CONTINUOUS DEPENDENCE ESTIMATES FOR NONLINEAR FRACTIONAL CONVECTION-DIFFUSION EQUATIONS

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ABSTRACT. We develop a general framework for finding error estimates for convection-diffusion equations with nonlocal, nonlinear, and possibly degenerate diffusion terms. The equations are nonlocal because they involve fractional diffusion operators that are generators of pure jump Lévy processes (e.g. the fractional Laplacian). As an application, we derive continuous dependence estimates on the nonlinearities and on the Lévy measure of the diffusion term. Estimates of the rates of convergence for general nonlinear nonlocal vanishing viscosity approximations of scalar conservation laws then follow as a corollary. Our results both cover, and extend to new equations, a large part of the known error estimates in the literature.

1. INTRODUCTION

This paper is concerned with the following Cauchy problem:

$$\begin{cases} \partial_t u(x,t) + \operatorname{div}\left(f(u)\right)(x,t) = \mathcal{L}^\mu[A(u(\cdot,t))](x) & \text{in} \quad Q_T := \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1.1}$$

where u is the scalar unknown function, div denotes the divergence with respect to (w.r.t.) x, and the operator $\mathcal{L}^\mu$ is defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ by

$$\mathcal{L}^\mu[\phi](x) := \int_{\mathbb{R}^d\setminus\{0\}} \left(\phi(x+z) - \phi(x) - z \cdot D\phi(x)\mathbf{1}_{|z|\leq 1}\right) \mathrm{d}\mu(z), \tag{1.2}$$

where $D\phi$ denotes the gradient of ϕ w.r.t. x and $\mathbf{1}_{|z|\leq 1} = 1$ for $|z| \leq 1$ and $= 0$ otherwise. Throughout the paper, the data (f, A, u_0, μ) is assumed to satisfy the following assumptions:

(1.3) $\quad f \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d)$ with $f(0) = 0$,

(1.4) $\quad A \in W^{1,\infty}(\mathbb{R})$ is nondecreasing with $A(0) = 0$,

(1.5) $\quad u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$,

and

(1.6) $\quad \mu$ is a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} (|z|^2 \wedge 1)\, \mathrm{d}\mu(z) < +\infty,$$

where we use the notation $a \wedge b = \min\{a, b\}$. The measure μ is a Lévy measure.

Remark 1.1.

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(1) Subtracting constants to f and A if necessary, there is no loss of generality in assuming that $f(0) = 0$ and $A(0) = 0$.
(2) Our results also hold for locally Lipschitz-continuous nonlinearities f and A since solutions will be bounded; see Remark 2.3 for more details.
(3) Assumption (1.6) and a Taylor expansion reveal that $\mathcal{L}^\mu[\phi]$ is well-defined for e.g. bounded C^2 functions ϕ:

$$|\mathcal{L}^\mu[\phi](x)| \leq \max_{|z|\leq 1} |D^2\phi(x+z)| \int_{0<|z|\leq 1} \frac{1}{2}|z|^2 \mathrm{d}\mu(z) + 2\|\phi\|_{L^\infty} \int_{|z|>1} \mathrm{d}\mu(z)$$

where $D^2\phi$ is the Hessian of ϕ. If in addition $D^2\phi$ is bounded on $\mathbb{R}^d$, then so is $\mathcal{L}^\mu[\phi]$.

Under (1.6), $\mathcal{L}^\mu$ is the generator of a pure jump Lévy process, and reversely, any pure jump Lévy process has a generator of like $\mathcal{L}^\mu$ (see e.g. [6, 58]). This class of diffusion processes contains e.g. the α-stable process whose generator is the fractional Laplacian $-(-\triangle)^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$. It can be defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ via the Fourier transform as

$$(-\triangle)^{\frac{\alpha}{2}} \phi = \mathcal{F}^{-1}\left(|\cdot|^\alpha \mathcal{F}\phi\right),$$

or in the form (1.2) with the following Lévy measure (see e.g. [6, 34, 36]):

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}z}{|z|^{d+\alpha}} \text{ (up to a positive multiplicative constant).} \tag{1.7}$$

Many other Lévy processes/operators of practical interest can be found in e.g. [6, 26]. Under assumption (1.4), $\mathcal{L}^\mu[A(\cdot)]$ is an example of a nonlinear nonlocal diffusion operator. For recent studies of this and similar type of operators, we refer the reader to [8, 9, 14, 19, 29] and the references therein.

Equation (1.1) appears in many different contexts such as overdriven gas detonations [23], mathematical finance [26], flow in porous media [29], radiation hydrodynamics [55, 56], and anomalous diffusion in semiconductor growth [61]. Equations of the form (1.1) constitute a large class of nonlinear degenerate parabolic integro-differential equations (integro-PDEs). Let us give some representative examples.

When $A = 0$ or $\mu = 0$, (1.1) is the well-known scalar conservation law (see e.g. [27] and references therein):

$$\partial_t u + \operatorname{div} f(u) = 0. \tag{1.8}$$

When $A(u) = u$ and $\mathcal{L}^\mu$ is the fractional Laplacian, (1.1) is the so-called fractal/fractional conservation law:

$$\partial_t u + \operatorname{div} f(u) = -(-\triangle)^{\frac{\alpha}{2}} u. \tag{1.9}$$

Equation (1.9) has been extensively studied since the nineties [1, 2, 4, 5, 7, 10, 11, 12, 16, 17, 21, 22, 30, 31, 32, 33, 34, 37, 40, 41, 42, 43, 48, 53, 54]. The case of more general Lévy diffusions, combined with nonlinear local diffusions,

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}(a(u)\nabla u) + \mathcal{L}^\mu[u], \tag{1.10}$$

can be found in [45].

When A is nonlinear, (1.1) can be seen as a generalization of the following classical convection-diffusion equation (possibly degenerate):

$$\partial_t u + \operatorname{div} f(u) = \triangle A(u); \tag{1.11}$$

see e.g. [13, 15, 18, 24, 44] for precise references on (1.11). Nonlinear nonlocal diffusions have been invetigated in [29] in the setting of nonlocal porous media equations and L^1 semi-group methods, and in [19] where an $L^\infty \cap L^1$ entropy solution theory is developed for more general degenerate equations of the form (1.1) along with connections to Hamilton-Jacobi-Bellman equations of stochastic control theory. Other interesting examples concern the class of nonsingular Lévy measures satisfying $\int_{\mathbb{R}^d\setminus\{0\}} \mathrm{d}\mu(z) < +\infty$. In that case, (1.1) can also be seen as a generalization of Rosenau's models [46, 47, 51, 52, 59, 60] and nonlinear radiation hydrodynamics models [55] of the form

$$\partial_t u + \operatorname{div} f(u) = g * A(u) - A(u), \tag{1.12}$$

where $*$ denotes the convolution product w.r.t. x and $g \in L^1(\mathbb{R}^d)$ is nonnegative with $\int_{\mathbb{R}^d} g(z)\,\mathrm{d}z = 1$.

Most of the results on these nonlocal equations concern Equation (1.9) with the α-stable linear diffusions, and convolution models (1.12) with nonlinear but nonsingular Lévy diffusions. It is known that shocks can occur in finite time [4, 30, 46, 47, 48, 52, 59], that weak solutions can be nonunique [2], and that the Cauchy problem is well-posed with the notion of entropy solutions in the sense of Kruzhkov [1, 51, 55]; see also the works [25, 39] for the related topic of time fractional derivatives. The entropy solution theory has been generalized in [45] to singular but linear Lévy diffusions along with nonlinear local diffusions. Very recently, it has been extended in [19] to cover the full problem (1.1) for general singular Lévy measures and nonlinear A.

The purpose of the present paper is to develop an abstract framework for finding error estimates for entropy solutions of (1.1). As applications, we focus in this paper on continuous dependence estimates and convergence rates for vanishing viscosity approximations. We refer the reader to [13, 18, 24, 44, 50] and the references therein for similar analysis on (1.11) and related local equations. As far as nonlocal equations are concerned, continuous dependence estimates for fully nonlinear integro-PDEs have already been derived in [38] in the context of viscosity solutions of Bellman-Isaacs equations; see also [34, 36, 38] for error estimates on nonlocal vanishing viscosity approximations.

To the best of our knowledge, the first and up to now only continuous dependence estimate for nonlocal conservation laws can be found in [45]; see also [1, 25, 31, 34, 51, 59] for convergence rates for vanishing viscosity approximations of Equations (1.9) and (1.12). The general estimate in [45] is established for Equation (1.10) for linear symmetric Lévy diffusions. Inspired by an early version of the present paper, a formal discussion on possible extensions to nonlinear nonlocal diffusions is also given. On the technical side, [45] employs so-called entropy defect measures while we do not.

To finish with the bibliography, let us also refer the reader to [20, 21, 22, 28, 32, 55] for the related topic of error estimates for numerical approximations.

Our main result is stated in Lemma 3.1, and it compares the entropy solution u of (1.1) with a general function v. Our main application consists in comparing u with the entropy solution v of

$$\begin{cases} \partial_t v + \operatorname{div} g(v) = \mathcal{L}^\nu[B(v)], \\ v(x,0) = v_0, \end{cases} \tag{1.13}$$

where the data set (g, B, v_0, ν) is assumed to satisfy (1.3)–(1.6). We obtain explicit continuous dependence estimates on the data stated in Theorems 3.3–3.4. Let us recall that when $B = 0$ or $\nu = 0$, (1.13) is the pure scalar conservation law in (1.8). Equation (1.1) can thus be seen as a nonlinear nonlocal vanishing viscosity approximation of (1.8) if A or μ vanishes. The rate of convergence is then obtained as a consequence of Theorems 3.3–3.4, see Theorem 3.9.

It is natural to compare Theorems 3.3–3.4 and Theorem 3.9 with the known error estimates for the different equations above. One can see that a quite important part of them are particular cases of our general results. We discuss this point in Section 3 by giving precise examples. Let us mention that we also give an example of a simple Hamilton-Jacobi equation where we show that Theorems 3.3–3.4 are in some sense the "conservation law version" of the results in [38]; see Example 3.2.

To finish, let us mention that in the case of fractional Laplacians of order $\alpha \geq 1$, Theorems 3.3–3.4 can be improved by taking advantage of the homogeneity of the measures in (1.7). In order not to make this paper too long, this special case (including $\alpha < 1$) is investigated in a second paper [3].

The rest of this paper is organized as follows. In Section 2 we recall the notion of entropy solution to (1.1). In Section 3, we state and discuss our main results. Sections 4–5 are devoted to the proofs of our main results; Section 4 states some preliminary results on the nonlocal operator.

Notation. Hereafter, $a \vee b := \max\{a, b\}$, while $\cdot$ and $|\cdot|$ denote the Euclidean inner product and norm. For $A \in \mathbb{R}^{d\times d}$, $|A| := \max\{A\,w : w \in \mathbb{R}^d,\ |w| \leq 1\}$. The symbols $\|\cdot\|$ and $|\cdot|$ are used for norms and semi-norms of functions respectively. The symbol supp is used for the support. The superscripts $^\pm$ are used for the positive and negative parts. The total variation of a Radon measure μ is denoted by $|\mu|$. Its tensor product with the Lebesgue measure $\mathrm{d}w$ is denoted by $\mathrm{d}\mu(z)\,\mathrm{d}w$.

