OPTIMAL CONTINUOUS DEPENDENCE ESTIMATES FOR FRACTIONAL DEGENERATE PARABOLIC EQUATIONS

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Abstract. We derive continuous dependence estimates for weak entropy solutions of degenerate parabolic equations with nonlinear fractional diffusion. The diffusion term involves the fractional Laplace operator, $\Delta^{\alpha/2}$ for $\alpha \in (0,2)$. Our results are quantitative and we exhibit an example for which they are optimal. We cover the dependence on the nonlinearities, and for the first time, the Lipschitz dependence on α in the BV-framework. The former estimate (dependence on nonlinearity) is robust in the sense that it is stable in the limits $\alpha \downarrow 0$ and $\alpha \uparrow 2$. In the limit $\alpha \uparrow 2$, $\Delta^{\alpha/2}$ converges to the usual Laplacian, and we show rigorously that we recover the optimal continuous dependence result of [24] for local degenerate parabolic equations (thus providing an alternative proof).

Contents

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1. Introduction

In this paper we consider the following Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{div} f(u) + (-\triangle)^{\frac{\alpha}{2}} \varphi(u) = 0 & \text{in} \quad Q_T := \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, $u = u(x,t)$ is the unknown function, div and $\triangle$ denote divergence and Laplacian with respect to x, and $(-\triangle)^{\frac{\alpha}{2}}$, $\alpha \in (0,2)$, is the fractional Laplacian e.g. defined as

$$(-\triangle)^{\frac{\alpha}{2}} \phi := \mathcal{F}^{-1} \left(|2\,\pi \cdot|^{\alpha} \mathcal{F} \phi \right) \tag{1.2}$$

with the Fourier transform $\mathcal{F}\phi(\xi) := \int_{\mathbb{R}^d} e^{-2\,i\,\pi\,x\cdot\xi} \, \phi(x) \, \mathrm{d}x$. Notice that (1.2) is compatible with the formula $-\triangle\phi = \mathcal{F}^{-1}\left(|2\,\pi \cdot|^2 \mathcal{F}\phi\right)$. Throughout the paper we assume that

$$u_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d), \tag{1.3}$$

$$f \in \left(W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \right)^d \text{ with } f(0) = 0, \tag{1.4}$$

$$\varphi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \text{ is nondecreasing with } \varphi(0) = 0. \tag{1.5}$$

Remark 1.1. Subtracting constants from f and φ if necessary, there is no loss of generality in assuming that $f(0) = 0$ and $\varphi(0) = 0$.

The fractional Laplacian is the generator of the symmetric α-stable process, the most famous pure jump Lévy process. There is a large literature on Lévy processes, we refer to e.g. [52] for more details, and they are important in many modern applications. Being very selective, we mention radiation hydrodynamics [51, 54, 50], anomalous diffusion in semiconductor growth [56], over-driven gas detonations [23], mathematical finance [26], and flow in porous media [29, 30].

Due to the second part of assumption (1.5), the term $(-\triangle)^{\frac{\alpha}{2}} \varphi(u)$ is a nonlinear and nonlocal diffusion term. It formally converges toward $\varphi(u)$ and $-\triangle\varphi(u)$ as $\alpha \downarrow 0$ and $\alpha \uparrow 2$ respectively. Hence, Equation (1.1) could be seen as a nonlocal "interpolation" between the hyperbolic equation

$$\partial_t u + \mathrm{div} f(u) + \varphi(u) = 0, \tag{1.6}$$

and the degenerate parabolic equation

$$\partial_t u + \mathrm{div} f(u) - \triangle\varphi(u) = 0. \tag{1.7}$$

Equation (1.1) is said to be supercritical if $\alpha < 1$, subcritical if $\alpha > 1$, and critical if $\alpha = 1$. The diffusion function φ is said to be strongly degenerate if φ' vanishes on a nontrivial interval. Equation (1.1) can therefore be of mixed hyperbolic parabolic type depending on the choice of α and φ. Note that in the mathematical community, interest in nonlinear nonlocal diffusions is in fact very recent, and only few results exist; cf. e.g. [9, 10, 15, 21, 29, 4, 30, 31] and the references therein.

Let us give the main references for the well-posedness of the Cauchy problems for (1.6) and (1.7). For a more complete bibliography, see the books [32, 28, 55] and the references in [40]. In the hyperbolic case where $\varphi' \equiv 0$, we get the scalar conservation law $\partial_t u + \mathrm{div} f(u) = 0$. The solutions of this equation could develop discontinuities in finite time and the weak solutions of the Cauchy problem are generally not unique. The most famous uniqueness result relies on the notion of entropy solutions introduced in [44]. In the pure diffusive case where $f' \equiv 0$, there is no more creation of shock and the initial-value problem for $\partial_t u - \triangle\varphi(u) = 0$ admits a unique weak solution, cf. [12]. Much later, the adequate notion of entropy solutions for mixed hyperbolic parabolic equations was introduced in [16]. This paper focuses on an initial-boundary value problem. For a general well-posedness result applying to both (1.6) and (1.7), see e.g. [40].

At the same time, there has been a large interest in nonlocal versions of these equations. The first work seems to be [25] on nonlocal time fractional derivatives, cf. also [39]. The study of nonlocal diffusion terms has probably been initiated by [8]. Now, the well-posedness is quite well-understood in the nondegenerate linear case where $\varphi(u) = u$. Smooth solutions exist and are unique for subcritical equations [8, 34], shocks could occur [5, 43] and weak solutions could be nonunique [3] for supercritical equations, entropy solutions exist and are always unique [2, 41]; cf. also [17, 18] for original regularizing effects. Very recently, the well-posedness theory of entropy solutions was extended in [21] to cover the full problem (1.1), even for strongly degenerate φ. See also [29, 30] on fractional porous medium type equations, and [31] on a logarithmic diffusion equation.

This paper is devoted to continuous dependence estimates for (1.1), i.e. explicit estimates on the difference of two entropy solutions u and v in terms of the difference of their respective data $(\alpha, u_0, f, \varphi)$ and (β, v_0, g, ψ). Let us point out that we investigate quantitative results which should be distinguished from qualitative ones. By qualitative, we mean stability results only stating that if $(\alpha_n, u_0^n, f_n, \varphi_n)$ converges toward $(\alpha, u_0, f, \varphi)$, then the associated entropy solutions u_n converge toward u. For scalar conservation laws, the first quantitative result on the continuous dependence on f appeared in [27] and also in [48] some years later. Roughly speaking, it states that for BV initial data $u_0 = v_0$,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} = O\left(\|f' - g'\|_\infty\right), \tag{1.8}$$

where throughout the L^∞-norm is always taken over the range of u_0. Next, the optimal error in $\sqrt{\epsilon}$ for the parabolic regularization $\partial_t u^\epsilon + \mathrm{div} f(u^\epsilon) - \epsilon \triangle u^\epsilon = 0$ of scalar conservation laws was established in [45]. In that paper, the author has developed a general method of error estimation based on the Kruzhkov's device of doubling the variables [44]. We use this method in the present paper. As far as degenerate parabolic equations are concerned, the continuous dependence on φ was first investigated in [7] for the equation $\partial_t u - \triangle \varphi(u) = 0$. Here the motivation was to obtain qualitative results under very general assumptions. Quantitative results were obtained in [11, 24] for the full equation (1.7). In [11], the authors established alternative estimates to (1.8) involving weaker norms, as roughly speaking an estimate in $\|f - g\|_\infty^{\frac{1}{2}}$. They gave different estimates for the φ-dependence with $\psi \equiv 0$. An estimate for nontrivial ψ was given in [24]. Roughly speaking, it states that if u has the same data as v except for $\varphi \neq \psi$, then

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1} = O\left(\|\sqrt{\varphi'} - \sqrt{\psi'}\|_\infty\right). \tag{1.9}$$

Recently, Estimates (1.8) and (1.9) were extended in [40, 19, 20] to anisotropic diffusions and (x, t)-dependent data; cf. also [6, 49] for recent qualitative results on local equations. For nonlocal equations, a number of papers were concerned with convergence rates for vanishing viscosity methods [54, 25, 33, 35, 2]. To the best of our knowledge, the first estimate on the "general continuous dependence on the data" was given in [41]. It concerns the case of linear nondegenerate Lévy diffusions. The main novelty was the explicit dependence in the Lévy measure, corresponding to the explicit dependence in α for the particular case of the fractional Laplacian. In [4], the authors of the present paper established a continuous dependence estimates for general nonlinear degenerate Lévy diffusions. For a qualitative result in the spirit of [7], see the very recent work [30] on the fractional porous medium equation $\partial_t u + (-\triangle)^{\alpha/2}(|u|^{m-1} u) = 0$, $m > 0$. In that paper, the continuous dependence on (α, m, u_0) is established under more general assumptions.

