, as far as the weak* converging sequences are concerned. The above theorem (especially *(ii)*) describes a "c_0-like" behavior of a Banach space X. The end of our argument is to show that these statements are equivalent. Thus, very naturally, it will rely on the Young's duality between the function N^{-1} and the "best" modulus of an equivalent w^*-UKK norm.

At this point we need to introduce more terminology and notation. Let f, g be continuous monotone increasing functions on $[0,1]$ which satisfy $f(0) = g(0) = 0$. We say that f *C-dominates* g if $f(\tau) \geq g(\tau/C)$ for every $0 \leq \tau \leq 1$

and that f,g are *C-equivalent* if f C-dominates g and g C-dominates f. For any such monotone increasing function f we denote by f^* its dual Young's function.

For the sequel, let X be a Banach space with a separable dual. We define for $0 \le \sigma \le 1$, $\rho(\sigma) = \rho_X(\sigma)$ to be the least constant so that

$$\limsup_{n\to\infty} \|x + x_n\| \le 1 + \rho_X(\sigma),$$

whenever $\|x\| = 1$, $x_n \xrightarrow{w} 0$ and $\limsup_{n\to\infty} \|x_n\| \le \sigma$.
We define $\eta(\tau) = \eta_X(\tau)$ for $0 \le \tau \le 1$ to be the greatest constant so that

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \ge 1 + \eta_X(\tau)$$

whenever $x^*, x_n^* \in X^*$, $\|x^*\| = 1$, $x_n^* \xrightarrow{w^*} 0$ and $\liminf_{n\to\infty} \|x_n^*\| \ge \tau$.
Note that η_X is equivalent to θ_X introduced earlier in this section.
We finally set

$$\varphi(\sigma) = \inf\{\rho_Y(\sigma) :\ d(X,Y) \le 2\} \ \text{ and } \ \psi(\tau) = \sup\{\eta_Y(\tau) :\ d(X,Y) \le 2\}.$$

We now consider $H(\varepsilon) = (\mathrm{Cz}(X,\varepsilon) - 1)^{-1}$. By applying Proposition 3.4, we easily get that H dominates ψ.
An other important step is to show that H^* dominates N^{-1}. The idea is to show that when $N(\sigma)$ is "small", then there exists a "large" weak*-null tree in B_{X^*} (we skip the rather technical argument). On the other hand, it follows from Theorem 8.8, that N^{-1} dominates φ. It follows from a duality argument similar to the forthcoming Lemma 9.13 that φ^* is equivalent to ψ. Therefore ψ dominates H^{**}. Then, it is easily seen that H is equivalent to a convex function and thus to H^{**}. Finally, we obtain that H is equivalent to ψ, which is the statement of Theorem 8.6.

For the proof of Theorem 8.5, we show the more precise estimate:

$$\exists C > 0,\ \ \forall \varepsilon \in (0,1]\ \ \mathrm{Cz}(X,\varepsilon) \le \sum_{\substack{k\ge 0\\ 2^k\varepsilon/C \le 1}} 2^k \mathrm{Sz}(X, 2^k\varepsilon/C).$$

For that purpose, we prove that K^* dominates N^{-1}, where

$$K(\varepsilon) = \Big(\sum_{\substack{k\ge 0\\ 2^k\varepsilon/C \le 1}} 2^k[\mathrm{Sz}(X, 2^k\varepsilon/C) - 1]\Big)^{-1}.$$

This again is done by constructing special weak*-null trees in B_{X^*}, depending on the value of $N(\sigma)$. Then, arguing as with H, we show that K is equivalent to K^{**}.

9. Applications to Lipschitz or uniform classification of Banach spaces

It follows from the work of S. Heinrich and P. Mankiewicz [34] that, for super-reflexive spaces, the best modulus of an equivalent uniformly rotund norm or of an equivalent uniformly smooth norm is invariant under uniform homeomorphisms. The results presented in this section, which are taken from [29] and [30], show in a rather quantitatively precise way, that the functions $\mathrm{Sz}(X,\varepsilon)$, $\mathrm{Cz}(X,\varepsilon)$ and $\theta_X(\varepsilon)$ are invariant under Lipschitz or uniform homeomorphisms. Then we explain some important consequences of these results such as: the class of subspaces of c_0 is invariant under Lipschitz homeomorphisms, a Banach space Lipschitz homeomorphic to c_0 is linearly isomorphic to c_0, the class of quotients of ℓ_p (for $1 < p < \infty$) is invariant under uniform homeomorphisms.