2. Entropy formulation and well-posedness

Let us recall the formal computations leading to the entropy formulation of (1.1). First we split $\mathcal{L}^\mu$ into 3 parts:

$$\mathcal{L}^\mu[\phi](x) = \mathcal{L}^\mu_r[\phi](x) + \mathrm{div}\left(b^\mu_r\,\phi\right)(x) + \mathcal{L}^{\mu,r}[\phi](x) \tag{2.1}$$

for $\phi \in C_c^\infty(\mathbb{R}^d)$, $r > 0$, and $x \in \mathbb{R}^d$, where

$$\mathcal{L}^\mu_r[\phi](x) := \int_{0<|z|\leq r} \left(\phi(x+z) - \phi(x) - z\cdot D\phi(x)\,\mathbf{1}_{|z|\leq 1}\right)\mathrm{d}\mu(z), \tag{2.2}$$

$$b^\mu_r := -\int_{|z|>r} z\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z), \tag{2.3}$$

$$\mathcal{L}^{\mu,r}[\phi](x) := \int_{|z|>r} \left(\phi(x+z) - \phi(x)\right)\mathrm{d}\mu(z). \tag{2.4}$$

Consider then the Kruzhkov [49] entropies $|\cdot - k|$, $k \in \mathbb{R}$, and entropy fluxes

$$q_f(u,k) := \mathrm{sgn}\,(u-k)\,(f(u) - f(k)) \in \mathbb{R}^d, \tag{2.5}$$

where we always use the following everywhere representative of the sign function:

$$\mathrm{sgn}\,(u) := \begin{cases} \pm 1 & \text{if } \pm u > 0, \\ 0 & \text{if } u = 0. \end{cases} \tag{2.6}$$

By (1.4) it is readily seen that for all $u, k \in \mathbb{R}$,

$$\mathrm{sgn}\,(u-k)\,(A(u) - A(k)) = |A(u) - A(k)|, \tag{2.7}$$

and we formally deduce from (2.1), (2.7), and the nonnegativity of μ that

$$\begin{aligned} &\mathrm{sgn}\,(u-k)\,\mathcal{L}^\mu[A(u)] \\ &\leq \mathcal{L}^\mu_r[|A(u) - A(k)|] + \mathrm{div}\,(b^\mu_r\,|A(u) - A(k)|) + \mathrm{sgn}\,(u-k)\,\mathcal{L}^{\mu,r}[A(u)]. \end{aligned}$$

Let u be a solution of (1.1), and multiply (1.1) by $\mathrm{sgn}\,(u-k)$. Formal computations then reveal that

$$\begin{aligned} &\partial_t|u-k| + \mathrm{div}\,\left(q_f(u,k) - b^\mu_r\,|A(u) - A(k)|\right) \\ &\qquad \leq \mathcal{L}^\mu_r[|A(u) - A(k)|] + \mathrm{sgn}\,(u-k)\,\mathcal{L}^{\mu,r}[A(u)]. \end{aligned}$$

The entropy formulation in Definition 2.1 below consists in asking that u satisfies this inequality for all entropy-flux pairs (i.e. for all $k \in \mathbb{R}$) and all $r > 0$. Roughly speaking one can give a sense to $\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]$ for bounded discontinuous u thanks to (1.6). But since μ may be singular at $z=0$, see Remark 1.1 (3), the other terms have to be interpreted in the sense of distributions: Multiply by test functions ϕ and integrate by parts to move singular operators onto test functions. For the nonlocal terms this can be done by change of variables: First take $(z,x,t)\to(-z,x,t)$ to see (formally) that

$$\int_{Q_T} \phi \operatorname{div}\left(b_r^{\mu}\,|A(u)-A(k)|\right)\mathrm{d}x\mathrm{d}t = \int_{Q_T} D\phi\cdot b_r^{\mu^*}\,|A(u)-A(k)|\,\mathrm{d}x\mathrm{d}t,$$

where μ^* is the Lévy measure (i.e. it satisfies (1.6)) defined for all $\phi\in C_c^\infty(\mathbb{R}^d\setminus\{0\})$ by

$$\int_{\mathbb{R}^d\setminus\{0\}}\phi(z)\,\mathrm{d}\mu^*(z) := \int_{\mathbb{R}^d\setminus\{0\}}\phi(-z)\,\mathrm{d}\mu(z). \tag{2.8}$$

In view of (2.2), we can take $(z,x,t)\to(-z,x+z,t)$ to find that

$$\int_{Q_T}\phi\,\mathcal{L}_r^{\mu}[|A(u)-A(k)|]\,\mathrm{d}x\mathrm{d}t = \int_{Q_T}|A(u)-A(k)|\,\mathcal{L}_r^{\mu^*}[\phi]\,\mathrm{d}x\mathrm{d}t.$$

This leads to the following definition introduced in [19].

Definition 2.1. (Entropy solutions) *Assume* (1.3)–(1.6). *We say that a function $u\in L^\infty(Q_T)\cap C\left([0,T];L^1\right)$ is an entropy solution of* (1.1) *provided that for all $k\in\mathbb{R}$, all $r>0$, and all nonnegative $\phi\in C_c^\infty(\mathbb{R}^{d+1})$,*

$$\begin{aligned}\int_{Q_T}&\Big\{|u-k|\,\partial_t\phi+\Big(q_f(u,k)+b_r^{\mu^*}\,|A(u)-A(k)|\Big)\cdot D\phi\Big\}\,\mathrm{d}x\mathrm{d}t\\ &+\int_{Q_T}\Big(|A(u)-A(k)|\,\mathcal{L}_r^{\mu^*}[\phi]+\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad-\int_{\mathbb{R}^d}|u(x,T)-k|\,\phi(x,T)\,\mathrm{d}x+\int_{\mathbb{R}^d}|u_0(x)-k|\,\phi(x,0)\,\mathrm{d}x\geq 0.\end{aligned} \tag{2.9}$$

Remark 2.1.

(1) Under assumptions (1.3)–(1.6), the entropy inequality (2.9) is well-defined independently of the a.e. representative of u. To see this, note that since μ^* satisfies (1.6), it easily follows that $\mathcal{L}_r^{\mu^*}[\phi]\in C_c^\infty(\mathbb{R}^{d+1})$. Since $\operatorname{sgn}(u-k)$, $q_f(u,k)$, and $A(u)$ belong to L^∞ by (2.6) and (1.3)–(1.4), it is then clear that all terms in (2.9) are well-defined except possibly the $\mathcal{L}^{\mu,r}$-term. Here it may look like we are integrating Lebesgue measurable functions w.r.t. a Radon measure μ. However, the integrand does have the right measurability by a classical approximation procedure, see Remark 5.1 in [19]. We therefore find that since $A(u)$ belongs to $C([0,T];L^1)$, so does also $\mathcal{L}^{\mu,r}[A(u)]$ and we are done.
(2) Another way to understand the measurability issue in (1), is simply to consider only Borel measurable a.e. representatives of the solutions. The reading of the paper would remain exactly the same, since our L^1-continuous dependence estimate do not depend on the representatives.
(3) In the definition of entropy solutions, it is possible to consider functions u only defined for a.e. $t\in[0,T]$ by taking test functions with compact support in Q_T and adding an explicit initial condition, see e.g. [19].
(4) One can check that classical solutions are entropy solutions, thus justifying the formal computations leading to Definition 2.1. Moreover entropy solution are weak solutions and hence smooth entropy solutions are classical solutions. We refer the reader to [19] for the proofs.

Here is a well-posedness result from [19].

Theorem 2.2. (Well-posedness) *Assume* (1.3)–(1.6)*. There exists a unique entropy solution u of* (1.1)*. This entropy solution belongs to $L^\infty(Q_T) \cap C\left([0,T];L^1\right) \cap L^\infty\left(0,T;BV\right)$ and*

$$\begin{cases} \|u\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \\ \|u\|_{C([0,T];L^1)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}, \\ |u|_{L^\infty(0,T;BV)} \leq |u_0|_{BV(\mathbb{R}^d)}. \end{cases} \tag{2.10}$$

Moreover, if v is the entropy solution of (1.1) *with $v(0) = v_0$ for another initial data v_0 satisfying* (1.5)*, then*

$$\|u - v\|_{C([0,T];L^1)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}. \tag{2.11}$$

Remark 2.3. By the L^∞-estimate in (2.10), all the results of this paper also holds for locally Lipschitz-continuous nonlinearities (f, A). Simply replace the data (f, A) by $(f, A)\,\psi_M$, where $\psi_M \in C_c^\infty(\mathbb{R})$ is such that $\psi_M = 1$ in $[-M, M]$ for $M = \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

3. Main results

Our first main result is a Kuznetsov type of lemma that measures the distance between the entropy solution u of (1.1) and an arbitrary function v.

Let $\epsilon, \delta > 0$ and $\phi^{\epsilon,\delta} \in C^\infty(Q_T^2)$ be the test function

$$\phi^{\epsilon,\delta}(x,t,y,s) := \theta_\delta(t-s)\,\bar{\theta}_\epsilon(x-y), \tag{3.1}$$

where $\theta_\delta(t) := \frac{1}{\delta}\,\tilde{\theta}_1\left(\frac{t}{\delta}\right)$ and $\bar{\theta}_\epsilon(x) := \frac{1}{\epsilon^d}\tilde{\theta}_d\left(\frac{x}{\epsilon}\right)$ are, respectively, time and space approximate units with kernel $\tilde{\theta}_n$ with $n = 1$ and $n = d$ satisfying

$$\tilde{\theta}_n \in C_c^\infty(\mathbb{R}^n), \quad \tilde{\theta}_n \geq 0, \quad \operatorname{supp}\tilde{\theta}_n \subseteq \{|x| < 1\}, \quad \text{and} \quad \int_{\mathbb{R}^n} \tilde{\theta}_n(x)\,\mathrm{d}x = 1. \tag{3.2}$$

We also let $\omega_u(\delta)$ be the modulus of continuity of $u \in C\left([0,T];L^1\right)$.

Lemma 3.1 (Kuznetsov type Lemma)**.** *Assume* (1.3)–(1.6)*. Let u be the entropy solution of* (1.1) *and $v \in L^\infty(Q_T) \cap C\left([0,T];L^1\right)$ with $v(0) = v_0$. Then for all $r > 0$, $\epsilon > 0$, and $0 < \delta < T$,*

(3.3)
$$\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde{\theta}}\, |u_0|_{BV(\mathbb{R}^d)} + 2(\omega_u(\delta) \vee \omega_v(\delta)) \\
&\quad - \iint_{Q_T^2} |v(x,t) - u(y,s)|\,\partial_t \phi^{\epsilon,\delta}(x,t,y,s)\,\mathrm{d}w \\
&\quad - \iint_{Q_T^2} \Big(q_f(v(x,t),u(y,s)) + b_r^{\mu^*}\,|A(v(x,t)) - A(u(y,s))|\Big) \cdot D_x\phi^{\epsilon,\delta}(x,t,y,s)\,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v(x,t)) - A(u(y,s))|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w \\
&\quad - \iint_{Q_T^2} \operatorname{sgn}\left(v(x,t) - u(y,s)\right) \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}(x,t,y,s)\,\mathrm{d}w \\
&\quad + \iint_{\mathbb{R}^d \times Q_T} |v(x,T) - u(y,s)|\,\phi^{\epsilon,\delta}(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \iint_{\mathbb{R}^d \times Q_T} |v_0(x) - u(y,s)|\,\phi^{\epsilon,\delta}(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

where $\mathrm{d}w := \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$, and $C_{\tilde{\theta}} := 2\int_{\mathbb{R}^d} |x|\tilde{\theta}_d(x)\,\mathrm{d}x$.

Remark 3.2.