Before explaining our main contributions, let us refer the reader to more or less related work. The theory of continuous dependence estimates for nonlocal equations was probably initiated in the context of viscosity solutions of fully nonlinear integro-PDEs, cf. [38] and the references therein. See also [37, 35] for error estimates for vanishing viscosity methods. The question of α-continuity has been raised earlier, e.g. when looking for a priori estimates that are robust or uniform as $\alpha \uparrow 2$. Such results can be found in e.g. [13, 14], see also [42] and the references therein.

The starting point of the present paper is the general theory of [4]. It is worth mentioning that different estimates could be difficult to compare, as e.g (1.8) with the estimate in $\|f-g\|_\infty^{\frac{1}{2}}$ of [11]. Hence, a remarkable feature is that the estimates in (1.8) and (1.9) are optimal for linear equations, cf. the discussion of Section 8. A natural question is whether the estimates of [4] applied to (1.1) possess such a property. The answer is positive only in the supercritical case $\alpha < 1$. In this paper, we obtain optimal estimates for all cases. To do so we restart the proofs from the beginning, by taking into account the homogeneity properties of the fractional Laplacian. The main ingredients are a new linearization argument a la Young measure theory/kinetic formulations, and for the linear case, a clever change of the (jump) z-variable in (2.1). This change of variable allows us to adapt ideas from viscosity solution theory developed in e.g. [38]. Let us also refer the reader to [53] for other applications of this change of variable in the context of viscosity solutions. Roughly speaking, we prove that

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1} = \begin{cases} O\left(\|(\varphi')^{\frac{1}{\alpha}}-(\psi')^{\frac{1}{\alpha}}\|_\infty\right), & \alpha > 1, \\ O\left(\|\varphi' \ln\varphi' - \psi' \ln\psi'\|_\infty\right), & \alpha = 1, \\ O\left(\|\varphi'-\psi'\|_\infty\right), & \alpha < 1, \end{cases} \tag{1.10}$$

with uniform constants in the limits $\alpha \downarrow 0$ and $\alpha \uparrow 2$. Note well that just as in [4], our proofs work directly with the entropy solutions without needing tools like entropy defect measures, etc.. And even though these tools play a key role in the local second-order theory, the arguments here really seem to be less technical relying only on basic convex inequalities and integral calculus. Hence, it seems interesting to mention that we recover the result (1.9) rigorously from (1.10) by passing to the limit. Another remarkable feature is that a simple rescaling transforms the Kuznetsov type estimate (1.10) into the following time continuity estimate:

$$\|u(\cdot,t)-u(\cdot,s)\|_{L^1} = \begin{cases} O\left(|t^{\frac{1}{\alpha}}-s^{\frac{1}{\alpha}}|\right), & \alpha > 1, \\ O\left(|t \ln t - s \ln s|\right), & \alpha = 1, \\ O\left(|t-s|\right), & \alpha < 1. \end{cases}$$

This result is optimal and strictly better than earlier results in [22], see Remark 3.7. E.g. for positive times, we get Lipschitz regularity in time with values in $L^1(\mathbb{R}^d)$. This is a regularizing effect in time when $\alpha \geq 1$ and u not more than BV initially.

In the second main contribution of this paper, we focus on the continuous dependence on α. By stability arguments, it is possible to show that the unique entropy solution $u =: u^\alpha$ is continuous in $\alpha \in [0,2]$ with values in L^1_{loc}. In this paper, we prove that in the BV-framework, it is in fact locally Lipschitz continuous in $\alpha \in (0,2)$ with values in $C([0,T];L^1)$. To the best of our knowledge, such an α-regularity result has never been obtained before. More precisely, the theory of [4] implies the result for $\alpha \in (0,1)$ but not for $\alpha \in [1,2)$. For the latter range of exponents, all the results cited above are either qualitative or suboptimal. The new ingredient to get the Lipschitz regularity is again a change of (the jump) variable. It seems interesting to recall that the type of Equation (1.1) could change from parabolic when $\alpha > 1$ to hyperbolic when $\alpha < 1$. As a consequence, quite different behaviors are observed in the φ- and t-continuity when α crosses 1, cf. the continuous dependence estimates above. A natural question is thus whether such kind of phenomena arises in the α-regularity? We prove that the answer is positive by carefully estimating the best Lipschitz constant of the function $\alpha \mapsto u^\alpha$ with respect to the position of α and the other data. More precisely, for $\lambda \in (0,2)$ we define

$$\operatorname{Lip}_\alpha(u;\lambda) := \limsup_{\alpha,\beta\to\lambda} \frac{\|u^\alpha - u^\beta\|_{C([0,T];L^1)}}{|\alpha-\beta|},$$

and roughly speaking we prove that

$$\operatorname{Lip}_\alpha(u;\lambda) = \begin{cases} O\left(M^{\frac{1}{\lambda}} \,|\ln M|\right), & \lambda > 1,\\ O\left(M \ln^2 M\right), & \lambda = 1,\\ O\left(M\right), & \lambda < 1,\end{cases}$$

for $M := T\,\|\varphi'\|_\infty$, and

$$\operatorname{Lip}_\alpha(u;\lambda) = \begin{cases} O\left(|u_0|_{BV}\right), & \lambda > 1,\\ O\left(|u_0|_{BV} \ln^2 \frac{\|u_0\|_{L^1}}{|u_0|_{BV}}\right), & \lambda = 1,\\ O\left(\|u_0\|_{L^1}^{1-\lambda}\, |u_0|_{BV}^{\lambda} \left|\ln \frac{\|u_0\|_{L^1}}{|u_0|_{BV}}\right|\right), & \lambda < 1.\end{cases}$$

We also exhibit an example of an equation for which these estimates are optimal in the regimes where M is sufficiently small or $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}}$ is sufficiently large.

Another natural question is whether $\alpha \mapsto u^\alpha$ is Lipschitz continuous up to the boundaries $\alpha = 0$ and $\alpha = 2$. The answer is negative for $\alpha = 0$ and remains open for $\alpha = 2$. For the reader's convenience, more details and open questions are given at the end of Section 3.

To conclude, note that even if we adapt some ideas from viscosity solution theory, the definition of relevant generalized solution and the mathematical arguments are very different from the ones in e.g. [38]. Moreover we obtain optimal results here, and, in an a work in progress, we adapt ideas of this paper to obtain new results in the viscosity solution setting.

The rest of the paper is organized as follows. In Section 2, we recall the well-posedness theory for fractional degenerate parabolic equations. In Section 3, we state our main results: continuous dependence with respect to the nonlinearities and the order of the fractional Laplacian. In Section 4, we recall the general continuous dependence estimates of [4] along with a general Kuznetsov type of Lemma. Sections 5–7 are devoted to the proofs of our main results. In Section 8, we exhibit an example of an equation for which we rigorously show that our estimates are optimal. Finally, there is an appendix containing technical lemmas and computations from the different proofs.

Notation. The symbols ∇ and ∇^2 denote the x-gradient and x-Hessian. The symbols $\|\cdot\|$ and $|\cdot|$ are used for norms and semi-norms, respectively. The symbol $\sim$ is used for asymptotic equality "up to a constant." The symbols $\wedge$ and $\vee$ are used for the minimum and maximum between two reals. For any $a, b \in \mathbb{R}$, we use the shorthand notation $\operatorname{co}\{a,b\}$ to design the interval $(a \wedge b, a \vee b)$. The surface measure of the unit sphere of $\mathbb{R}^d$ is denoted by S_d.