9.1. **Statements of the main results.**

Theorem 9.1. *Suppose X and Y are separable Banach spaces which are uniformly homeomorphic. Then there is a constant $C \geq 1$ such that for any $\eta > 0$ there exists a C-equivalent norm $|\ |_\eta$ on Y satisfying*

$$\forall 0 < \varepsilon \leq 1 \quad \theta_{|\ |_\eta}(\varepsilon) \geq \theta_X(\varepsilon/C) - \eta.$$

If $\|\ \|_{X^*}$ is w^*-UKK and Y is uniformly homeomorphic to X, the above result does not provide us with an equivalent norm $|\ |$ on Y such that $\theta_{|\ |}$ is equivalent to θ_X. However, by considering for instance

$$\forall y^* \in Y^* \quad |y^*| = \sum_{k=1}^{\infty} \frac{1}{k^2} |y^*|_{2^{-k}},$$

one can easily deduce

Corollary 9.2. *Suppose X and Y are separable Banach spaces which are uniformly homeomorphic and $\|\ \|_{X^*}$ is w^*-UKK. Then Y admits an equivalent norm $|\ |$ such that*

$$\exists C \geq 1 \quad \forall 0 < \varepsilon \leq 1, \quad \theta_{|\ |}(\varepsilon) \geq \frac{\theta_X(\varepsilon/C)}{|\ln(\varepsilon/C)|^2}.$$

In particular, $p_X = p_Y$, where p_X is the Szlenk power type of X.

The most interesting consequence is that the convex Szlenk index turns out to be (up to equivalence) a perfect invariant under uniform homeomorphisms. More precisely, we have

Corollary 9.3. *Suppose X and Y are uniformly homeomorphic. Then*

(i) There exists a constant C so that

$$\forall 0 < \varepsilon \leq 1, \quad Cz(X, C\varepsilon) \leq Cz(Y, \varepsilon) \leq Cz(X, \varepsilon/C).$$

(ii) $Sz(X) \leq \omega_0$ if and only if $Sz(Y) \leq \omega_0$.

Proof. It follows from Theorem 3.11 and a standard back and forth separable saturation argument, that we may assume X and Y to be separable. Then one can easily obtain *(i)* by combining Theorem 9.1 and Theorem 8.6, and *(ii)* is a consequence of *(i)* and Corollary 8.4. □

Remark 9.4. It is important at this point, to mention a fundamental counterexample due to M. Ribe (see [57] or its exposition in [7]). Suppose (p_n) is a sequence in $(1, +\infty)$ that is strictly decreasing to 1. Then let X denote the ℓ_2-sum of the spaces ℓ_{p_n} and $Y = X \oplus \ell_1$. M. Ribe showed that X and Y are uniformly homeomorphic. It follows that uniform homeomorphisms do not preserve reflexivity or the separability of the dual. Moreover, it is not difficult to show that $\mathrm{Sz}(X) = \omega^2$. So, only the values of the Szlenk index not exceeding ω are preserved under uniform homeomorphisms.

We now turn to the case of Lipschitz homeomorphisms, where we obtain a quantitatively better estimate.

Theorem 9.5. *Suppose X and Y are separable Banach spaces which are Lipschitz homeomorphic. Then there is an equivalent norm $|\ |$ on Y such that*

$$\exists C \geq 1 \ \ \forall 0 < \varepsilon \leq 1, \ \ \theta_{|\ |}(\varepsilon) \geq \theta_X(\varepsilon/C).$$

Remark 9.6. It is proved in [5] that the separability of the dual is preserved under Lipschitz homeomorphisms. Using the tools from descriptive set theory described in section 6, Y. Dutrieux showed in [21] that there is a universal function $\lambda : \omega_1 \to \omega_1$ such that $\mathrm{Sz}(Y) \leq \lambda(\mathrm{Sz}(X))$, whenever X and Y are two separable Asplund spaces which are Lipschitz homeomorphic.
We do not know if the function λ is the identity.