(1) The error in time only depends on the moduli of continuity of u and v at $t = 0$ and $t = T$. Here we simply take the global-in-time moduli of continuity $\omega_u(\delta)$ and $\omega_v(\delta)$, since this is sufficient in our settings.
(2) When $A = 0$ or $\mu = 0$ this lemma reduces to the well-known Kuznetsov lemma [50] for multidimensional scalar conservation laws.
(3) Notice that the $\mathcal{L}_r^{\mu^*}$-term vanishes when $r \to 0$, see Lemma 4.5.
(4) Lemma 3.1 has many applications. In this paper and in [3] we focus on continuous dependence results and error estimates for the vanishing viscosity method. Then in [20], we will use the lemma to obtain error estimates for numerical approximations of (1.1).

In this paper we apply Lemma 3.1 to compare the entropy solution u of (1.1) with the entropy solution v of (1.13). This is our second main result, and we present it in the two theorems below. The first focuses on the dependence on the nonlinearities (with $\mu = \nu$) and the second one on the Lévy measure (with $A = B$).

Theorem 3.3. (Continuous dependence on the nonlinearities) *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* $(g, B, v_0, \nu = \mu)$ *satisfying* (1.3)–(1.6). *Then for all* $T, r > 0$,

$$
\begin{aligned}
\|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)} \sqrt{c_d\, T \int_{0<|z|\leq r} |z|^2 \,\mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})}} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)}\, T \left| \int_{r\wedge 1<|z|\leq r\vee 1} z \,\mathrm{d}\mu(z) \right| \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&\quad + T \int_{|z|>r} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})},
\end{aligned} \tag{3.4}
$$

where $c_d = \frac{4d^2}{d+1}$.

Theorem 3.4. (Continuous dependence on the Lévy measure) *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* $(g, B = A, v_0, \nu)$ *satisfying* (1.3)–(1.6). *Then for all* $T, r > 0$,

$$
\begin{aligned}
\|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)} \sqrt{c_d\, T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{0<|z|\leq r} |z|^2 \,\mathrm{d}|\mu - \nu|(z)} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \left| \int_{r\wedge 1<|z|\leq r\vee 1} z \,\mathrm{d}(\mu - \nu)(z) \right| \\
&\quad + T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{|z|>r} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}|\mu - \nu|(z),
\end{aligned} \tag{3.5}
$$

where $c_d = \frac{4d^2}{d+1}$.

Remark 3.5. In the error estimates of Theorems 3.3 and 3.4, there are 3 terms accounting for the dependence on the fractional diffusion term in (1.1): One term accounts for the behavior near the singularity of μ at $z = 0$ (the integral over $0 < |z| \leq r$), another term accounts for the behavior near infinity (the integral over $|z| \geq r$), and the last term (the integral over $r \wedge 1 < |z| \leq r \vee 1$) is a drift term that is only present for nonsymmetric measures μ. The square root estimate for the singular term is similar to estimates for 2nd derivative terms in the local case and for non-local equations with different structure, cf. e.g. [18, 38, 45].

Remark 3.6. Since the initial data is $L^1 \cap BV$, an application of Fubini's theorem shows that for any $\hat{r} > r > 0$,

$$\begin{aligned}&\int_{|z|>r} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \, \mathrm{d}\mu(z) \\ &\leq |u_0|_{BV(\mathbb{R}^d)} \int_{r<|z|\leq \hat{r}} |z| \, \mathrm{d}\mu(z) + 2\|u_0\|_{L^1(\mathbb{R}^d)} \int_{|z|>\hat{r}} \mathrm{d}\mu(z).\end{aligned}$$

From Theorems 3.3 and 3.4 we can easily find a general continuous dependence estimate when both A and μ are different from B and ν, respectively. E.g. we can take an intermediate solution w of $w_t + \operatorname{div} f(w) = \mathcal{L}^\mu[B(w)]$ and $w(0) = u_0$, and use the triangle inequality. Using this idea we can show that the following estimates always have to hold:

Corollary 3.7. *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets (f, A, u_0, μ) and (g, B, v_0, ν) satisfying* (1.3)–(1.6). *Then for all $T > 0$*

$$\begin{aligned}&\|u - v\|_{C([0,T];L^1)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)} \, T \, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\ &\quad + C\,(T^{\frac{1}{2}} \vee T) \left(\sqrt{\|A' - B'\|_{L^\infty(\mathbb{R})}} + \sqrt{\int_{\mathbb{R}^d\setminus\{0\}} (|z|^2 \wedge 1) \, \mathrm{d}|\mu - \nu|(z)} \right)\end{aligned} \tag{3.6}$$

where C only depends on d and the data. Moreover, if in addition

$$\int_{\mathbb{R}^d\setminus\{0\}} (|z| \wedge 1) \, \mathrm{d}\mu(z) + \int_{\mathbb{R}^d\setminus\{0\}} (|z| \wedge 1) \, \mathrm{d}\nu(z) < +\infty,$$

then we have the better estimate

$$\begin{aligned}&\|u - v\|_{C([0,T];L^1)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)} \, T \, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\ &\quad + CT \left(\|A' - B'\|_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}^d\setminus\{0\}} (|z| \wedge 1) \, \mathrm{d}|\mu - \nu|(z) \right),\end{aligned} \tag{3.7}$$

where C only depends on the data.

Outline of proof. To prove (3.6), we use Theorems 3.3 and 3.4 with $r = 1$ and the triangle inequality. We also use estimates like $|a - b| \leq \sqrt{|a| + |b|}\sqrt{|a - b|}$, $|\mu - \nu| \leq |\mu| + |\nu|$ etc. To prove (3.7), we also use Remark 3.6 and set $r = 0$ and $\hat{r} = 1$. □

Remark 3.8.

(1) All these estimates hold for arbitrary Lévy measures μ, ν and even for strongly degenerate diffusions where A, B may vanish on large sets. They are consistent (at least for the $|\mu - \nu|$ term) with general results for nonlocal Hamilton-Jacobi-Bellman equations in [38]. When μ, ν have the special form (1.7) (with possibly different α's), then it is possible to use the extra symmetry and homogeneity properties to obtain better estimates, see [3].
(2) The optimal choice of the $r, \hat{r}$ in Remark 3.6 depends on the behavior of the Lévy measures at zero and infinity, see the discussion above and at the end of this section for more details.

Let us now consider the nonlocal vanishing viscosity problem

$$\begin{cases} \partial_t u^\epsilon + \operatorname{div} f(u^\epsilon) = \epsilon \, \mathcal{L}^\mu[A(u^\epsilon)], \\ u^\epsilon(0) = u_0, \end{cases} \tag{3.8}$$

i.e. problem (1.8) with a perturbation term $\epsilon \, \mathcal{L}^\mu[A(u^\epsilon)]$. When $\epsilon > 0$ tend to zero, u^ϵ is expected to converge toward the solution u of (1.8). As an immediate application of Theorem 3.3 or 3.4, we have the following result:

Theorem 3.9 (Vanishing viscosity)**.** *Assume* (1.3)–(1.6). *Let u and u^ϵ be the entropy solutions of* (1.8) *and* (3.8) *respectively. Then for every $T, \epsilon > 0$ and all $\hat{r} > r > 0$,*

$$\begin{aligned} \|u - u^\epsilon\|_{C([0,T];L^1)} \leq C \min_{\hat{r} > r > 0} \Bigg\{ & d^{\frac{1}{2}} T^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \sqrt{\int_{0<|z|\leq r} |z|^2 \, \mathrm{d}\mu(z)} \\ & + T\epsilon \left[\int_{r<|z|\leq \hat{r}} |z| \, \mathrm{d}\mu(z) + \Big| \int_{r\wedge 1<|z|\leq r\vee 1} z \, \mathrm{d}\mu(z) \Big| + \int_{|z|>\hat{r}} \mathrm{d}\mu(z) \right] \Bigg\}, \end{aligned} \tag{3.9}$$

where C only depends on $\|u_0\|_{L^1(\mathbb{R}^d)\cap BV(\mathbb{R}^d)}$ and $\|A'\|_{L^\infty(\mathbb{R})}$.

Outline of proof. Note that u can be seen as the entropy solution of (1.1) with $A = 0$ and μ as Lévy measure. Hence we can estimate $\|u - u^\epsilon\|_{C([0,T];L^1)}$ from Theorem 3.3. The error coming from the difference of the derivatives of the nonlinearities is equal to $\epsilon \|A'\|_{L^\infty(\mathbb{R})}$. Inequality (3.9) then follows from (3.4) and Remark 3.6. □

Corollary 3.10. *Assume* (1.3)–(1.6). *Let u and u^ϵ be the entropy solutions of* (1.8) *and* (3.8) *respectively. Then for all $T, \epsilon > 0$*

$$\|u - u^\epsilon\|_{C([0,T];L^1)} \leq C \, (T^{\frac{1}{2}} \vee T) \, \epsilon^{\frac{1}{2}},$$

where C only depends on d and the data. Moreover, if in addition

$$\int_{\mathbb{R}^d\setminus\{0\}} (|z| \wedge 1) \, \mathrm{d}\mu(z) < +\infty, \tag{3.10}$$

then we have the better estimate

$$\|u - u^\epsilon\|_{C([0,T];L^1)} \leq CT\epsilon,$$

where C depends on the data.

This corollary follows immediately from Theorem 3.9 or Corollary 3.7.

Remark 3.11.

(1) Our estimates are just as good or better than the standard $\mathcal{O}(\epsilon^{\frac{1}{2}})$ estimate for the classical vanishing viscosity method ((1.11) with $A(u) = \epsilon \, u$).
(2) Our estimates hold for arbitrary Lévy measures μ and even for strongly degenerate diffusions where A may vanish on a large set! This is consistent with general results for nonlocal Hamilton-Jacobi-Bellman equations [38].
(3) As for the classical (local) vanishing viscosity method, better rates could be obtained if the solutions are more regular. E.g. if $A(u^\epsilon)$ is uniformly (in ϵ) bounded in $W^{2,1}$, then the error estimate should be $O(\epsilon)$ even without assumption (3.10). Such a result can not be derived from (3.9), but should be proved directly.
(4) Corollary 3.10 contains less information than Theorem 3.9; indeed, if μ is as in (1.7), the additional symmetry and homogeneity can be used to obtain better estimates which can be proved to be optimal. See Example 3.3 below.
(5) The error estimates above trivially also holds for the more general vanishing viscosity equation

$$\begin{cases} \partial_t u^\epsilon + \mathrm{div} f(u^\epsilon) = \mathcal{L}^\nu[B(u^\epsilon)] + \epsilon \, \mathcal{L}^\mu[A(u^\epsilon)], \\ u^\epsilon(0) = u_0. \end{cases}$$

Further discussion. We now make a more precise comparison of the results above with known estimates from the literature. We begin with continuous dependence estimates and finish with convergence rates for vanishing viscosity approximations.

Let u and v denote the entropy solutions of (1.1) and (1.13), respectively. To simplify, we take the same data sets $(f, A, u_0) = (g, B, v_0)$ and we only allow the Lévy measures μ and ν to be different. We also let C denote a constant only depending on T, d and the data.

Example 3.1. Let us consider Equation (1.10) with $a = 0$. Let us also consider the class of Lévy operators satisfying

$$\begin{cases} \int_{\mathbb{R}^d\setminus\{0\}} (|z|^2 \wedge |z|)\,\mathrm{d}\mu(z) < +\infty, \\ \mu = \mu^*. \end{cases}$$

For such kind of equations, the following continuous dependence estimate on the Lévy measure has been established in [45]:

$$\|u - v\|_{C([0,T];L^1)} \leq C\sqrt{\int_{0<|z|\leq 1} |z|^2\,\mathrm{d}|\mu - \nu|(z)} + C\int_{|z|>1} |z|\,\mathrm{d}|\mu - \nu|(z).$$

This estimate follows from Theorem 3.4 and Remark 3.6 by taking $r = 1$ and $\hat{r} = +\infty$ in (3.5).