2. Preliminaries

In this section we recall some basic facts on the fractional Laplacian and fractional degenerate parabolic equations. We start by a Lévy–Khinchine type representation formula. For $\alpha \in (0,2)$ and all $\phi \in C_c^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $r > 0$,

$$
\begin{aligned}
-(-\triangle)^{\frac{\alpha}{2}}\phi(x) &= G_d(\alpha)\int_{|z|<r} \frac{\phi(x+z)-\phi(x)-\nabla\phi(x)\cdot z}{|z|^{d+\alpha}}\,\mathrm{d}z \\
&\quad + G_d(\alpha)\int_{|z|>r} \frac{\phi(x+z)-\phi(x)}{|z|^{d+\alpha}}\,\mathrm{d}z \\
&=: \mathcal{L}_r^\alpha[\phi](x) + \mathcal{L}^{\alpha,r}[\phi](x),
\end{aligned} \tag{2.1}
$$

where

$$
G_d(\alpha) := \frac{2^{\alpha-1}\,\alpha\,\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{2-\alpha}{2}\right)}.
$$

The result is standard, see e.g. [46, 37, 35] and the references therein. Here are some properties on the coefficient that will be needed later:

$$
\begin{cases} G_d(\alpha) > 0 \text{ is smooth (and analytic) with respect to } \alpha \in (0,2); \\ \lim_{\alpha\downarrow 0} \frac{S_d\, G_d(\alpha)}{\alpha} = 1 \text{ and } \lim_{\alpha\uparrow 2} \frac{S_d\, G_d(\alpha)}{d\,(2-\alpha)} = 1, \end{cases} \tag{2.2}
$$

where S_d is the surface measure of the unit sphere of $\mathbb{R}^d$.

We then proceed to define entropy solutions of (1.1). For each $k \in \mathbb{R}$, we consider the Kruzhkov [44] entropy $u \mapsto |u-k|$ and entropy flux

$$
u \mapsto q_f(u,k) := \operatorname{sgn}(u-k)\,(f(u)-f(k)),
$$

where throughout this paper we always consider the following everywhere representation of the sign function:

$$
\operatorname{sgn}(u) := \begin{cases} \pm 1 & \text{if } \pm u > 0, \\ 0 & \text{if } u = 0. \end{cases} \tag{2.3}
$$

By monotonicity (1.5) of φ,

$$
\operatorname{sgn}(u-k)\,(\varphi(u)-\varphi(k)) = |\varphi(u)-\varphi(k)|, \tag{2.4}
$$

and then we formally deduce from (2.1) that for any function $u = u(x,t)$,

$$
\operatorname{sgn}(u-k)\,(-(-\triangle)^{\frac{\alpha}{2}})\,\varphi(u) \le \mathcal{L}_r^\alpha[|\varphi(u)-\varphi(k)|] + \operatorname{sgn}(u-k)\,\mathcal{L}^{\alpha,r}[\varphi(u)].
$$

This Kato type inequality is the starting point of the entropy formulation from [21].

Definition 2.1 (Entropy solutions)**.** *Let $\alpha \in (0,2)$, $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, and* (1.4)–(1.5) *hold. We say that $u \in L^\infty(Q_T) \cap L^\infty\left(0,T;L^1\right)$ is an entropy solution of* (1.1) *provided that for all $k \in \mathbb{R}$, $r > 0$, and all nonnegative $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$,*

$$
\begin{aligned}
&\int_{Q_T} \Big(|u-k|\,\partial_t\phi + q_f(u,k)\cdot\nabla\phi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{Q_T} \Big(|\varphi(u)-\varphi(k)|\,\mathcal{L}_r^\alpha[\phi] + \operatorname{sgn}(u-k)\,\mathcal{L}^{\alpha,r}[\varphi(u)]\,\phi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\mathbb{R}^d} |u_0(x)-k|\,\phi(x,0)\,\mathrm{d}x \ge 0.
\end{aligned} \tag{2.5}
$$

Remark 2.1. Under our assumptions, the entropy solutions are continuous in time with values in $L^1(\mathbb{R}^d)$ (cf. Theorem 2.2 below). Hence we get an equivalent definition if we take $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ and add the term $-\int_{\mathbb{R}^d}|u(x,T)-k|\,\phi(x,T)\,\mathrm{d}x$ to (2.5); see [21] for more details.

Here is the well-posedness result from [21].

Theorem 2.2. (Well-posedness) *Let* $\alpha \in (0,2)$, $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, *and* (1.4)–(1.5) *hold. Then there exists a unique entropy solution* $u \in L^\infty(Q_T) \cap C\left([0,T];L^1\right)$ *of* (1.1), *satisfying*

$$\begin{cases} \operatorname{ess\,inf} u_0 \le u \le \operatorname{ess\,sup} u_0, \\ \|u\|_{C([0,T];L^1)} \le \|u_0\|_{L^1}, \\ |u|_{L^\infty(0,T;BV)} \le |u_0|_{BV}. \end{cases} \tag{2.6}$$

Moreover, if v *is an entropy solution of* (1.1) *with* $v(\cdot,0) = v_0(\cdot) \in L^\infty \cap L^1(\mathbb{R}^d)$, *then*

$$\|u - v\|_{C([0,T];L^1)} \le \|u_0 - v_0\|_{L^1}. \tag{2.7}$$

3. The main results

We state our main results in this section. They compare the entropy solution u of (1.1) to the entropy solution v of

$$\begin{cases} \partial_t v + \operatorname{div} g(v) + (-\triangle)^{\frac{\beta}{2}} \psi(v) = 0, \\ v(\cdot,0) = v_0(\cdot), \end{cases} \tag{3.1}$$

under the assumptions that

$$\begin{cases} \alpha, \beta \in (0,2), \\ u_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d),\ v_0 \in L^\infty \cap L^1(\mathbb{R}^d), \\ f, g \in \left(W^{1,\infty}_{\text{loc}}(\mathbb{R})\right)^d \text{ with } f(0) = 0 = g(0), \\ \varphi, \psi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \text{ are nondecreasing with } \varphi(0) = 0 = \psi(0). \end{cases} \tag{3.2}$$

From now on, we will use the shorthand notation

$$\begin{aligned} \|f' - g'\|_\infty &:= \operatorname*{ess\,sup}_{I(u_0)} |f' - g'|, \\ \|\varphi' - \psi'\|_\infty &:= \operatorname*{ess\,sup}_{I(u_0)} |\varphi' - \psi'|, \end{aligned}$$

where $I(u_0) := (\operatorname{ess\,inf} u_0, \operatorname{ess\,sup} u_0)$. We will also define

$$\mathrm{E}_i(u_0) := |u_0|_{BV} \left\{ 1 + \left(\ln \frac{\|u_0\|_{L^1}}{|u_0|_{BV}} \right)^i \right\} \mathbf{1}_{\frac{\|u_0\|_{L^1}}{|u_0|_{BV}} > 1}, \tag{3.3}$$

with the convention that $\mathrm{E}_i(u_0) = 0$ if $|u_0|_{BV} = 0$ $(i = 1,2)$. These quantities will appear when computing the optimal constants in our main estimates. Notice that we always have $0 \le \mathrm{E}_i(u_0) \le \|u_0\|_{L^1}$.

Here is our first main result.

Theorem 3.1. (Continuous dependence on the nonlinearities) *Let* (3.2) *hold with* $\alpha = \beta$, *and let* u *and* v *be the entropy solutions of* (1.1) *and* (3.1) *respectively. Then we have*

$$\|u - v\|_{C([0,T];L^1)} \le \|u_0 - v_0\|_{L^1} + T\,|u_0|_{BV}\,\|f' - g'\|_\infty + C\,\mathcal{E}^{\varphi-\psi}_{T,\alpha,u_0}, \tag{3.4}$$

with $C = C(d,\alpha)$ and

$$\mathcal{E}^{\varphi-\psi}_{T,\alpha,u_0} = \begin{cases} T^{\frac{1}{\alpha}}\,|u_0|_{BV}\,\|(\varphi')^{\frac{1}{\alpha}} - (\psi')^{\frac{1}{\alpha}}\|_\infty, & \alpha \in (1,2), \\ T\ \mathrm{E}_1(u_0)\,\|\varphi' - \psi'\|_\infty \\ \qquad + T\,(1+|\ln T|)\,|u_0|_{BV}\,\|\varphi' - \psi'\|_\infty \\ \qquad\qquad + T\,|u_0|_{BV}\,\|\varphi'\,\ln\varphi' - \psi'\,\ln\psi'\|_\infty, & \alpha = 1, \\ T\,\|u_0\|_{L^1}^{1-\alpha}\,|u_0|_{BV}^{\alpha}\,\|\varphi' - \psi'\|_\infty, & \alpha \in (0,1). \end{cases} \tag{3.5}$$

The proof of this result can be found in Sections 5 and 6.