9.2. **Applications.** We start with our applications to the non linear classification of the subspaces of c_0.

Theorem 9.7. *(i) The class of all Banach spaces linearly isomorphic to a subspace of c_0 is stable under Lipschitz homeomorphisms.*

(ii) If a Banach space is Lipschitz homeomorphic to c_0, then it is linearly isomorphic to c_0.

(iii) If a Banach space X is uniformly homeomorphic to c_0, then X^* is linearly isomorphic to ℓ_1.

Proof. The arguments will consist in combining our result with various deep theorems on the linear structure of these Banach spaces.

(i) It follows from Theorem 9.5 and the following result: a Banach space X is isomorphic to a subspace of c_0 if and only if it admits an equivalent norm $|\ |$ such that

$$\exists C \geq 1 \ \ \forall 0 < \varepsilon \leq 1, \ \ \theta_{|\ |}(\varepsilon) \geq \varepsilon/C.$$

This last result follows from the techniques developed by N.J. Kalton and D. Werner in [41]. A proof written in this spirit can be found in [30]. A simpler argument, but yielding a larger isomorphism constant, is given in [37].

(ii) It is known (see [34] by Heinrich and Mankiewicz) that the class of all $\mathcal{L}^\infty$-spaces is stable under uniform homeomorphisms. On the other hand, it has been proved by W.B. Johnson and M. Zippin in [40] that any $\mathcal{L}^\infty$-subspace of c_0 is linearly isomorphic to c_0.

(iii) Assume that X is uniformly homeomorphic to c_0. Corollary 9.2 insures that X^* is separable. Then we apply a result of D.R. Lewis and C. Stegall ([49]) asserting that if X is a $\mathcal{L}^\infty$-space with separable dual, then X^* is isomorphic to ℓ_1. □

Remarks 9.8. It is not known if a Banach space uniformly homeomorphic to c_0 is linearly isomorphic to c_0.

The question of the non linear classification of the $C(K)$ spaces, with K countable and compact, is still widely open.

Y. Dutrieux ([20]) proved that if a Banach space X is Lipschitz homeomorphic to a quotient of c_0 and X^* has the approximation property, then X is isomorphic to a quotient of c_0.

In the non separable case, we still have the following characterization:

Theorem 9.9. *Let K be a compact space. The following assertions are equivalent:*

(i) $Sz(C(K)) \leq \omega$.

(ii) $K^{(\omega)} = \emptyset$.

(iii) $C(K)$ is Lipschitz homeomorphic to $c_0(\Gamma)$, where Γ is the density character of $C(K)$.

(iv) $C(K)$ is uniformly homeomorphic to $c_0(\Gamma)$.

(v) $C(K)$ admits an equivalent norm $|\ |$ such that:

$$\exists C \geq 1 \quad \forall \varepsilon > 0, \quad \theta_{|\ |}(\varepsilon) \geq \frac{\varepsilon}{C}.$$

Proof. *(i)* ⇒ *(ii)* follows from the fact that for any ordinal α and any x in K, the Dirac mass δ_x belongs to $s_1^\alpha(B_{C(K)^*})$, whenever x is in $K^{(\alpha)}$. *(ii)* ⇒ *(iii)* is due to Deville, Godefroy and Zizler ([16]), *(iii)* ⇒ *(iv)* is trivial and the argument for *(iv)* ⇒ *(ii)* is in [38]. A simple and direct proof of *(ii)* ⇒ *(v)* is given in [47], but it can now be deduced from *(ii)* ⇒ *(iii)* and Theorem 9.5 where the separability is not an issue. Finally, *(v)* ⇒ *(i)* is clear.