Example 3.2. Consider the following one-dimensional Hamilton-Jacobi equation

$$U_t + f(U_x) = \mathcal{L}^\mu[U]$$

with initial data $U_0(x) := \int_{-\infty}^x u_0(y)\,\mathrm{d}y$. This particular equation is related to the nonlocal conservation law (1.8), since its solution is $U(x,t) = \int_{-\infty}^x u(y,t)\,\mathrm{d}y$ where u solves (1.8), see [19]. It is also an example of an integro-PDE for which the general theory of [38] applies, and this theory allows us to establish the following continuous dependence estimate on the Lévy measure:

$$\sup_{\mathbb{R}\times[0,T]} |U - V| \leq C\sqrt{\int_{\mathbb{R}\setminus\{0\}} (|z|^2 \wedge 1)\,\mathrm{d}|\mu - \nu|(z)},$$

where $V(x,t) := \int_{-\infty}^x v(y,t)\,\mathrm{d}y$. (This result is a version of Theorem 4.1 in [38] which follows from Theorem 3.1 by setting $p_0, \ldots, p_4, p_s = 0$ and $\rho = |z| \wedge 1$ in (A0)). Since

$$\sup_{\mathbb{R}\times[0,T]} |U - V| \leq \|u - v\|_{C([0,T];L^1)},$$

this estimate also follows from (3.6) in Corollary 3.7 when $(A, f, u_0) = (B, g, v_0)$.

Let us now compare Theorem 3.9 with known convergence rates. We keep the same notation for u and u^ϵ as in Theorem 3.9.

Example 3.3. Let us consider the case where $A(u^\epsilon) = u^\epsilon$ and $\mathcal{L}^\mu = -(-\triangle)^{\frac{\alpha}{2}}$, $\alpha \in (0, 2)$. Then the following optimal rates have been derived in [1, 31]:

$$\|u - u^\epsilon\|_{C([0,T];L^1)} = \begin{cases} \mathcal{O}\left(\epsilon^{\frac{1}{\alpha}}\right) & \text{if } \alpha > 1, \\ \mathcal{O}\left(\epsilon\,|\ln\epsilon|\right) & \text{if } \alpha = 1, \\ \mathcal{O}\left(\epsilon\right) & \text{if } \alpha < 1. \end{cases} \tag{3.11}$$

Let us explain how these results can be deduced from (3.9). First we use (1.7) to explicitly compute the integrals in (3.9) and obtain

$$\|u - u^\epsilon\|_{C([0,T];L^1)} = \mathcal{O}\left(\min_{\hat{r}>r>0}\left\{\sqrt{\epsilon\,\frac{r^{2-\alpha}}{2-\alpha}} + \epsilon\int_r^{\hat{r}} \frac{\mathrm{d}\tau}{\tau^\alpha} + \epsilon\,\hat{r}^{-\alpha}\right\}\right).$$

We then deduce (3.11) by taking $r = \epsilon^{\frac{1}{\alpha}}$ and $\hat{r} = +\infty$ if $\alpha > 1$, $r = \epsilon$ and $\hat{r} = 1$ if $\alpha = 1$, and $r = 0$ and $\hat{r} = 1$ if $\alpha < 1$.

Example 3.4. Let us finally consider the vanishing approximation (3.8) with the viscous term

$$\partial_t u^\epsilon + \operatorname{div} f(u^\epsilon) = \frac{1}{\epsilon} \left(g_\epsilon * u^\epsilon - u^\epsilon \right),$$

where $g_\epsilon(z) := \frac{1}{\epsilon^d}\, g\left(\frac{z}{\epsilon}\right)$ with an even and nonnegative kernel $g \in L^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |z|^2\, g(z)\, \mathrm{d}z < +\infty.$$

This is the Rosenau's regularization of the Chapman-Enskog expansion for hydrodynamics [56]; see also Equations (1.1) and (2.3) of [59]. Its convergence toward (1.8) has been established in [51, 59]. In Corollary 5.2 of [59] the following rate of convergence has been derived:

$$\|u - u^\epsilon\|_{C([0,T];L^1)} = \mathcal{O}\left(\epsilon^{\frac{1}{2}}\right).$$

This result can be recovered from Theorems 3.3 or 3.4. Indeed, we can choose e.g. $A(u^\epsilon) = u^\epsilon$, $\mathrm{d}\mu(z) = \frac{g_\epsilon(z)}{\epsilon}\, \mathrm{d}z$ and $\nu = 0$, to get the desired equations. Next, we apply (3.5) with $r = +\infty$ and rescale the z-variable to show that the error term is bounded above by

$$\begin{aligned} C \sqrt{\int_{\mathbb{R}^d \setminus \{0\}} |z|^2\, \mathrm{d}|\mu - \nu|(z)} &= C \sqrt{\int_{\mathbb{R}^d} |z|^2\, \frac{g\left(\frac{z}{\epsilon}\right)}{\epsilon^{d+1}}\, \mathrm{d}z} \\ &= C\, \epsilon^{\frac{1}{2}} \sqrt{\int_{\mathbb{R}^d} |z|^2\, g(z)\, \mathrm{d}z.} \end{aligned}$$

4. Auxiliary results

Before proving our main results in the next section, we state several technical lemmas.

Lemma 4.1. *Assume* (1.6) *and* $r > 0$*. Then for all* $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\|\mathcal{L}_r^\mu[\phi]\|_{L^1(\mathbb{R}^d)} \leq \int_{0<|z|\leq r} |z|^2\, \mathrm{d}\mu(z)\, \|\phi\|_{W^{2,1}(\mathbb{R}^d)}.$$

The proof easily follows from a Taylor expansion and Fubini's theorem.

Remark 4.2. Similarly $\mathcal{L}^{\mu,r}$ is (linear and) bounded from L^1 into itself, thanks to Remark 3.6 with $\hat{r} = r$.

In the next result, we establish a Kato type inequality for $\mathcal{L}^{\mu,r}[A(u)]$.

Lemma 4.3. *Assume* (1.4) *and* (1.6)*. Then for all* $u \in L^1(\mathbb{R}^d)$, $k \in \mathbb{R}$, $r > 0$, *and all* $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\, \mathcal{L}^{\mu,r}[A(u)]\, \phi\, \mathrm{d}x \leq \int_{\mathbb{R}^d} |A(u) - A(k)|\, \mathcal{L}^{\mu^*,r}[\phi]\, \mathrm{d}x.$$

Proof. Note first that $A(u)$ is L^1 by (1.4), and hence $\mathcal{L}^{\mu,r}[A(u)]$ is well-defined in L^1 by Remark 4.2. Easy computations then reveal that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x,\\
&= \int_{\mathbb{R}^d}\int_{|z|>r} \operatorname{sgn}(u(x)-k)\,\Big(A(u(x+z))-A(u(x))\Big)\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x,\\
&= \int_{\mathbb{R}^d}\int_{|z|>r} \operatorname{sgn}(u(x)-k)\\
&\qquad\qquad \Big\{(A(u(x+z))-A(k))-(A(u(x))-A(k))\Big\}\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x,\\
&\le \int_{\mathbb{R}^d}\int_{|z|>r} \Big(|A(u(x+z))-A(k)|-|A(u(x))-A(k)|\Big)\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x \quad \text{by (2.7)},\\
&= \underbrace{\int_{\mathbb{R}^d}\int_{|z|>r} |A(u(x+z))-A(k)|\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x}_{=:I}\\
&\qquad\qquad -\underbrace{\int_{\mathbb{R}^d}\int_{|z|>r} |A(u(x))-A(k)|\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x}_{=:J}.
\end{aligned}
$$

Note that all these integrals are well-defined, thanks to (1.6) ([1]).

By the respective changes of variable $(z,x)\to(-z,x+z)$ and $(z,x)\to(-z,x)$, we find that

$$
\begin{aligned}
I &= \int_{\mathbb{R}^d}\int_{|z|>r} \phi(x+z)\,|A(u(x))-A(k)|\,\mathrm{d}\mu^*(z)\,\mathrm{d}x,\\
J &= \int_{\mathbb{R}^d}\int_{|z|>r} \phi(x)\,|A(u(x))-A(k)|\,\mathrm{d}\mu^*(z)\,\mathrm{d}x.
\end{aligned}
$$

Here the measure μ^* in (2.8) appears because of the relabelling of z. This measure has the same properties as μ. Hence we can conclude that

$$
\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x \le I-J = \int_{\mathbb{R}^d} |A(u)-A(k)|\,\mathcal{L}^{\mu^*,r}[\phi]\,\mathrm{d}x,
$$

and the proposition follows. □

The next lemma is a consequence of the Kato inequality, and it plays a key role in the doubling of variables arguments throughout this paper and in the uniqueness proof of [1, 19].

Lemma 4.4. *Assume* (1.4) *and* (1.6)*, and let* $u,v\in L^\infty(Q_T)\cap C([0,T];L^1)$*,* $0\le\psi\in L^1(\mathbb{R}^d\times(0,T)^2)$*, and* $r>0$*. Then*

$$
\begin{aligned}
&\iint_{Q_T^2} \operatorname{sgn}(u(y,s)-v(x,t))\\
&\qquad\cdot\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\,\psi(x-y,t,s)\,\mathrm{d}w\le 0
\end{aligned}
$$

(where $\mathrm{d}w=\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$*).*

[1]The measurability is immediate if the reader only consider Borel measurable representatives of u as suggested in Remark 2.1 (2).

Proof. Note that

$$\begin{aligned}
&\operatorname{sgn}(u(y,s)-v(x,t))\Big(A(u(y+z,s))-A(u(y,s))\Big)\\
&\qquad -\operatorname{sgn}(u(y,s)-v(x,t))\Big(A(v(x+z,t))-A(v(x,t))\Big)\\
&=\operatorname{sgn}(u(y,s)-v(x,t))\\
&\qquad\cdot\Big\{\Big(A(u(y+z,s))-A(v(x+z,t))\Big)-\Big(A(u(y,s))-A(v(x,t))\Big)\Big\}\\
&\le |A(u(y+z,s))-A(v(x+z,t))|-|A(u(y,s))-A(v(x,t))|
\end{aligned}$$

where these functions are both defined. By an integration w.r.t. $\mathbf{1}_{|z|>r}\,\mathrm{d}\mu(z)$, we find that for all $(t,s)\in(0,T)^2$ and a.e. $(x,y)\in\mathbb{R}^{2d}$,

$$\begin{aligned}
&\operatorname{sgn}(u(y,s)-v(x,t))\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\\
&\le\int_{|z|>r}\big(|A(u(y+z,s))-A(v(x+z,t))|-|A(u(y,s))-A(v(x,t))|\big)\,\mathrm{d}\mu(z).
\end{aligned}$$

After another integration, this time w.r.t. $\psi(x-y,t,s)\,\mathrm{d}w$, we then get that

$$\begin{aligned}
&\iint_{Q_T^2}\operatorname{sgn}(u(y,s)-v(x,t))\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\,\psi\,\mathrm{d}w\\
&\le\iint_{Q_T^2}\int_{|z|>r}|A(u(y+z,s))-A(v(x+z,t))|\;\psi(x-y,t,s)\,d\mu(z)\,\mathrm{d}w\\
&\quad-\iint_{Q_T^2}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x-y,t,s)\,d\mu(z)\,\mathrm{d}w,\\
&=: I+J.
\end{aligned}$$

Note that these integrals are finite since $\|A(u)\|_{C([0,T];L^1)}\le\|A'\|_{L^\infty}\|u\|_{C([0,T];L^1)}$ (A is Lipschitz-continuous and 0 at 0) and by Fubini (note the convolution integrals in x and y),

$$I,J\le\Big(\|A(u)\|_{C([0,T];L^1)}+\|A(v)\|_{C([0,T];L^1)}\Big)\|\psi\|_{L^1(\mathbb{R}^d\times(0,T)^2)}\int_{|z|>r}\mathrm{d}\mu(z).$$

We then change variables $(z,x,t,y,s)\to(z,x+z,t,y+z,s)$ in I,

$$I=\iint_{Q_T}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x-z-(y-z),t,s)\,\mathrm{d}\mu(z)\,\mathrm{d}w,$$

to find that $I+J=0$ and the proof is complete. □

Lemma 4.5. *Under the assumptions of Lemma 3.1,*

$$I=\iint_{Q_T^2}|A(v(x,t))-A(u(y,s))|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w\le C_\epsilon\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z),$$

where $C_\epsilon>0$ does not depend on $r>0$.