Remark 3.2. We emphasize that this result is optimal with respect to the modulus in φ. In the regimes where T is sufficiently small or $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}}$ is sufficiently large, it is also optimal with respect to the dependence of T and u_0. See the discussion of Section 8 for more details. In particular, see Proposition 8.1 and Remark 8.2.

Note that our result is robust in the sense that the constant $C = C(d,\alpha)$ in Theorem 3.1 has finite limits as $\alpha \downarrow 0$ or $\alpha \uparrow 2$. This will be seen during the proof, cf. Remarks 5.1(1) and 6.2(1). Hence, we can recover the known continuous dependence estimates of the limiting cases $\alpha = 0$ and $\alpha = 2$ (cf. (1.9)), i.e. for Equations (1.6) and (1.7).

To show this we start by identifying the limits of the solutions u^α of (1.1) as $\alpha \downarrow 0$ and $\alpha \uparrow 2$.

Theorem 3.3 (Limiting equations)**.** *Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, (1.4)–(1.5) hold, and for each $\alpha \in (0,2)$, let u^α denote the entropy solution of (1.1). Then u^α converges in $C([0,T];L^1_{\mathrm{loc}})$, as $\alpha \downarrow 0$ (resp. $\alpha \uparrow 2$), to the unique entropy solution $u \in L^\infty(Q_T) \cap C([0,T];L^1)$ of (1.6) (resp. (1.7)) with initial condition u_0.*

Let us recall that under our assumptions there are unique entropy solutions of (1.6) and (1.7) with initial data u_0; cf. [44, 16, 40]. The proof of Theorem 3.3 can be found in Section 7, as well as the definitions of entropy solutions of [44, 16, 40].

Now we prove that the estimates hold in the limiting cases $\alpha = 0$ and $\alpha = 2$.

Corollary 3.4 (Limiting estimates)**.** *Theorem 3.1 holds with $\alpha \in [0,2]$.*

Proof. We only do the proof for $\alpha = 2$, the case $\alpha = 0$ being similar. Let u and v denote the entropy solutions of (1.1) and (3.1) with $\alpha = 2$ respectively. Moreover, for each $\alpha \in (0,2)$, we denote by u^α and v^α the entropy solutions of (1.1) and (3.1) respectively, and $\mathcal{E}(\alpha)$ the right-hand side of (3.4). Then

$$u - v = (u - u^\alpha) + (u^\alpha - v^\alpha) + (v^\alpha - v),$$

and the triangle inequality and Theorems 3.1 and 3.3 imply that for all $R > 0$,

$$\|(u-v)\,\mathbf{1}_{|x|<R}\|_{C([0,T];L^1)} \leq o(1) + \mathcal{E}(\alpha) + o(1)$$

as $\alpha \uparrow 2$ and R is fixed. By the monotone convergence theorem, Remark 6.2(1), and α-continuity of $\mathcal{E}^{\varphi-\psi}_{T,\alpha,u_0}$ at $\alpha = 2$, the result follows by first sending $\alpha \uparrow 2$ and then sending $R \to +\infty$. □

Remark 3.5. By our results for $\alpha = 2$, we get back the modulus of [24],

$$\mathcal{E}^{\varphi-\psi}_{T,\alpha=2,u_0} = \sqrt{T}\,|u_0|_{BV}\,\|\sqrt{\varphi'} - \sqrt{\psi'}\|_\infty.$$

Our approach also gives an alternative proof of this result.

Optimal time regularity for (1.1) is another corollary of Theorem 3.3.

Corollary 3.6 (Modulus of continuity in time)**.** *Let $\alpha \in [0,2]$ and (1.3)–(1.5) hold. Let u be the entropy solution of (1.1). Then for all $t, s \geq 0$,*

$$\|u(\cdot,t) - u(\cdot,s)\|_{L^1} \leq |u_0|_{BV} \, \|f'\|_\infty \, |t-s| + C \, \mathcal{E}^{t-s}_{\alpha,u_0,\varphi}, \tag{3.6}$$

with $C = C(d,\alpha)$,

$$\mathcal{E}^{t-s}_{\alpha,u_0,\varphi} = \begin{cases} |u_0|_{BV} \, \|(\varphi')^{\frac{1}{\alpha}}\|_\infty \, |t^{\frac{1}{\alpha}} - s^{\frac{1}{\alpha}}|, & \alpha \in (1,2], \\ \mathrm{E}_1(u_0) \, \|\varphi'\|_\infty \, |t-s| & \\ \qquad + |u_0|_{BV} \, \|\varphi'\|_\infty \, (1 + \|\ln \varphi'\|_\infty) \, |t-s| & \\ \qquad\qquad + |u_0|_{BV} \, \|\varphi'\|_\infty \, |t \ln t - s \ln s|, & \alpha = 1, \\ \|u_0\|^{1-\alpha}_{L^1} \, |u_0|^\alpha_{BV} \, \|\varphi'\|_\infty \, |t-s|, & \alpha \in [0,1), \end{cases}$$

and where $\mathrm{E}_1(u_0)$ is defined in (3.3).

Remark 3.7. This result is optimal with respect to the modulus in time, and also with respect to the dependence of φ and u_0 in the regimes where $\|\varphi'\|_\infty$ is sufficiently small or the ratio $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}}$ is sufficiently large, cf. Remark 8.5. The result improves earlier results by the two last authors in [22] where the modulus was given as

$$\mathcal{E}^{t-s}_{\alpha,u_0,\varphi} = C(\alpha,u_0,\varphi) \begin{cases} |t-s|^{\frac{1}{\alpha}}, & \alpha > 1, \\ |t-s| \, (1 + |\ln|t-s||), & \alpha = 1, \\ |t-s|, & \alpha < 1. \end{cases}$$

The optimal new results give a strictly better modulus of continuity when $\alpha \in [1,2]$ at the initial time[1] and for positive times $u \in W^{1,\infty}_{\mathrm{loc}}((0,+\infty];L^1)$. The Lipschitz in time result is a regularizing effect when the solution is no more than BV initially.

Proof. We fix $t, s > 0$ and introduce the rescaled solutions $v(x,\tau) := u(x, t\,\tau)$ and $w(x,\tau) := u(x, s\,\tau)$. These are solutions of (1.1) with initial data u_0, new respective fluxes $t\,f$ and $s\,f$, and new respective diffusion functions $t\,\varphi$ and $s\,\varphi$. The result immediately follows from the preceding corollary applied at time $\tau = 1$. □

Next we consider the continuous dependence on α. Given $\lambda \in (0,2)$, we define "the best Lipschitz constant" of $\alpha \mapsto u^\alpha$ at the position $\alpha = \lambda$ as follows:

$$\mathrm{Lip}_\alpha(u;\lambda) := \limsup_{\alpha,\beta \to \lambda} \frac{\|u^\alpha - u^\beta\|_{C([0,T];L^1)}}{|\alpha - \beta|}, \tag{3.7}$$

where u^α denotes the unique entropy solution of (1.1).

Theorem 3.8. (Lipschitz continuity in α) *Let $\lambda \in (0,2)$ and (1.3)–(1.5) hold. Then*

$$\mathrm{Lip}_\alpha(u;\lambda) \leq C \begin{cases} M^{\frac{1}{\lambda}} \, (1 + |\ln M|) \, |u_0|_{BV}, & \lambda \in (1,2), \\ M \, \mathrm{E}_2(u_0) + M \, (1 + \ln^2 M) \, |u_0|_{BV}, & \lambda = 1, \\ M \, \|u_0\|^{1-\lambda}_{L^1} \, |u_0|^\lambda_{BV} \left(1 + \left|\ln \frac{\|u_0\|_{L^1}}{|u_0|_{BV}}\right|\right), & \lambda \in (0,1), \end{cases} \tag{3.8}$$

where $C = C(d,\lambda)$, $M := T\,\|\varphi'\|_\infty$ and $\mathrm{E}_2(u_0)$ is defined in (3.3). In particular, the function $\alpha \in (0,2) \mapsto u^\alpha \in C([0,T];L^1)$ is locally Lipschitz continuous.

The proof of Theorem 3.8 can be found in Sections 5 and 6.