□

Remarks 9.10. 1) Ciesielski and Pol have constructed in [13] a (non metrizable) compact space K such that $K^{(3)}$ is empty but there is no weak-to-weak continuous injective map, in particular no bounded linear injective map, from

$C(K)$ into any $c_0(\Gamma)$; while the previous theorem insures that $C(K)$ is Lipschitz homeomorphic to some $c_0(\Gamma)$.
Then, W. Marciszewski provides in [50] a simple characterization of compact spaces K such that $C(K)$ is linearly isomorphic to some $c_0(\Gamma)$. This implies in particular that such a compact space is Eberlein. The converse is not true as a counterexample of M. Bell and W. Marciszewski shows ([6]).

2) Y. Dutrieux and N. Kalton have obtained in [23] many important results on the non linear classification of $C(K)$ spaces. Let K and L be two Hausdorff compact spaces. They show that K and L are homeomorphic whenever the Gromov-Hausdorff distance between $C(K)$ and $C(L)$ is less than $1/10$ (in particular when the uniform distance is less than $11/10$). They also prove that $C(K)$ and $C(L)$ have the same Szlenk index (and are therefore linearly isomorphic in the separable case), when their Kadets distance is less than 1. We refer the reader to the original paper for the definitions of these non linear distances between Banach spaces.

Next,we turn to the non linear classification of quotients of ℓ_p.

Theorem 9.11. *Let p in $(2, +\infty)$ and let X and Y be two Banach spaces which are uniformly homeomorphic. If X is a quotient (resp. subspace) of ℓ_p, then Y is linearly isomorphic to a quotient (resp. subspace) of ℓ_p.*

In the quotient case, the proof, which appeared in [30], uses Corollary 9.2 and the work of W.B. Johnson ([36]) on quotients of L^p that are quotients of ℓ_p. Similarly, the subspace version combines Corollary 9.2 and the results of Johnson and Odell ([39]) on subspaces of L^p which embed in ℓ_p. However, this later case can also simply be deduced from the methods of [38].

Let us mention, that the question, whether the class of quotients, or the class of subspaces, of ℓ_p is closed under uniform homeomorphisms is open for $1 \leq p < 2$.

9.3. **Proofs of the main results.** In this section, we will give the proofs of Theorems 9.1 and 9.5. They are taken from [29] and [30]. The argument is partly based on a renorming technique (see Lemma 9.14) that we find interesting to explain here. Therefore, we have chosen to include these proofs, despite some technical difficulties. It will also require the use of an important tool from non linear analysis, known as the "Gorelik principle". This principle first appeared in [38] and was inspired to the authors by the earlier work of E. Gorelik [31]. We will use a slightly different version, whose proof (very similar to the original one) can be found in [29].

Theorem 9.12. (The Gorelik principle) *Let X and Y be two Banach spaces and U be a homeomorphism from X onto Y with uniformly continuous inverse. Let b and d two positive constants and let X_0 be a subspace of finite* *codimension of X. If $d > \omega(U^{-1}, b)$ ($\omega(U^{-1}, .)$ is the modulus of uniform continuity of U^{-1}), then there exists a compact subset K of Y such that*

$$bB_Y \subset K + U(2dB_{X_0}).$$

We will also need the following duality lemma.

Lemma 9.13. *Let X be a separable Banach space and $0 < \sigma, \varepsilon < 1$. Suppose X satisfies the following property:*

$$\liminf \|x^* + x_n^*\| \geq 1 + \sigma\varepsilon, \quad \textit{whenever } x^* \in S_{X^*},\ x_n^* \xrightarrow{w^*} 0 \textit{ and } \|x_n^*\| \geq \varepsilon.$$

Then

$$\limsup \|x + x_n\| \leq 1 + \sigma\varepsilon, \quad \textit{whenever } x \in S_X,\ x_n \xrightarrow{w} 0 \textit{ and } \|x_n\| \leq \sigma.$$

Proof. Assume $\|x\| = 1$, $\|x_n\| \leq \sigma$, $x_n \xrightarrow{w} 0$ and $\lim \|x + x_n\| > 1 + \sigma\varepsilon$. Then, pick y_n^* in B_{X^*} such that $\lim y_n^*(x + x_n) > 1 + \sigma\varepsilon$. Passing to a subsequence, we may assume that $y_n^* \xrightarrow{w^*} x^* \in B_{X^*}$ and $\|y_n^* - x^*\| \to l$. Then $\lim y_n^*(x + x_n) \leq 1 + l\sigma$ and therefore $l > \varepsilon$. But the assumption in the statement of the lemma yields

$$\liminf \Big\| \frac{lx^*}{\varepsilon\|x^*\|} - x^* + y_n^* \Big\| \geq \frac{l}{\varepsilon} + l\sigma.$$