Proof. Easy computations show that

$$\begin{aligned}
&\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\\
&=\theta_\delta(t-s)\int_{0<|z|\le r}\big(\bar\theta_\epsilon(x-y-z)-\bar\theta_\epsilon(x-y)+z\cdot D\bar\theta_\epsilon(x-y)\,\mathbf{1}_{|z|\le1}\big)\,\mathrm{d}\mu^*(z)\\
&=\theta_\delta(t-s)\int_{0<|z|\le r}\big(\bar\theta_\epsilon(x-y+z)-\bar\theta_\epsilon(x-y)-z\cdot D\bar\theta_\epsilon(x-y)\,\mathbf{1}_{|z|\le1}\big)\,\mathrm{d}\mu(z)\\
&=\theta_\delta(t-s)\,\mathcal{L}_r^{\mu}[\bar\theta_\epsilon](x-y),
\end{aligned}$$

and by Fubini (there are again convolution integrals in I!),

$$
\begin{aligned}
I &\leq \iint_{Q_T^2} |A(u(y,s)) - A(v(x,t))|\, \theta_\delta(t-s) \left|\mathcal{L}_r^\mu[\bar{\theta}_\epsilon](x-y)\right| \,\mathrm{d}w \\
&\leq \Big(\|A(u)\|_{L^1(Q_T)} + \|A(v)\|_{L^1(Q_T)}\Big) \|\theta_\delta\, \mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^{d+1})} \\
&\leq T\|A'\|_{L^\infty} \Big(\|u\|_{C([0,T];L^1)} + \|v\|_{C([0,T];L^1)}\Big) \|\theta_\delta\, \mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^{d+1})}.
\end{aligned}
$$

By classical properties of approximate units and Lemma 4.1,

$$
\begin{aligned}
\|\theta_\delta\, \mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^{d+1})} &= \underbrace{\|\theta_\delta\|_{L^1(\mathbb{R})}}_{=1} \|\mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^d)} \\
&\leq \|\bar{\theta}_\epsilon\|_{W^{2,1}(\mathbb{R}^d)} \int_{0<|z|\leq r} |z|^2 \,\mathrm{d}\mu(z).
\end{aligned}
$$

The proof is complete. □

5. Proofs of the main results

The proofs of this section use the so-called doubling of variables technique of Kruzhkov [49] along with ideas from [38, 50]; for other relevant references, see also e.g. [1, 19, 45] for nonlocal equations. It consists in considering u as a function of the new variables (y,s) and using the approximate units $\phi^{\epsilon,\delta}$ in (3.1) as test functions. For brevity, we do not specify anymore the variables of $u = u(y,s)$, $v = v(x,t)$ and $\phi^{\epsilon,\delta} = \phi^{\epsilon,\delta}(x,t,y,s)$ when the context is clear; recall also that $\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$ is denoted by $\mathrm{d}w$.

5.1. Proof of Lemma 3.1.

Let $(x,t) \in Q_T$ be fixed and $u = u(y,s)$, $k = v(x,t)$, and $\phi(y,s) := \phi^{\epsilon,\delta}(x,t,y,s)$. The entropy inequality for u (see (2.9)) then takes the form

$$
\begin{aligned}
&\int_{Q_T} \Big\{ |u-v|\, \partial_s \phi^{\epsilon,\delta} + \Big(q_f(u,v) + |A(u)-A(v)|\, b_r^{\mu^*} \Big) \cdot D_y \phi^{\epsilon,\delta} \Big\} \,\mathrm{d}y\,\mathrm{d}s \\
&+ \int_{Q_T} |A(u)-A(v)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) \,\mathrm{d}y\,\mathrm{d}s \\
&+ \int_{Q_T} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta} \,\mathrm{d}y\,\mathrm{d}s \\
&- \int_{\mathbb{R}^d} |u(y,T) - v(x,t)|\, \phi^{\epsilon,\delta}(x,t,y,T) \,\mathrm{d}y \\
&+ \int_{\mathbb{R}^d} |u_0(y) - v(x,t)|\, \phi^{\epsilon,\delta}(x,t,y,0) \,\mathrm{d}y \geq 0.
\end{aligned}
$$

We integrate this inequality w.r.t. $(x,t) \in Q_T$, noting that q_f in (2.5) is symmetric, and that $\partial_s \phi^{\epsilon,\delta} = -\partial_t \phi^{\epsilon,\delta}$ and $D_y \phi^{\epsilon,\delta} = -D_x \phi^{\epsilon,\delta}$ by (3.1). Consequently we find that

$$
\begin{aligned}
&I_1+\cdots+I_5\\
&:= -\iint_{Q_T^2}\Big\{|u-v|\,\partial_t\phi^{\epsilon,\delta}+\Big(q_f(v,u)+|A(u)-A(v)|\,b_r^{\mu^*}\Big)\cdot D_x\phi^{\epsilon,\delta}\Big\}\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}|A(u)-A(v)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&\quad-\iint_{Q_T\times\mathbb{R}^d}|u(y,T)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\\
&\quad+\iint_{Q_T\times\mathbb{R}^d}|u_0(y)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\geq 0.
\end{aligned}
\tag{5.1}
$$

Note that the terms in the inequality above are well-defined since they are all essentially of the form of convolution integrals of L^1-functions. See Lemma 4.1, Remark 4.2, and the discussions in the proofs of Lemmas 4.4 and 4.5 for more details.

A classical computation from [50] reveals that

$$
\begin{aligned}
&I_4+I_5-\iint_{\mathbb{R}^d\times Q_T}|u(y,s)-v(x,T)|\,\phi^{\epsilon,\delta}(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad+\iint_{\mathbb{R}^d\times Q_T}|u(y,s)-v_0(x)|\,\phi^{\epsilon,\delta}(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\leq -\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)}+\|u_0-v_0\|_{L^1(\mathbb{R}^d)}\\
&\quad+\epsilon\,C_{\tilde\theta}\,|u_0|_{BV(\mathbb{R}^d)}+2(\omega_u(\delta)\vee\omega_v(\delta)),
\end{aligned}
$$

where $C_{\tilde\theta}$ is as in Lemma 3.1. Lemma 3.1 now follows from (5.1) and the above estimates on I_4 and I_5.

5.2. **Proof of Theorem 3.3.** The proof uses the Kuznetsov lemma, and morally speaking it amounts to subtracting the $u(y,s)$ and $v(x,t)$ equations, multiplying by $\operatorname{sgn}(u-v)$, and then applying both new and classical tricks to arrive at an L^1-estimate of $|u-v|$. We expect to see terms involving

$$\operatorname{sgn}(u-v)\Big(\mathcal{L}^{\mu,r}[A(u)]-\mathcal{L}^{\mu,r}[B(v)]\Big),$$

and naively we can write this as

$$\operatorname{sgn}(u-v)\mathcal{L}^{\mu,r}[(A-B)(u)]+\operatorname{sgn}(u-v)\mathcal{L}^{\mu,r}[B(u)-B(v)].$$

These terms are estimated by Kato type inequalities (see Lemmas 4.3 and 4.4), the first term should give the dependence on $A-B$ while the second term is a nonpositive term that also appears in the uniqueness proof. *The problem with this approach is that we can not apply Kato for the first term because $A-B$ then have to be monotone!*

There are different ways to overcome this monotonicity problem, and we have chosen to adapt ideas from [38] – a paper on continuous dependence estimates for fully nonlinear Bellman-Isaacs type of equations via viscosity solution techniques. We consider the region where $A'\geq B'$ and its complementary. Let $E_\pm$ be sets satisfying:

$$
\begin{cases}
E_\pm\subseteq\mathbb{R}\text{ are Borel sets;}\\
\cup_\pm E^\pm=\mathbb{R}\text{ and }\cap_\pm E_\pm=\emptyset;\\
\mathbb{R}\setminus\operatorname{supp}(A'-B')^\mp\subseteq E_\pm.
\end{cases}
\tag{5.2}
$$

For all $u \in \mathbb{R}$, we define

$$
\begin{aligned}
A_\pm(u) &:= \int_0^u A'(\tau)\, \mathbf{1}_{E_\pm}(\tau)\, \mathrm{d}\tau, \\
B_\pm(u) &:= \int_0^u B'(\tau)\, \mathbf{1}_{E_\pm}(\tau)\, \mathrm{d}\tau, \\
C_\pm(u) &:= \pm(A_\pm(u) - B_\pm(u)).
\end{aligned} \tag{5.3}
$$

These functions satisfy the following properties:

Lemma 5.1. *Under the assumptions of Theorem 3.3,*

(i) $A = A_+ + A_-$ *and* $B = B_+ + B_-$*;*
(ii) $A_\pm, B_\pm, C_\pm$ *satisfy* (1.4)*, in particular, they are monotone;*
(iii) $\sum_\pm |C_\pm(u)|_{L^1(0,T;BV)} \leq \|A' - B'\|_{L^\infty(\mathbb{R})}\, |u|_{L^1(0,T;BV)}$*;*
(iv) *for all* $z \in \mathbb{R}^d \setminus \{0\}$*,*

$$
\sum_\pm \|C_\pm(u(\cdot + z, \cdot)) - C_\pm(u)\|_{L^1(Q_T)} \leq \|A' - B'\|_{L^\infty(\mathbb{R})}\, \|u(\cdot + z, \cdot) - u\|_{L^1(Q_T)}.
$$

The proofs of (i) and (ii) are immediate, whereas (iii) and (iv) follow from standard arguments for Lipschitz-continuous and BV-functions (see e.g. [13, 35, 57]); the details are left to the reader.

In the proof below, $A^\pm - B^\pm$ will be the monotone functions replacing the nonmonotone function $A - B$ of the formal argument above.

Proof of Theorem 3.3. Let us divide the proof into several steps.