---

[1] Since $\liminf_{t,s\downarrow 0} \frac{|t^{\frac{1}{\alpha}} - s^{\frac{1}{\alpha}}|}{|t-s|^{\frac{1}{\alpha}}} = 0 = \liminf_{t,s\downarrow 0} \frac{|t \ln t - s \ln s|}{|t-s| \, |\ln|t-s||}$ (take $t_n, s_n \downarrow 0$ and $\frac{t_n}{s_n} \to 1$).

Remark 3.9. This result is optimal with respect to the dependence of M and u_0 in the regimes where M is sufficiently small or $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}}$ is sufficiently large. An example is given in Section 8, cf. Proposition 8.3 and Remark 8.4.

Remark 3.10. With Theorem 3.1, Corollary 3.6 and Theorem 3.8 in hands, we can easily get an explicit continuous dependence estimate of u with respect to the quintuplet $(t, \alpha, u_0, f, \varphi)$ under (3.2).

Further comments and open problems.

A. *Robustness of the Lipschitz estimates in α as $\alpha \downarrow 0$ or $\alpha \uparrow 2$.* In Theorem 3.8, $C = C(d, \lambda)$ blows up as $\lambda \downarrow 0$ or $\lambda \uparrow 2$, and we do not get Lipschitz regularity in α up to the boundaries $\alpha = 0$ and $\alpha = 2$.

At $\alpha = 0$, we can do no better because the entropy solutions of (1.1) may not even converge toward the entropy solution of (1.7) in L^1 as $\alpha \downarrow 0$. The reason is that the mass preserving property could be lost at the limit. This was already observed in Section 11 of [30] for the fractional porous medium equation (3.9) below. Note that the convergence always holds in L^1_{loc} by Theorem 3.3, so that an interesting question is whether it holds in L^p for any $p \in (1, +\infty)$. To the best of our knowledge, this problem is still open at least for the full equation (1.1).

At $\alpha = 2$, it is an open problem whether $\alpha \mapsto u^\alpha$ is Lipschitz with values in L^1 or not. This problem is related to the following problems: Do the entropy solutions of (1.1) converge toward the entropy solution of (1.7) in L^1 or L^p as $\alpha \uparrow 2$? If yes, what is the optimal rate of convergence? Note that here again the convergence holds in L^1_{loc} by Theorem 3.3, and it moreover holds in L^1 for Equation (3.9) by [30].

B. *Implications for the fractional porous medium equation.* In [30], the following Cauchy problem is studied:

$$\partial_t u + (-\triangle)^{\alpha/2}(|u|^{m-1}\, u) = 0 \quad \text{and} \quad u(\cdot, 0) = u_0(\cdot), \tag{3.9}$$

where $\alpha \in (0, 2)$ and $m > 0$. The authors prove that if $u_0 \in L^1(\mathbb{R}^d)$, there exists a unique mild solution which under further assumptions ($m \geq 1$ is sufficient) is the (unique) strong solution. By Theorems 10.1 and 10.3 of [30], this solution is continuous in the data $(\alpha, m, u_0) \in D \times L^1(\mathbb{R}^d)$ with values in $C([0, +\infty); L^1)$, where

$$D := \left\{(\alpha, m) : 0 < \alpha \leq 2,\, m > \frac{(d-\alpha)^+}{d}\right\}.$$

We will now show that this dependence is locally Lipschitz in some cases.

Let us first establish the equivalence between entropy and strong solutions.

Lemma 3.11. *Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, $m \geq 1$, and u be the unique entropy solution of* (3.9) *given by Theorem 2.2 (with $T = +\infty$). Then u coincides with the unique strong solution of* (3.9) *(cf. Definition 3.5 in* [30]*).*

Proof. Note that $u \in L^\infty(\mathbb{R}^d \times (0, +\infty)) \cap C([0, +\infty); L^1)$. By Lemma 7.4, we also have $|u|^{m-1}\, u \in L^2(0, +\infty; H^{\frac{\alpha}{2}})$. Here $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ is the usual fractional Sobolev space defined in (7.5). Let us also recall that u satisfies the equation in $\mathcal{D}'(\mathbb{R}^d \times (0, +\infty))$ and the initial condition $u(\cdot, 0) = u_0(\cdot)$ almost everywhere, cf. [21]. It follows that u is a weak solution in the sense of Definition 3.1 in [30]. Since u is bounded, Corollary 8.3 of [30] completes the proof. $\square$

Theorems 3.1 and 3.8 and Lemma 3.11 then imply the following result:

Corollary 3.12. *For all $T > 0$, the unique strong solution u to* (3.9) *is locally Lipschitz continuous in $(\alpha, m, u_0) \in \tilde{D} \times \left(L^\infty \cap L^1 \cap BV(\mathbb{R}^d)\right)$ with values in $C([0, T]; L^1)$, where*

$$\tilde{D} := \{(\alpha, m) : 0 < \alpha < 2,\, m > 1\}.$$

If $u_0 \notin L^\infty \cap BV(\mathbb{R}^d)$, it is possible to find an explicit (non-Lipschitz) modulus of continuity for the function $(\alpha, m) \in \tilde{D} \mapsto u \in C([0,T]; L^1)$. To do so, it suffices to use an approximation argument and the L^1-contraction principle. It is an open problem whether this would give an optimal modulus or not. It is also an open problem to find an explicit modulus when $(\alpha, m) \notin \tilde{D}$.

4. Two general results from [4]

In this section we recall two key results developed in [4] for the more general case where the diffusion operator can be the generator of an arbitrary pure jump Lévy process. First we state the Kuznetsov type lemma of [4] that measures the L^1-distance between u and an arbitrary function v. From now on, let ϵ and ν be positive parameters and $\phi^{\epsilon,\nu} \in C^\infty(\mathbb{R}^{2d+2})$ denote the test function

$$\phi^{\epsilon,\nu}(x,t,y,s) := \theta_\nu(t-s)\,\rho_\epsilon(x-y) := \frac{1}{\nu}\,\theta\left(\frac{t-s}{\nu}\right)\frac{1}{\epsilon^d}\,\rho\left(\frac{x-y}{\epsilon}\right), \tag{4.1}$$

where

$$\begin{cases} \theta \in C_c^\infty(\mathbb{R}), \quad \theta \geq 0, \quad \operatorname{supp}\theta \subseteq [-1,1], \quad \int \theta = 1, \\ \rho \in C_c^\infty(\mathbb{R}^d), \quad \rho \geq 0, \quad \text{and } \int \rho = 1. \end{cases}$$

We also let $m_u(\nu)$ denote the modulus of continuity in time of $u \in C\left([0,T]; L^1\right)$.

Lemma 4.1 (Kuznetsov type Lemma). *Let $\alpha \in (0,2)$, $u_0 \in L^\infty \cap L^1 \cap BV(\mathbb{R}^d)$, and let us assume (1.4)–(1.5). Let u be the entropy solution of (1.1) and let $v \in L^\infty(Q_T) \cap C\left([0,T]; L^1\right)$ be such that $v(\cdot,0) = v_0(\cdot)$. Then for all $r, \epsilon > 0$ and $T > \nu > 0$,*

$$\begin{aligned} &\|u(\cdot,T) - v(\cdot,T)\|_{L^1} \\ &\leq \|u_0 - v_0\|_{L^1} + C_\rho\,|u_0|_{BV}\,\epsilon + 2\,m_u(\nu) \vee m_v(\nu) \\ &\quad - \int_{Q_T^2} |v(x,t) - u(y,s)|\,\partial_t \phi^{\epsilon,\nu}(x,t,y,s)\,\mathrm{d}w \\ &\quad - \int_{Q_T^2} q_f(v(x,t), u(y,s)) \cdot \nabla_x \phi^{\epsilon,\nu}(x,t,y,s)\,\mathrm{d}w \\ &\quad + \int_{Q_T^2} |\varphi(v(x,t)) - \varphi(u(y,s))|\,\mathcal{L}_r^\alpha[\phi^{\epsilon,\nu}(x,t,\cdot,s)](y)\,\mathrm{d}w \\ &\quad - \int_{Q_T^2} \operatorname{sgn}(v(x,t) - u(y,s))\,\mathcal{L}^{\alpha,r}[\varphi(u(\cdot,s))](y)\,\phi^{\epsilon,\nu}(x,t,y,s)\,\mathrm{d}w \\ &\quad + \int_{\mathbb{R}^d \times Q_T} |v(x,T) - u(y,s)|\,\phi^{\epsilon,\nu}(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \int_{\mathbb{R}^d \times Q_T} |v_0(x) - u(y,s)|\,\phi^{\epsilon,\nu}(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{4.2}$$

where $\mathrm{d}w := \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$ and the constant C_ρ only depends on ρ.