So $\liminf \|y_n^*\| \geq \|x^*\| + l\sigma$. Hence $\|x^*\| \leq 1 - l\sigma$ and

$$\lim y_n^*(x + x_n) = \lim (x^* + y_n^* - x^*)(x + x_n) \leq 1 - l\sigma + l\sigma = 1,$$

which is a contradiction.

□

Proof of Theorem 9.1. Let U be a uniform homeomorphism from X onto Y. Note that U and U^{-1} are Lipschitz for large distances. So, there is a constant $C > 0$ such that:

$$\|Ux_1 - Ux_2\| \leq M \max(\|x_1 - x_2\|, 1) \qquad x_1, x_2 \in X \quad \text{and}$$

$$\|U^{-1}y_1 - U^{-1}y_2\| \leq M \max(\|y_1 - y_2\|, 1) \qquad y_1, y_2 \in Y.$$

We now define a decreasing sequence of dual norms $\{|\ |_k\}_{k=1}^{\infty}$ on Y^* by

$$|y^*|_k = \sup \left\{ \frac{|y^*(Ux_1 - Ux_2)|}{\|x_1 - x_2\|}; \quad x_1, x_2 \in X,\ \|x_1 - x_2\| \geq 2^k \right\}.$$

It is clear that for all $k \in \mathbb{N}$, we have $M^{-1}\|\ \| \leq |\ |_k \leq M\|\ \|$.

We will need the following lemma.

Lemma 9.14. *There exist $C > 0$ and $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, any $\varepsilon \in (0,1)$ and any y^*, y_n^* in Y^* satisfying $\|y^*\| \leq M$, $\|y_n^*\| \geq \varepsilon/M$ and $y_n^* \xrightarrow{w^*} 0$, we have:*

$$\liminf_{n\to\infty} |y^* + y_n^*|_k \geq 2|y^*|_{k+1} - |y^*|_k + \theta_X(\frac{\varepsilon}{C}). \tag{9.12}$$

Let us first conclude the proof of Theorem 9.1. We set

$$\forall N \in \mathbb{N} \ \ \forall y^* \in Y^* \ \ \|y^*\|_N = \frac{1}{N}\sum_{k=k_0+1}^{k=k_0+N} |y^*|_k,$$

which is a dual norm on Y^* with $M^{-1}\|\ \| \leq \|\ \|_N \leq M\|\ \|$.
If $\|y^*\|_N = 1$, $\|y_n^*\|_N \geq \varepsilon$ and $y_n^* \xrightarrow{w^*} 0$, we can apply the above lemma and by summing over k we get:

$$\liminf \|y^* + y_n^*\|_N \geq \|y^*\|_N - \frac{2}{N}|y^*|_{k_0+1} + \frac{2}{N}|y^*|_{k_0+N+1} + \theta_X(\frac{\varepsilon}{C}).$$

We fix now $\eta > 0$ and get for N big enough:

$$\liminf \|y^* + y_n^*\|_N \geq \|y^*\|_N - \eta + \theta_X(\frac{\varepsilon}{C}).$$

□

Proof of Lemma 9.14. Let $k \in \mathbb{N}$. For a small $\delta > 0$, to be chosen later, we can pick $x, x' \in X$ so that $\|x - x'\| \geq 2^{k+1}$ and $y^*(Ux - Ux') \geq (1-\delta)\|x - x'\||y^*|_{k+1}$. We may assume that $x' = -x$ and $Ux' = -Ux$. So $\|x\| \geq 2^k$ and $y^*(Ux) \geq (1-\delta)\|x\||y^*|_{k+1}$.
Let $C > 0$. We apply Lemma 9.13 to deduce the existence of a finite codimensional subspace X_0 of X so that

$$\|x + z\| \geq \|x\| \geq 2^k \qquad \forall z \in X_0 \tag{9.13}$$

and

$$\|x + z\| \leq (1 + 2\theta_X(\frac{\varepsilon}{C}))\|x\| \qquad \forall z \in C\varepsilon^{-1}\theta_X(\frac{\varepsilon}{C})\|x\|B_{X_0}. \tag{9.14}$$