1. We argue as in the beginning of the proof of Lemma 3.1 changing the roles of u and v. We fix (y, s) and take $k = u(y, s)$ and $\phi^{\epsilon,\delta} = \phi^{\epsilon,\delta}(x, t, y, s)$ in the entropy inequality for $v = v(x, t)$ to find that

$$
\begin{aligned}
&\iint_{Q_T^2} \Big\{ |v - u|\, \partial_t \phi^{\epsilon,\delta} + \Big(q_g(v, u) + |B(v) - B(u)|\, b_r^{\mu^*} \Big) \cdot D_x \phi^{\epsilon,\delta} \Big\}\, \mathrm{d}w \\
&+ \iint_{Q_T^2} |B(v) - B(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(\cdot, t, y, s)](x)\, \mathrm{d}w \\
&+ \iint_{Q_T^2} \operatorname{sgn}(v - u)\, \mathcal{L}^{\mu,r}[B(v(\cdot, t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&- \iint_{\mathbb{R}^d \times Q_T} |v(x, T) - u(y, s)|\, \phi^{\epsilon,\delta}(x, T, y, s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \\
&+ \iint_{\mathbb{R}^d \times Q_T} |v_0(x) - u(y, s)|\, \phi^{\epsilon,\delta}(x, 0, y, s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \geq 0.
\end{aligned}
$$

Then we add this inequality and inequality (3.3) in Lemma 3.1,

$$
\begin{aligned}
&\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)}\\
&\le \|u_0-v_0\|_{L^1(\mathbb{R}^d)}+\epsilon\, C_{\tilde\theta}\,|u_0|_{BV(\mathbb{R}^d)}+2(\omega_u(\delta)\vee\omega_v(\delta))\\
&\quad+\underbrace{\iint_{Q_T^2}(q_g-q_f)(v,u)\cdot D_x\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_1}\\
&\quad+\underbrace{\iint_{Q_T^2}|B(v)-B(u)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(\cdot,y,t,s)](x)\,\mathrm{d}w}_{=:I_2}\\
&\quad+\underbrace{\iint_{Q_T^2}|A(v)-A(u)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w}_{=:I_2'}\\
&\quad+\underbrace{\iint_{Q_T^2}\Big(|B(v)-B(u)|-|A(v)-A(u)|\Big)\,b_r^{\mu^*}\cdot D_x\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_3}\\
&\quad+\underbrace{\iint_{Q_T^2}\operatorname{sgn}(v-u)\Big(\mathcal{L}^{\mu,r}[B(v(\cdot,t))](x)-\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\Big)\,\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_4},
\end{aligned}
\tag{5.4}
$$

where $r,\epsilon>0$, $0<\delta<T$, and $C_{\tilde\theta}>0$ only depends on the kernel $\tilde\theta_d$ from (3.2).

2. It is standard to estimate I_1 (cf. e.g. [27, 50]), and I_2+I_2' can be estimated by Lemma 4.5,

$$
I_1\le |u_0|_{BV(\mathbb{R}^d)}\,T\,\|f'-g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)},
\tag{5.5}
$$

$$
I_2+I_2'\le C_\epsilon\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z),
\tag{5.6}
$$

where C_ϵ does not depend on $r>0$. Now we focus on I_3 and I_4.

3. *Cutting w.r.t. $E_\pm$.* We split I_3 and I_4 into four new terms using the sets $E_\pm$, see (5.2)–(5.3). By Lemma 5.1 (i), I_4 can be written as

$$
I_4=\sum_{\pm}\iint_{Q_T^2}\operatorname{sgn}(v-u)\Big(\mathcal{L}^{\mu,r}[B_\pm(v(\cdot,t))](x)-\mathcal{L}^{\mu,r}[A_\pm(u(\cdot,s))](y)\Big)\,\phi^{\epsilon,\delta}\,\mathrm{d}w.
$$

By Lemma 5.1 (ii), we can apply twice Lemma 4.4 with B_+ and A_- instead of A, followed by the definitions of $C_\pm$, see (5.3), to show that

$$
\begin{aligned}
I_4&\le\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r}\Big[B_+(u(\cdot,s))-A_+(u(\cdot,s))\Big](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r}\Big[B_-(v(\cdot,t))-A_-(v(\cdot,t))\Big](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&=\iint_{Q_T^2}\operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r}\big[C_+(u(\cdot,s))\big](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r}\big[C_-(v(\cdot,t))\big](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&=:I_4^+ + I_4^-.
\end{aligned}
\tag{5.7}
$$

Note that it is crucial to have u in the first term and v in the second – otherwise we will not be able to apply the Kato inequality later on!

We now consider I_3. By (2.7), Lemma 5.1 (i)–(ii), the formula $D_x\phi^{\epsilon,\delta} = -D_y\phi^{\epsilon,\delta}$, and the definitions $D_+ = D_y$ and $D_- = D_x$, it follows that

$$\begin{aligned}
&\Big(|B(v)-B(u)| - |A(v)-A(u)|\Big) D_x\phi^{\epsilon,\delta}\\
&= \operatorname{sgn}(u-v)\Big\{ (A(u)-B(u)) - (A(v)-B(v)) \Big\} D_y\phi^{\epsilon,\delta}\\
&= \sum_{\pm} \operatorname{sgn}(u-v)\Big\{ \pm (A_\pm(u)-B_\pm(u)) \mp (A_\pm(v)-B_\pm(v)) \Big\} D_\pm\phi^{\epsilon,\delta}\\
&= \sum_{\pm} |C_\pm(u) - C_\pm(v)|\, D_\pm\phi^{\epsilon,\delta}.
\end{aligned}$$

We can then rewrite I_3 as

$$I_3 = \sum_{\pm} \underbrace{\iint_{Q_T} |C_\pm(u)-C_\pm(v)|\, b_r^{\mu^*}\cdot D_\pm\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_3^\pm}. \tag{5.8}$$

4. *Cutting w.r.t. z.* We decompose $\mathcal{L}^{\mu,r}$ into two new terms using a new cutting parameter $r_1 > r$. Let $\mu = \mu_1 + \mu_{|_{|z|>r_1}}$ for

$$\mu_1 := \mu_{|_{0<|z|\le r_1}},$$

and note that by (2.4), $\mathcal{L}^{\mu,r} = \mathcal{L}^{\mu_1,r} + \mathcal{L}^{\mu,r_1}$. Then

$$\begin{aligned}
I_4^+ = &\underbrace{\iint_{Q_T^2} \operatorname{sgn}(u-v)\,\mathcal{L}^{\mu_1,r}[C_+(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_5^+}\\
&+ \iint_{Q_T^2} \operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r_1}[C_+(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w.
\end{aligned}\tag{5.9}$$

Since C_+ satisfies (1.4) by Lemma 5.1 (ii) and μ_1 clearly satisfies (1.6), we can apply the Kato type inequality in Lemma 4.3 (with $k = v(x,t)$ and $A = C_+$) to show that

$$\begin{aligned}
I_5^+ &= \int_{Q_T}\int_{Q_T} \operatorname{sgn}(u(y,s)-v(x,t))\,\mathcal{L}^{\mu_1,r}[C_+(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{Q_T}\int_{Q_T} |C_+(u(y,s)) - C_+(v(x,t))|\,\mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Adding I_3^+ in the form (5.8) then gives

$$I_3^+ + I_5^+ \le \iint_{Q_T^2} |C_+(u)-C_+(v)|\,\Big(b_r^{\mu^*}\cdot D_y\phi^{\epsilon,\delta} + \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\Big)\,\mathrm{d}w. \tag{5.10}$$

Now easy computations show that

$$D_y\phi^{\epsilon,\delta} = -\theta_\delta(t-s)\,D\bar\theta_\epsilon(x-y), \quad \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) = \theta_\delta(t-s)\,\mathcal{L}^{\mu_1,r}[\bar\theta_\epsilon](x-y).$$

Hence by adding and subtracting $z\cdot D\bar\theta_\epsilon(x-y)$, we get that

$$\begin{aligned}
&b_r^{\mu^*}\cdot D_y\phi^{\epsilon,\delta} + \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\\
&= \theta_\delta(t-s)\int_{r<|z|\le r_1} \Big(\bar\theta_\epsilon(x-y+z) - \bar\theta_\epsilon(x-y) - z\cdot D\bar\theta_\epsilon(x-y)\Big)\,\mathrm{d}\mu(z)\\
&\quad + \theta_\delta(t-s)\,D\bar\theta_\epsilon(x-y)\cdot \underbrace{\left(-b_r^{\mu^*} + \int_{r<|z|\le r_1} z\,\mathrm{d}\mu(z)\right)}_{=\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z)},
\end{aligned}\tag{5.11}$$

where the last equality comes from (2.3) and the change of variable $z \to -z$. We insert (5.11) into (5.10) and combine the resulting inequality with (5.9),

$$
\begin{aligned}
&I_3^+ + I_4^+ \leq \\
&\iint_{Q_T^2} |C_+(u) - C_+(v)| \\
&\quad \cdot \theta_\delta(t-s) \int_{r<|z|\leq r_1} \big(\bar\theta_\epsilon(x-y+z) - \bar\theta_\epsilon(x-y) - z \cdot D\bar\theta_\epsilon(x-y)\big)\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&+ \iint_{Q_T^2} |C_+(u) - C_+(v)| \\
&\qquad \cdot \theta_\delta(t-s)\, D\bar\theta_\epsilon(x-y) \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&+ \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu,r_1}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\,\mathrm{d}w \\
&=: J_1^+ + J_2^+ + J_3^+.
\end{aligned} \tag{5.12}
$$

Similar arguments show that we can bound $I_3^- + I_4^-$ (see (5.7)–(5.8)) as follows,

$$
\begin{aligned}
&I_3^- + I_4^- \leq \\
&\iint_{Q_T^2} |C_-(v) - C_-(u)| \\
&\quad \cdot \theta_\delta(t-s) \int_{r<|z|\leq r_1} \big(\bar\theta_\epsilon(x-y-z) - \bar\theta_\epsilon(x-y) + z \cdot D\bar\theta_\epsilon(x-y)\big)\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&- \iint_{Q_T^2} |C_-(v) - C_-(u)| \\
&\qquad \cdot \theta_\delta(t-s)\, D\bar\theta_\epsilon(x-y) \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&+ \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{\mu,r_1}[C_-(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\,\mathrm{d}w}_{\leq \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{\mu,r_1}[C_-(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\,\mathrm{d}w \text{ by Lemma 4.4}} \\
&=: J_1^- + J_2^- + J_3^-.
\end{aligned} \tag{5.13}
$$

5. *$L^1 \cap BV$-regularity.* It remains to estimate $J_i^\pm$ for $i = 1, \ldots, 3$ in (5.12)–(5.13). For $J_1^\pm$ and $J_2^\pm$, we integrate by parts to take advantage of the BV-regularity of u. After some technical computations detailed in Appendix A, we find that

$$
|J_1^\pm| \leq \frac{1}{2\epsilon} \int_{\mathbb{R}^d} |D\tilde\theta_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z)\, |C_\pm(u)|_{L^1(0,T;BV)}, \tag{5.14}
$$

$$
|J_2^\pm| \leq \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z) \right| |C_\pm(u)|_{L^1(0,T;BV)}, \tag{5.15}
$$

and hence

$$
\begin{aligned}
\sum_\pm (J_1^\pm + J_2^\pm) \leq{}& \frac{1}{2\epsilon} \int_{\mathbb{R}^d} |D\tilde\theta_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z) \sum_\pm |C_\pm(u)|_{L^1(0,T;BV)} \\
&+ \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z) \right| \sum_\pm |C_\pm(u)|_{L^1(0,T;BV)}.
\end{aligned}
$$

By Lemma 5.1 (iii) and a priori estimates for u, cf. (2.10), we see that

$$\sum_{\pm}(J_1^{\pm}+J_2^{\pm}) \leq \frac{1}{2\epsilon}\int_{\mathbb{R}^d}|D\tilde{\theta}_d|\mathrm{d}x\,\underbrace{|u|_{L^1(0,T;BV)}}_{\leq |u_0|_{BV(\mathbb{R}^d)}\,T}\int_{r<|z|\leq r_1}|z|^2\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R})}$$

$$+\underbrace{|u|_{L^1(0,T;BV)}}_{\leq |u_0|_{BV(\mathbb{R}^d)}\,T}\left|\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right|\,\|A'-B'\|_{L^\infty(\mathbb{R})}. \tag{5.16}$$