Proof. This is Lemma 3.1 of [4] with the particular diffusion operator (2.1). □

In the setting of this paper, the general continuous dependence estimates of [4] take the following form:

Theorem 4.2. *Let us assume (3.2) and let u and v be the respective entropy solutions of (1.1) and (3.1). Then for all $r > 0$,*

$$\|u - v\|_{C([0,T];L^1)} \leq \|u_0 - v_0\|_{L^1} + T\,|u_0|_{BV}\,\|f' - g'\|_\infty + \mathcal{E}_{T,\alpha,\beta,u_0,\varphi,r}^{\alpha-\beta,\varphi-\psi}$$

with

$$\mathcal{E}^{\alpha-\beta,\varphi-\psi}_{T,\alpha,\beta,u_0,\varphi,r} = \tag{4.3}$$
$$\begin{cases} T\int_{|z|>r} \|u_0(\cdot+z)-u_0(\cdot)\|_{L^1}\,\mathrm{d}\mu_\alpha(z)\,\|\varphi'-\psi'\|_\infty \\ \qquad +c_d\sqrt{T}\,|u_0|_{BV}\sqrt{\int_{|z|<r}|z|^2\,\mathrm{d}\mu_\alpha(z)\,\|\varphi'-\psi'\|_\infty}, & \alpha=\beta, \\ M\int_{|z|>r}\|u_0(\cdot+z)-u_0(\cdot)\|_{L^1}\,\mathrm{d}|\mu_\alpha-\mu_\beta|(z) \\ \qquad +c_d\sqrt{M}\,|u_0|_{BV}\sqrt{\int_{|z|<r}|z|^2\,\mathrm{d}|\mu_\alpha-\mu_\beta|(z)}, & \varphi=\psi, \end{cases}$$

where $\mathrm{d}\mu_\alpha(z)=\frac{G_d(\alpha)}{|z|^{d+\alpha}}\,\mathrm{d}z$, $M=T\,\|\varphi'\|_\infty$ *and* $c_d=\sqrt{\frac{4\,d^2}{d+1}}$.

Proof. This is Theorems 3.3 and 3.4 of [4] with the special choice of diffusion (2.1) and Lévy measure $\frac{G_d(\alpha)}{|z|^{d+\alpha}}\,\mathrm{d}z$. □

5. Continuous dependence in the supercritical case

In this section we use Theorem 4.2 to prove Theorems 3.1 and 3.8 for supercritical diffusions.

Proof of Theorem 3.1 when $\alpha<1$. We use Estimate (4.3) with $\beta=\alpha$. The worst term $\sqrt{\int_{|z|<r}|z|^2\,\mathrm{d}\mu_\alpha(z)\,\|\varphi'-\psi'\|_\infty}$ vanishes when $r\downarrow 0$, and hence

$$\mathcal{E}^{\alpha-\beta,\varphi-\psi}_{T,\alpha,\beta,u_0,\varphi,r} \underset{r\downarrow 0}{\longrightarrow} I:=T\int \|u_0(\cdot+z)-u_0(\cdot)\|_{L^1}\,\mathrm{d}\mu_\alpha(z)\,\|\varphi'-\psi'\|_\infty.$$

To estimate this integral, we consider separately the domains $|z|>\tilde r$ and $|z|<\tilde r$ for arbitrary $\tilde r>0$. In the second domain, we use the inequality $\|u_0(\cdot+z)-u_0(\cdot)\|_{L^1}\le |u_0|_{BV}\,|z|$. A direct computation using the fact that $\alpha<1$, then leads to

$$I\le 2\,T\,\|u_0\|_{L^1}\,\|\varphi'-\psi'\|_\infty\,S_d\,\frac{G_d(\alpha)}{\alpha}\,\tilde r^{-\alpha}+T\,|u_0|_{BV}\,\|\varphi'-\psi'\|_\infty\,S_d\,\frac{G_d(\alpha)}{1-\alpha}\,\tilde r^{1-\alpha}$$

(where S_d is the surface measure of the unit sphere of $\mathbb{R}^d$). We complete the proof by taking $\tilde r=\|u_0\|_{L^1}\,|u_0|^{-1}_{BV}$. □

Remark 5.1. (1) From the proof, we have $C\le S_d\left(\frac{2\,G_d(\alpha)}{\alpha}+\frac{G_d(\alpha)}{1-\alpha}\right)$ in (3.4) when $\alpha<1$. By (2.2), $\lim_{\alpha\downarrow 0}C(d,\alpha)$ is finite and independent of d.
(2) We also have $C\le S_d\left(\frac{2\,G_d(\alpha)}{\alpha}+\frac{G_d(\alpha)}{1-\alpha}\right)$ when $\alpha<1$ in (3.6), since we have seen that this estimate is a simple rewriting of the preceding one by rescaling the time variable.

Proof of Theorem 3.8 when $\lambda\in(0,1)$. Given $\alpha,\beta\in(0,2)$, we use Theorem 4.2 with $u=u^\alpha$ and $v=u^\beta$, i.e. with $(u_0,f,\varphi)=(v_0,g,\psi)$. As in the preceding proof, we pass to the limit as $r\downarrow 0$ in (4.3) and we cut the remaining integral in two parts. We find that

$$\begin{aligned} &\|u^\alpha-u^\beta\|_{C([0,T];L^1)} \\ &\le 2\,M\,\|u_0\|_{L^1}\underbrace{\int_{|z|>\tilde r}\mathrm{d}|\mu_\alpha-\mu_\beta|(z)}_{=:J_1}+M\,|u_0|_{BV}\underbrace{\int_{|z|<\tilde r}|z|\,\mathrm{d}|\mu_\alpha-\mu_\beta|(z)}_{=:J_2}. \end{aligned} \tag{5.1}$$

In the rest of the proof we use the letter C to denote various constants $C=C(d,\lambda)$.

We have

$$J_1 = \int_{|z|>\tilde r} \left|G_d(\alpha)\,|z|^{-d-\alpha} - G_d(\beta)\,|z|^{-d-\beta}\right| \mathrm{d}z \tag{5.2}$$
$$\leq |G_d(\alpha) - G_d(\beta)| \max_{\sigma=\alpha,\beta} \int_{|z|>\tilde r} \frac{\mathrm{d}z}{|z|^{d+\sigma}}$$
$$+ (G_d(\alpha) \vee G_d(\beta)) \underbrace{\int_{|z|>\tilde r} \left||z|^{-d-\alpha} - |z|^{-d-\beta}\right| \mathrm{d}z}_{=:\tilde J_1},$$

where $\tilde J_1 \leq S_d \left|\frac{\tilde r^{-\alpha}}{\alpha} - \frac{\tilde r^{-\beta}}{\beta}\right| + 2\,S_d \left|\frac{1}{\alpha} - \frac{1}{\beta}\right| \mathbf{1}_{\tilde r<1}$. We have estimated $\tilde J_1$ using the fact that $|z|^{-d-\alpha} - |z|^{-d-\beta}$ has a sign both inside and outside the unit ball. By (2.2) and a simple passage to the limit under the integral sign,

$$\limsup_{\alpha,\beta\to\lambda} \frac{J_1}{|\alpha-\beta|} \leq \underbrace{C\,(\tilde r^{-\lambda} + \mathbf{1}_{\tilde r<1})}_{\leq C\,\tilde r^{-\lambda}} + C \underbrace{\limsup_{\alpha,\beta\to\lambda} \frac{1}{|\alpha-\beta|} \left|\frac{\tilde r^{-\alpha}}{\alpha} - \frac{\tilde r^{-\beta}}{\beta}\right|}_{=:\tilde{\tilde J}_1}.$$

By the Taylor formula with integral remainder,

$$\tilde{\tilde J}_1 = \limsup_{\alpha,\beta\to\lambda} \left|\int_0^1 \frac{\alpha_\tau\,\tilde r^{-\alpha_\tau}\,\ln\tilde r + \tilde r^{-\alpha_\tau}}{\alpha_\tau^2}\,\mathrm{d}\tau\right| \leq C\,\tilde r^{-\lambda}\,(1 + |\ln\tilde r|),$$

where $\alpha_\tau := \tau\,\alpha + (1-\tau)\,\beta$. We deduce the following estimate:

$$\limsup_{\alpha,\beta\to\lambda} \frac{J_1}{|\alpha-\beta|} \leq C\,\tilde r^{-\lambda}\,(1+|\ln\tilde r|). \tag{5.3}$$