Now, $\omega(U^{-1}, b) < 2Mb$ for all $b \geq 1$. Then we apply the Gorelik principle (Theorem 9.12) for $b = \frac{\sigma\|x\|}{4M}$ and $d = \frac{\sigma\|x\|}{2}$, where $\sigma = C\varepsilon^{-1}\theta_X(\frac{\varepsilon}{C})$. Note that for k large enough (say $k \geq k_0$), we have $b > 1$. Hence there is a compact subset K of Y so that:

$$\frac{\sigma\|x\|}{4M}B_Y \subset K + U(\sigma\|x\|B_{X_0}). \tag{9.15}$$

Since $y_n^* \to 0$ uniformly on K, (9.15) yields the existence of $(z_n) \subset \sigma\|x\|B_{X_0}$ so that

$$\liminf_{n\to\infty} y_n^*(-Uz_n) \geq \frac{\sigma\varepsilon}{4M^2}\|x\|.$$

Now $\|x + z_n\| = \|z_n - x'\|$ so, by (9.13) and (9.14), we have

$$y^*(Ux + Uz_n) = y^*(Uz_n - Ux') \leq (1 + 2\theta_X(\frac{\varepsilon}{C}))|y^*|_k\|x\|$$

But

$$y^*(Ux) = \frac{1}{2}y^*(Ux - Ux') \geq (1 - \delta)|y^*|_{k+1}\|x\|.$$

So

$$y^*(Uz_n) \leq \big((1 + 2\theta_X(\frac{\varepsilon}{C}))|y^*|_k - (1 - \delta)|y^*|_{k+1}\big)\|x\|.$$

Combining these estimates and letting δ tend to 0 gives

$$\liminf_{n\to\infty}(y^*+y_n^*)(Ux-Uz_n) \geq |y^*|_{k+1}\|x\|-|y^*|_k\|x\|+\frac{\sigma\varepsilon}{4M^2}\|x\|-2\theta_X(\frac{\varepsilon}{C})|y^*|_k\|x\|.$$

But

$$2^k \leq \|Ux - Uz_n\| \leq (1 + 2\theta_X(\frac{\varepsilon}{C}))\|x\| \leq \frac{\|x\|}{1 - 2\theta_X(\frac{\varepsilon}{C})}.$$

So

$$\liminf_{n\to\infty} |y^* + y_n^*|_k \geq (1 - 2\theta_X(\frac{\varepsilon}{C}))(2|y^*|_{k+1} - |y^*|_k + \frac{\sigma\varepsilon}{4M^2} - 2\theta_X(\frac{\varepsilon}{C})M^2)$$

Therefore, for C big enough, chosen before k_0:

$$\liminf_{n\to\infty} |y^* + y_n^*|_k \geq 2|y^*|_{k+1} - |y^*|_k - \theta_X(\frac{\varepsilon}{C}).$$

□

Proof of Theorem 9.5. If U is a Lipschitz homeomorphism from X onto Y, we define at once

$$|y^*| = \sup\left\{\frac{|y^*(Ux_1) - y^*(Ux_2)|}{\|x_1 - x_2\|} \qquad x_1 \neq x_2\right\}.$$

This will give us the desired norm. The computations are similar and simpler than for the previous case. □

Remark 9.15. As we already mentionned, the question whether a Banach space uniformly homeomorphic to c_0 is linearly isomorphic to c_0 is still open. The main obstacle is that, unlike the Lipschitz case, we cannot build a single norm that does the job of Lemma 9.14. A natural suggestion would be to consider the limit of the decreasing sequence $(\| \ \|)_N$. Unfortunately, we are then confronted to the problem of exchanging the weak*-limit and $\lim_N \| \ \|_N$.

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