Let us now estimate J_3^+ in (5.12). Easy computations (see the proofs of Lemmas 4.4–4.5) show that

$$J_3^+ \leq \|\theta_\delta\,\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^{d+1})}\,\|\mathcal{L}^{\mu,r_1}[C_+(u)]\|_{L^1(Q_T)}.$$

Let us recall that $\|\theta_\delta\,\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^{d+1})}=\|\theta_\delta\|_{L^1(\mathbb{R})}\,\|\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^d)}=1$, and then

$$J_3^+ \leq \int_0^T\int_{|z|>r_1}\|C_+(u(\cdot+z,s))-C_+(u(\cdot,s))\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\mu(z)\mathrm{d}s.$$

Since $C_+(u)\in L^\infty\cap C([0,T];L^1)$, $(z,s)\to\|C_+(u(\cdot+z,s))-C_+(u(\cdot,s))\|_{L^1(\mathbb{R}^d)}$ is a continuous function, hence Borel and $\mathrm{d}\mu(z)\,\mathrm{d}s$-measurable. Thus, we may change the order of the integration to find

$$J_3^+ \leq \int_{|z|>r}\|C_+(u(\cdot+z,\cdot))-C_+(u)\|_{L^1(Q_T)}\,\mathrm{d}\mu(z).$$

We get a similar estimates for J_3^- and find by Lemma 5.1 (iii)–(iv) and (2.10) that

$$\begin{aligned}\sum_{\pm}J_3^{\pm} &\leq \int_{|z|>r_1}\sum_{\pm}\|C_\pm(u(\cdot+z,\cdot))-C_\pm(u)\|_{L^1(Q_T)}\,\mathrm{d}\mu(z),\\ &\leq \int_{|z|>r_1}\underbrace{\|u(\cdot+z,\cdot))-u\|_{L^1(Q_T)}}_{\leq T\,\|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}}\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}\tag{5.17}$$

The last inequality (under the bracket) comes from (2.11) applied to the solution $u(\cdot+z,\cdot)$ of (1.1) with initial data $u_0(\cdot+z)$.

6. *Conclusion.* By (5.7)–(5.8) and (5.12)–(5.13), $I_3+I_4\leq\sum_{\pm}\sum_{i=1}^3 J_i^{\pm}$. Therefore we may estimate (5.4) by (5.5)–(5.6) and (5.16)–(5.17). For all $r_1>r>0$, $\epsilon>0$, and $T>\delta>0$, we find that

$$\begin{aligned}&\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)}\\ &\leq \|u_0-v_0\|_{L^1(\mathbb{R}^d)}+|u_0|_{BV(\mathbb{R}^d)}\,T\,\|f'-g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)}\\ &\quad+\epsilon\,C_{\tilde{\theta}}\,|u_0|_{BV(\mathbb{R}^d)}+2(\omega_u(\delta)\vee\omega_v(\delta))+C_\epsilon\int_{0<|z|\leq r}|z|^2\,\mathrm{d}\mu(z)\\ &\quad+\frac{1}{2\epsilon}\int_{\mathbb{R}^d}|D\tilde{\theta}_d|\mathrm{d}x\,|u_0|_{BV(\mathbb{R}^d)}\,T\int_{r<|z|\leq r_1}|z|^2\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}\\ &\quad+|u_0|_{BV(\mathbb{R}^d)}\,T\left|\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1}z\,\mathrm{d}\mu(z)\right|\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}\\ &\quad+T\int_{|z|>r_1}\|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)},\end{aligned}\tag{5.18}$$

where $C_\epsilon>0$ does not depend on $r>0$.

To finish, we first pass to the limit as $r\to 0$ in (5.18). By the dominated convergence theorem, the result is equivalent to setting $r=0$ in each term, and in particular the term $C_\epsilon\int_{0<|z|\leq r}|z|^2\,\mathrm{d}\mu(z)$ vanishes. Secondly, we pass to the limit as $\delta \to 0$ to get rid of the term $2(\omega_u(\delta) \vee \omega_v(\delta))$. Finally, we optimize the remaining terms w.r.t. $\epsilon > 0$ by using the formula $\min_{\epsilon>0}\left(\epsilon\, a + \frac{b}{\epsilon}\right) = 2\sqrt{ab}$ (for $a, b \geq 0$). This gives us the following continuous dependence estimate: For all $r_1 > 0$,

$$
\begin{aligned}
\|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|g' - f'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
&\quad + 2\sqrt{\frac{1}{2}\, C_{\tilde\theta} \int_{\mathbb{R}^d} |D\tilde\theta_d| \mathrm{d}x\, |u_0|^2_{BV(\mathbb{R}^d)}\, T \int_{0<|z|\leq r_1} |z|^2\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})}} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)}\, T \left| \int_{r_1\wedge 1<|z|\leq r_1 \vee 1} z\, \mathrm{d}\mu(z) \right| \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&\quad + T \int_{|z|\geq r_1} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)}\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})},
\end{aligned}
\tag{5.19}
$$

where $\tilde\theta_d$ is an arbitrary approximate unit (3.2) and $C_{\tilde\theta} = 2\int_{\mathbb{R}^d} |x|\, \tilde\theta_d(x)\, \mathrm{d}x$ by Lemma 3.1.

Let $\tilde\theta_d = \theta_n$ where $\{\theta_n\}_{n\in\mathbb{N}}$ is a sequence of kernels s.t. θ_n satisfies (3.2), $\theta_n \to {\omega_d}^{-1}\mathbf{1}_{|\cdot|<1}$ in L^1, and $\int_{\mathbb{R}^d} |D\theta_n|\, \mathrm{d}x \to {\omega_d}^{-1}|\mathbf{1}_{|\cdot|<1}|_{BV(\mathbb{R}^d)}$. Here ω_d is the volume of the unit ball in $\mathbb{R}^d$. Note that the BV-semi-norm of the indicator function of the unit ball is equal to the surface area of the unit sphere, i.e. $|\mathbf{1}_{|\cdot|<1}|_{BV(\mathbb{R}^d)} = d\omega_d$. Moreover, we have

$$
\int_{\mathbb{R}^d} |x||\theta_n(x)|\, \mathrm{d}x \to \frac{1}{\omega_d} \int_{|x|<1} |x|\, \mathrm{d}x = \frac{d}{d+1}.
$$

The proof of (3.4) is then complete after passing to the limit as $n \to +\infty$ in (5.19). □

5.3. **Proof of Theorem 3.4.** We argue step by step as in the proof of Theorem 3.3. This time, $E_\pm$ are taken such as

$$
\begin{cases}
E_\pm \subseteq \mathbb{R}^d \setminus \{0\} \text{ are Borel sets;} \\
\cup_\pm E^\pm = \mathbb{R}^d \setminus \{0\} \text{ and } \cap_\pm E_\pm = \emptyset; \\
\big(\mathbb{R}^d \setminus \{0\}\big) \setminus \operatorname{supp}(\mu - \nu)^\mp \subseteq E_\pm.
\end{cases}
\tag{5.20}
$$

Let $\mu_\pm$ and $\nu_\pm$ denote the restrictions of μ and ν to $E_\pm$. It is clear that

$$
\begin{cases}
\mu = \sum_\pm \mu_\pm \quad \text{and} \quad \nu = \sum_\pm \nu_\pm, \\
\pm(\mu_\pm - \nu_\pm) = (\mu - \nu)^\pm, \\
\mu_\pm, \nu_\pm, \text{ and } \pm(\mu_\pm - \nu_\pm) \text{ all satisfy (1.6).}
\end{cases}
\tag{5.21}
$$

Proof of Theorem 3.4.

1. We apply Lemma 3.1 with $A = B$, but different Lévy measures μ and ν, along with the entropy inequality for v to show that for all $r, \epsilon > 0$, $0 < \delta < T$

$$
\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde{\theta}}\, |u_0|_{BV(\mathbb{R}^d)} + 2(\omega_u(\delta) \vee \omega_v(\delta)) \\
&\quad + \iint_{Q_T^2} (q_g - q_f)(v,u) \cdot D_x \phi^{\epsilon,\delta} \, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}_r^{\nu^*}[\phi^{\epsilon,\delta}(\cdot, y, t, s)](x) \, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x, t, \cdot, s)](y) \, \mathrm{d}w \\
&\quad + \underbrace{\iint_{Q_T^2} |A(v) - A(u)| \left(b_r^{\nu^*} - b_r^{\mu^*}\right) \cdot D_x \phi^{\epsilon,\delta} \, \mathrm{d}w}_{=: I_3} \\
&\quad + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v - u) \left(\mathcal{L}^{\nu,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\right) \phi^{\epsilon,\delta} \, \mathrm{d}w}_{=: I_4},
\end{aligned}
\tag{5.22}
$$

where $C_\epsilon > 0$ does not depend on $r > 0$. Except for I_3 and I_4, the other terms were estimated in the proof of Theorem 3.3.

2. *Cutting w.r.t. $E_\pm$.* We use the notation introduced in (5.20). We apply Lemma 4.4 twice with ν_+ and μ_- instead of μ, along with linearity of $\mathcal{L}^{\mu,r}$ in μ, see (2.2), to see that

$$
\begin{aligned}
I_4 &= \sum_{\pm} \iint_{Q_T^2} \operatorname{sgn}(v - u) \left(\mathcal{L}^{\nu_\pm,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu_\pm,r}[A(u(\cdot,s))](y)\right) \phi^{\epsilon,\delta} \, \mathrm{d}w \\
&\leq \iint_{Q_T^2} \operatorname{sgn}(v - u) \left(\mathcal{L}^{\nu_+,r}[A(u(\cdot,s))](y) - \mathcal{L}^{\mu_+,r}[A(u(\cdot,s))](y)\right) \phi^{\epsilon,\delta} \, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v - u) \left(\mathcal{L}^{\nu_-,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu_-,r}[A(v(\cdot,t))](x)\right) \phi^{\epsilon,\delta} \, \mathrm{d}w \\
&= \iint_{Q_T^2} \operatorname{sgn}(u - v)\, \mathcal{L}^{\mu_+ - \nu_+,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta} \, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v - u)\, \mathcal{L}^{-(\mu_- - \nu_-),r}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta} \, \mathrm{d}w \\
&=: I_4^+ + I_4^- .
\end{aligned}
$$

Again, it is crucial to have u in I_4^+ and v in I_4^- in order to use Kato's inequality later on.

Let us now consider I_3. By (2.3) and (2.8), b_r^μ and μ^* are linear w.r.t μ. Easy computations using (5.21) then leads to

$$
\left(b_r^{\nu^*} - b_r^{\mu^*}\right) \cdot D_x \phi^{\epsilon,\delta} = \sum_{\pm} b_r^{\pm(\mu_\pm - \nu_\pm)^*} \cdot D_\pm \phi^{\epsilon,\delta}
$$

where $D_+ = D_y$ and $D_- = D_x$, and hence

$$
I_3 = \sum_{\pm} \iint_{Q_T} |A(u) - A(v)|\, b_r^{\pm(\mu_\pm - \nu_\pm)^*} \cdot D_\pm \phi^{\epsilon,\delta} \, \mathrm{d}w =: I_3^+ + I_3^- .
$$

3. *Cutting w.r.t. z.* The computations of this step are similar to the ones in the proof of Theorem 3.3. For the reader's convenience, we estimate $I_3^- + I_4^-$, the terms that was left to the reader in the preceding proof.