Let us notice that this estimate works for all $\lambda \in (0,2)$. By similar arguments, we also have

$$\limsup_{\alpha,\beta\to\lambda} \frac{J_2}{|\alpha-\beta|} \leq C\,\tilde r^{1-\lambda}\,(1+|\ln\tilde r|),$$

but this time we have to use that $\lambda < 1$. Inserting these inequalities into (5.1), we find that for all $\tilde r > 0$,

$$\mathrm{Lip}_\alpha(u;\lambda) \leq C\,M\,(1+|\ln\tilde r|)\,(\|u_0\|_{L^1}\,\tilde r^{-\lambda} + |u_0|_{BV}\,\tilde r^{1-\lambda}).$$

To conclude we take $\tilde r = \|u_0\|_{L^1}\,|u_0|_{BV}^{-1}$. □

Remark 5.2. (1) When $\alpha \geq 1$, the estimate in $\varphi - \psi$ of Theorem 4.2 is not optimal. Indeed, let $\alpha = \beta$, u_0 be such that $\|u_0(\cdot + z) - u_0(\cdot)\|_{L^1} \sim |z|$ as $z \to 0$, and $\omega_{\varphi-\psi} := \inf_{r>0} \mathcal{E}^{\alpha-\beta,\varphi-\psi}_{T,\alpha,\beta,u_0,\varphi,r}$ be the best modulus given by Theorem 4.2. Then

$$\omega_{\varphi-\psi} \sim \begin{cases} \|\varphi'-\psi'\|_\infty^{\frac{1}{\alpha}}, & \alpha > 1,\\ \|\varphi'-\psi'\|_\infty\,\left|\ln\|\varphi'-\psi'\|_\infty\right|, & \alpha = 1,\end{cases}$$

as $\|\varphi'-\psi'\|_\infty \to 0$, thanks to the minimization giving $r \sim \|\varphi'-\psi'\|_\infty^{\frac{1}{\alpha}}$. These moduli are strictly worse than those in (3.5) e.g. when $\varphi' \equiv a$, $\psi' \equiv b$, $a,b > 0$.[2]

[2]Indeed $\lim_{a,b\to c} \frac{|a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}|}{|a-b|^{\frac{1}{\alpha}}} = 0 = \lim_{a,b\to c} \frac{|a\,\ln a - b\,\ln b|}{|a-b|\,|\ln|a-b||}$ for any $c > 0$ and even for $c = 0^+$ by taking liminfs.

(2) Theorem 4.2 does not imply the local Lipschitz continuity in $\alpha \in [1,2)$. Indeed, let $\varphi = \psi$ be nontrivial and u_0 be as above. Then the modulus $\omega_{\alpha-\beta} := \inf_{r>0} \mathcal{E}^{\alpha-\beta,\varphi-\psi}_{T,\alpha,\beta,u_0,\varphi,r}$ is worse than any Lipschitz modulus since $\lim_{\alpha,\beta\to\lambda} \frac{\omega_{\alpha-\beta}}{|\alpha-\beta|} = +\infty$ for all $\lambda \in [1,2)$.[3]

6. Continuous dependence in the critical and subcritical cases

Since we can not use Theorem 4.2 any more, we start from Lemma 4.1 and take advantage of the homogeneity of the fractional Laplacian. We thus use the Kruzhkov type doubling of variables techniques introduced in [44] along with ideas from [45]; see also [54, 25, 33, 39, 35, 2, 50, 21, 41, 4, 22] for other applications of this technique to nonlocal equations. We recall that the idea is to consider v to be a function of (x,t), u to be a function of (y,s), and use the approximate unit $\phi^{\epsilon,\nu}(x,t,y,s)$ in (4.1) as a test function. For brevity, we do not specify the variables of u, v, and $\phi^{\epsilon,\nu}$ when the context is clear. Finally, we recall that $\mathrm{d}w = \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$.

6.1. A technical lemma.

In order to adapt the ideas of [45] to the nonlocal case, we need the following Kato type of inequality. The reader could skip this technical subsection at the first reading.

Lemma 6.1. *Let $\alpha \in (0,2)$, $c, \tilde{c} \in \mathbb{R}$, $\gamma, \tilde{\gamma} \in \mathbb{R}$ and I be a real interval with a positive lower bound. Let $u, v \in L^1(Q_T)$, φ satisfy (1.5) and $\phi^{\epsilon,\nu}$ be the test function in (4.1). Then*

$$
\begin{aligned}
&\mathcal{E}\\
&:= \int_{Q_T^2} \int_{|z|\in I} \operatorname{sgn}(v(x,t)-u(y,s))\\
&\quad \cdot \frac{\left\{\varphi\left(v(x+\tilde{c}\,|z|^{\tilde{\gamma}-1}\,z,t)\right) - \varphi\left(u(y+c\,|z|^{\gamma-1}\,z,s)\right)\right\} - \left\{\varphi(v(x,t)) - \varphi(u(y,s))\right\}}{|z|^{d+\alpha}}\\
&\quad \cdot \phi^{\epsilon,\nu}(x,t,y,s)\,\mathrm{d}z\,\mathrm{d}w\\
&\le \int_{Q_T^2} \int_{|z|\in I} |\varphi(v(x,t)) - \varphi(u(y,s))|\,\theta_\nu(t-s)\frac{\rho_\epsilon\left(x-y+h(z)\right) - \rho_\epsilon(x-y)}{|z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}w,
\end{aligned}
$$

with $h(z) := (\tilde{c}\,|z|^{\tilde{\gamma}-1} - c\,|z|^{\gamma-1})\,z$. In particular, if $c = \tilde{c}$ and $\gamma = \tilde{\gamma}$, then $\mathcal{E} \le 0$.

Proof. Note that $\mathcal{E}$ is well-defined as "convolution-like integral of L^1-functions." Indeed, $\phi^{\epsilon,\nu}(x,t,y,s) = \theta_\nu(t-s)\,\rho_\epsilon(x-y)$, where θ_ν and ρ_ϵ are approximate units,

[3] If not, there are $\alpha_n, \beta_n \to \lambda$ and $r_n \to r_* \in [0,+\infty]$ such that $\lim \frac{\omega_{\alpha_n-\beta_n}}{|\alpha_n-\beta_n|} < +\infty$ and

$$
\frac{\omega_{\alpha_n-\beta_n}}{|\alpha_n-\beta_n|} = o(1) + \underbrace{\int_{|z|>r_n} \|u_0(\cdot+z)-u_0(\cdot)\|_{L^1}\,\frac{\mathrm{d}|\mu_{\alpha_n}-\mu_{\beta_n}|(z)}{|\alpha_n-\beta_n|}}_{=:I_n} + \underbrace{\sqrt{\int_{|z|<r_n} |z|^2\,\frac{\mathrm{d}|\mu_{\alpha_n}-\mu_{\beta_n}|(z)}{|\alpha_n-\beta_n|^2}}}_{=:J_n}
$$

($M = c_d\sqrt{M}\,|u_0|_{BV} = 1$ to simplify). By Fatou's lemma $\liminf J_n^2 \ge \int_{|z|<r_*} |z|^2\,(+\infty)\,\mathrm{d}z$ and $\liminf I_n \ge \int_{|z|>r_*} \|u_0(\cdot+z)-u_0(\cdot)\|_{L^1}\,|G_d'(\lambda) - G_d(\lambda)\ln|z||\,|z|^{-d-\lambda}\,\mathrm{d}z$. This is not possible since these integrals can not be both finite at the same time.

so that by Fubini,

$$\int_{Q_T^2} \int_{|z|\in I} \phi^{\epsilon,\nu} \cdot \left| \frac{\left\{\varphi\left(v(x+\tilde{c}\,|z|^{\tilde{\gamma}-1}\,z,t)\right) - \varphi\left(u(y+c\,|z|^{\gamma-1}\,z,s)\right)\right\} - \{\varphi(v)-\varphi(u)\}}{|z|^{d+\alpha}} \right| \mathrm{d}z\,\mathrm{d}w$$
$$\leq 2\left(\|\varphi(u)\|_{L^1(Q_T)} + \|\varphi(v)\|_{L^1(Q_T)}\right) \int_{|z|\in I} \frac{\mathrm{d}z}{|z|^{d+\alpha}} < +\infty,$$

since u and v are $L^\infty \cap L^1$, φ is $W^{1,\infty}_{\mathrm{loc}}$ with $\varphi(0)=0$, and $\inf I > 0$.