For any measure $\tilde{\mu}$ we let $\tilde{\mu}_1 = \tilde{\mu}_{|0<|z|\leq r_1}$ and write $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_{||z|>r_1}$ for $r_1 > r$. Then

$$I_4^- \leq \underbrace{\iint_{Q_T} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-)_1, r}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w}_{=:I_5^-} + \iint_{Q_T} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-), r_1}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w.$$

Recall that $-(\mu_- - \nu_-)_1$ is a positive Lévy measure by (5.21), so we can apply Lemma 4.3 with $-(\mu_- - \nu_-)_1$ instead of μ and $k = u(y,s)$ to find that

$$I_5^- \leq \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x)\, \mathrm{d}w$$

and

$$I_3^- + I_5^- \leq \iint_{Q_T^2} |A(v) - A(u)| \left(b_r^{-(\mu_- - \nu_-)^*} \cdot D_x \phi^{\epsilon,\delta} + \mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \right) \mathrm{d}w.$$

Easy computations then leads to

$$\begin{aligned} &\mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \\ &= \theta_\delta(t-s) \int_{r<|z|\leq r_1} \left(\bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y) \right) \mathrm{d}(\nu_- - \mu_-)(z), \end{aligned}$$

and we can rewrite the nonlocal operator as follows,

$$\begin{aligned} &b_r^{-(\mu_- - \nu_-)^*} \cdot D_x \phi^{\epsilon,\delta} + \mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \\ &= \theta_\delta(t-s) \int_{r<|z|\leq r_1} \left(\bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y) + z \cdot D\bar{\theta}_\epsilon(x-y) \right) \mathrm{d}(\nu_- - \mu_-)(z) \\ &\quad - \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \cdot \underbrace{\left(-b_r^{-(\mu_- - \nu_-)^*} + \int_{r<|z|\leq r_1} z\, \mathrm{d}(\nu_- - \mu_-)(z) \right)}_{=\operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1 \vee r) < |z| \leq r_1 \vee 1} z\, \mathrm{d}(\nu_- - \mu_-)(z)}. \end{aligned}$$

Compare this expression with (5.11) that appear when I_3^+ and I_4^+ are considered.

We add the different estimates and find that for all $r_1 > r$,

$$
\begin{aligned}
& I_3^- + I_4^- \\
& \leq \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s) \\
& \qquad \cdot \int_{r<|z|\leq r_1} \left(\bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y) + z \cdot D\bar{\theta}_\epsilon(x-y)\right) \mathrm{d}(\nu_- - \mu_-)(z)\, \mathrm{d}w \\
& \quad - \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \\
& \qquad \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r) < |z| \leq r_1 \vee 1} z\, \mathrm{d}(\nu_- - \mu_-)(z)\, \mathrm{d}w \\
& \quad + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-), r_1}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w}_{\leq \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-), r_1}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \text{ by Lemma 4.4}} \\
& = J_1^- + J_2^- + J_3^-.
\end{aligned}
$$

Similar arguments also lead to

$$
\begin{aligned}
& I_3^+ + I_4^+ \\
& \leq \iint_{Q_T^2} |A(u) - A(v)| \theta_\delta(t-s) \\
& \qquad \cdot \int_{r<|z|\leq r_1} \left(\bar{\theta}_\epsilon(x-y+z) - \bar{\theta}_\epsilon(x-y) - z \cdot D\bar{\theta}_\epsilon(x-y)\right) \mathrm{d}(\mu_+ - \nu_+)(z)\, \mathrm{d}w \\
& \quad + \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \\
& \qquad \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r) < |z| \leq r_1 \vee 1} z\, \mathrm{d}(\mu_+ - \nu_+)(z)\, \mathrm{d}w \\
& \quad + \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{(\mu_+ - \nu_+), r_1}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w, \\
& =: J_1^+ + J_2^+ + J_3^+.
\end{aligned}
$$

4. *$L^1 \cap BV$-regularity.* We estimate $J_i^\pm$ ($i = 1, \ldots, 3$). Almost all the computations have already been done in the preceding proof. Indeed, $J_1^\pm$ are of the same form as in (5.12)–(5.13), with the new nonlinearity A and the new measures $\pm(\mu_\pm - \nu_\pm)$. Arguing as for (5.14) thus gives

$$
\begin{aligned}
\sum_\pm J_1^\pm \leq & \frac{1}{2\epsilon} \int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x \\
& \cdot \underbrace{|A(u)|_{L^1(0,T;BV)}}_{\leq |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \text{ by (2.10)}} \int_{r<|z|\leq r_1} |z|^2\, \mathrm{d} \underbrace{\sum_\pm \pm(\mu_\pm - \nu_\pm)}_{=|\mu-\nu| \text{ by (5.21)}}(z).
\end{aligned}
$$

Moreover, $\sum_\pm (\mu_\pm - \nu_\pm) = \mu - \nu$ and hence

$$
\begin{aligned}
\sum_\pm J_2^\pm = & \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \\
& \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r) < |z| \leq r_1 \vee 1} z\, \mathrm{d}(\mu - \nu)(z)\, \mathrm{d}w.
\end{aligned}
$$

This term is of the same form than J_2^+ in (5.12) (or J_2^- in (5.13)) with the new non-linearity A and the new flux $\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}(\mu-\nu)(z)$. Arguing as for (5.15) and using (2.10) thus give

$$\sum_{\pm} J_2^{\pm} \leq |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}(\mu-\nu)(z) \right|.$$

Finally,

$$\sum_{\pm} J_3^{\pm} \leq T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{|z|\geq r_1} \|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}|\mu-\nu|(z).$$

5. *Conclusion.* The rest of the proof is the same as for Theorem 3.3; i.e. we use the estimates on $J_i^{\pm}$ to estimate $I_3+I_4\leq \sum_{i=1}^3\sum_{\pm} J_i^{\pm}$ in (5.22) and pass to limit and/or optimizes w.r.t. the parameters $r,\epsilon,\delta>0$. The proof is complete. □

Appendix A. Technical computations

Proof of (5.14) *and* (5.15). We start by proving (5.14) in the + case. Similar arguments give the proof also in the − case. From Taylor's formula

$$\bar\theta_\epsilon(x-y+z)-\bar\theta_\epsilon(x-y)-z\cdot D\bar\theta_\epsilon(x-y) \;=\int_0^1 (1-\tau)\,D^2\bar\theta_\epsilon(x-y+\tau\, z)\,z\cdot z\,\mathrm{d}\tau,$$

and hence by Fubini's theorem, J_1^+ in (5.12) can be written as

$$\text{(A.1)}\quad J_1^+ = \iint_{(0,T)^2}\int_{r<|z|\leq r_1}\int_0^1 \theta_\delta(t-s)\,(1-\tau)$$
$$\cdot\underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |C_+(v(x,t))-C_+(u(y,s))|\,D^2\bar\theta_\epsilon(x-y+\tau\, z)\,z\cdot z\,\mathrm{d}y\,\mathrm{d}x}_{=:J}\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s.$$

For any $k\in\mathbb{R}$, it is classical that $\eta_k(C_+(u(\cdot,s)))=|k-C_+(u(\cdot,s))|\in BV$ with

$$|D\eta_k(C_+(u(\cdot,s))|\leq |DC_+(u(\cdot,s))|,$$

since it is the composition of a Lipschitz-continuous function with a BV-function; see e.g. [13, 35, 57]. Integration by parts w.r.t. y (for fixed z,x,t,s), then leads to

$$\begin{aligned}|J| &= \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D\bar\theta_\epsilon(x-y+\tau\, z)\cdot z\, z\cdot \mathrm{d}D\eta_{C_+(v(x,t))}(C_+(u(\cdot,s)))(y)\,\mathrm{d}x\right|,\\ &\leq |z|^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |D\bar\theta_\epsilon(x-y+\tau\, z)|\,\mathrm{d}|DC_+(u(\cdot,s))|(y)\,\mathrm{d}x.\end{aligned}$$

Note that by the definition of $\bar\theta_\epsilon$ (just below (3.1)), we have

$$\int_{\mathbb{R}^d}|D\bar\theta_\epsilon(x)|\,\mathrm{d}x=\frac{1}{\epsilon}\int_{\mathbb{R}^d}|D\tilde\theta_d|\mathrm{d}x.$$

Hence, we change the order of integration (using Fubini) to see that

$$|J|\leq |z|^2\,|C_+(u(s))|_{BV(\mathbb{R}^d)}\int_{\mathbb{R}^d}|D\bar\theta_\epsilon(x)|\,\mathrm{d}x\leq |z|^2\,|C_+(u(s))|_{BV(\mathbb{R}^d)}\frac{1}{\epsilon}\int_{\mathbb{R}^d}|D\tilde\theta_d|\mathrm{d}x,$$

and then from (A.1) that

$$|J_1^+|\leq\frac{1}{\epsilon}\int_{\mathbb{R}^d}|D\tilde\theta_d|\mathrm{d}x$$
$$\cdot\iint_{(0,T)^2}\int_{r<|z|\leq r_1}\int_0^1 \theta_\delta(t-s)\,(1-\tau)\,|z|^2\,|C_+(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s.$$

Let us recall that the integrand above is $\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s$-measurable since $s \to |u(s)|_{BV(\mathbb{R}^d)}$ is lower semi-continuous. By Fubini we then integrate first w.r.t. t and use that $\int_0^T \theta_\delta(t-s)\,\mathrm{d}t \leq 1$ to see that

$$|J_1^+| \leq \frac{1}{\epsilon} \int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x \int_0^1 (1-\tau)\,\mathrm{d}\tau \int_{r<|z|\leq r_1} |z|^2 \mathrm{d}\mu(z) \int_0^T |C_+(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}s,$$

and the proof of (5.14) is complete.

We prove (5.15) by similar arguments. Define

$$q(v,u) := |v-u| \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r)<|z|\leq r_1 \vee 1} z\,\mathrm{d}\mu(z), \tag{A.2}$$

and note that it is Lipschitz-continuous. Again by Fubini's theorem, J_2^+ in (5.12) can be written as

$$J_2^+ = \iint_{(0,T)^2} \theta_\delta(t-s) \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D\bar{\theta}_\epsilon(x-y) \cdot q(C_+(v(x,t), C_+(u(y,s)))\,\mathrm{d}y\,\mathrm{d}x}_{=:J}\,\mathrm{d}t\,\mathrm{d}s. \tag{A.3}$$

For fixed (x,t,s), $q(C_+(v(x,t),\cdot)$ is Lipschitz-continuous and $C_+(u(\cdot,s))$ is BV; hence, the composition $q(C_+(v(x,t)), C_+(u(\cdot,s)))$ is in $BV(\mathbb{R}^d,\mathbb{R}^d)$ with

$$|\mathrm{div}_y q(C_+(v(x,t)), C_+(u(\cdot,s)))| \leq |DC_+(u(\cdot,s))|\,\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)},$$

where $\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)}$ denotes the Lipschitz constant of q w.r.t. its second variable. We thus may integrate by parts in y to see that

$$|J| \leq \|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bar{\theta}_\epsilon(x-y)\,\mathrm{d}|DC_+(u(\cdot,s))|(y)\,\mathrm{d}x.$$

Changing the order of integration, we find that

$$J \leq |C_+(u(s))|_{BV(\mathbb{R}^d)} \|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)},$$

and hence by (A.3) and integrating first w.r.t. t, we get that

$$|J_2^+| \leq \|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \int_0^T |C_+(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}s.$$

The proof of (5.15) is now complete since by (A.2),

$$\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} = \left| \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z) \right|.$$

□

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