Then by (2.4) and the nonnegativity of $\phi^{\epsilon,\nu}$,

$$\mathcal{E} \leq \int_{Q_T^2} \int_{|z|\in I} \phi^{\epsilon,\nu} \cdot \frac{\left|\varphi\left(v(x+\tilde{c}\,|z|^{\tilde{\gamma}-1}\,z,t)\right) - \varphi\left(u(y+c\,|z|^{\gamma-1}\,z,s)\right)\right| - |\varphi(v)-\varphi(u)|}{|z|^{d+\alpha}} \,\mathrm{d}z\,\mathrm{d}w$$
$$= \int_{Q_T^2} \int_{|z|\in I} |\varphi(v)-\varphi(u)| \cdot \underbrace{\left\{\phi^{\epsilon,\nu}(x+\tilde{c}\,|z|^{\tilde{\gamma}-1}\,z,t,y+c\,|z|^{\gamma-1}\,z,s) - \phi^{\epsilon,\nu}\right\}}_{=\theta_\nu(t-s)\{\rho_\epsilon(x-y+(\tilde{c}\,|z|^{\tilde{\gamma}-1}-c\,|z|^{\gamma-1})\,z)-\rho_\epsilon(x-y)\}} \frac{\mathrm{d}z}{|z|^{d+\alpha}}\,\mathrm{d}w;$$

the last line has been obtained by splitting the integral in two pieces and using the change of variable $(x+\tilde{c}\,|z|^{\tilde{\gamma}-1}\,z,t,y+c\,|z|^{\gamma-1}\,z,s,-z) \mapsto (x,t,y,s,z)$. The proof is complete. □

6.2. **Proof of Theorem 3.1.** During the proof we freeze the nonlinear diffusion functions and use a sort of linearization procedure. The techniques could look a little bit like the ones in Young measure theory and kinetic formulations [47, 11, 20].

Proof of Theorem 3.1.

1. *Initial reduction.* We first reduce the proof to the case where

$$\begin{cases} v_0 = u_0, \\ \varphi' \text{ and } \psi' \text{ vanish outside } I(u_0) \text{ and take values in } [\,\underline{\Lambda},\overline{\Lambda}\,], \end{cases} \tag{6.1}$$

with $I(u_0) = (\operatorname{ess\,inf} u_0, \operatorname{ess\,sup} u_0)$ and for some $\overline{\Lambda} \geq \underline{\Lambda} > 0$. Let us justify that we can do this without loss of generality.

Since u takes its values in $I(u_0)$ by (2.6), we can redefine φ to be constant outside this interval without changing the solutions of the initial-value problem (1.1). Hence $\overline{\Lambda}$ could be taken as a Lipschitz constant of φ on $I(u_0)$. In a similar way, we could also modify ψ outside $I(u_0)$ if $v_0 = u_0$. The last assumption is no restriction. Indeed, by (2.7),

$$\|u-v\|_{C([0,T];L^1)} \leq \|u-w\|_{C([0,T];L^1)} + \underbrace{\|w-v\|_{C([0,T];L^1)}}_{\leq \|u_0-v_0\|_{L^1}}$$

for the entropy solution w of (3.1) with initial data u_0; hence, (3.4) of Theorem 3.1 holds for $u-v$ whenever it does for $u-w$. Finally, if $\underline{\Lambda}$ does not exist, we can always consider sequences $\varphi_n(\xi) := \varphi(\xi) + \frac{\xi}{n}$ and $\psi_n(\xi) := \psi(\xi) + \frac{\xi}{n}$ for which it does. The associated entropy solutions u_n and v_n respectively converge to u and v in $C([0,T];L^1)$ by e.g. Theorem 4.2. Consequently, if we could prove (3.4) for $u_n - v_n$, it would follow for $u-v$ by going to the limit.

In the rest of the proof we always assume (6.1).

2. *Applying Kuznetsov.* Let us use the entropy inequality (2.5) for $v = v(x,t)$ with $k = u(y,s)$ fixed and $\phi(x,t) := \phi^{\epsilon,\nu}(x,t,y,s)$. By Remark 2.1 and an integration of (y,s) over Q_T, we find that

$$\begin{aligned}
&\int_{Q_T^2} \Big(|v-u|\, \partial_t \phi^{\epsilon,\nu} + q_g(v,u)\cdot \nabla_x \phi^{\epsilon,\nu} \Big)\, \mathrm{d}w \\
&+ \int_{Q_T^2} |\psi(v)-\psi(u)|\, \mathcal{L}_r^\alpha[\phi^{\epsilon,\nu}(\cdot,t,y,s)](x)\, \mathrm{d}w \\
&+ \int_{Q_T^2} \mathrm{sgn}(v-u)\, \mathcal{L}^{\alpha,r}[\psi(v(\cdot,t))](x)\, \phi^{\epsilon,\nu}\, \mathrm{d}w \\
&- \int_{\mathbb{R}^d\times Q_T} |v(x,T)-u(y,s)|\, \phi^{\epsilon,\nu}(x,T,y,s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \\
&+ \int_{\mathbb{R}^d\times Q_T} |v_0(x)-u(y,s)|\, \phi^{\epsilon,\nu}(x,0,y,s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \geq 0.
\end{aligned}$$

Inserting this inequality into the Kuznetsov inequality (4.2), we obtain for all $r, \epsilon > 0$ and $T > \nu > 0$,

(6.2)
$$\begin{aligned}
&\|u(\cdot,T)-v(\cdot,T)\|_{L^1} \leq \\
&C(d)\,|u_0|_{BV}\,\epsilon + 2\,(m_u(\nu)\vee m_v(\nu)) \\
&\underbrace{+\int_{Q_T^2} (q_g-q_f)(v,u)\cdot\nabla_x\phi^{\epsilon,\nu}\,\mathrm{d}w}_{=:\mathcal{E}_1} \\
&\underbrace{+\int_{Q_T^2} \Big(|\psi(v)-\psi(u)|\,\mathcal{L}_r^\alpha[\phi^{\epsilon,\nu}(\cdot,t,y,s)](x) + |\varphi(v)-\varphi(u)|\,\mathcal{L}_r^\alpha[\phi^{\epsilon,\nu}(x,t,\cdot,s)](y)\Big)\,\mathrm{d}w}_{=:\mathcal{E}_2} \\
&\underbrace{+\int_{Q_T^2} \mathrm{sgn}(v-u)\,\left(\mathcal{L}^{\alpha,r}[\psi(v(\cdot,t))](x) - \mathcal{L}^{\alpha,r}[\varphi(u(\cdot,s))](y)\right)\phi^{\epsilon,\nu}\,\mathrm{d}w}_{=:\mathcal{E}_3}
\end{aligned}$$

where $C(d) = C_\rho$ from (4.2). During the proof, $C(d)$ will denote various constant depending only on d.

3. *Estimates of $\mathcal{E}_1$ and $\mathcal{E}_2$.* A standard estimate shows that

$$\mathcal{E}_1 \leq T\,|u_0|_{BV}\,\|f'-g'\|_\infty, \tag{6.3}$$

see e.g. [27, 48, 28]. Let us estimate $\mathcal{E}_2$. By Taylor's formula,

$$\rho_\epsilon(x+z)-\rho_\epsilon(x)-\nabla\rho_\epsilon(x)\cdot z = \int_0^1 (1-\tau)\,\nabla^2\rho_\epsilon(x+\tau\,z)\cdot z^2\,\mathrm{d}\tau$$

for all $x, z \in \mathbb{R}^d$. Since $\rho_\epsilon \in C_c^\infty(\mathbb{R}^d)$, we infer that $\mathcal{L}_r^\alpha[\rho_\epsilon] \in L^1(\mathbb{R}^d)$ with

$$\begin{aligned}
\|\mathcal{L}_r^\alpha[\rho_\epsilon]\|_{L^1} &\leq G_d(\alpha)\int_{|z|<r}\int_0^1 (1-\tau)\,|z|^{-d+2-\alpha}\int_{\mathbb{R}^d}|\nabla^2\rho_\epsilon(x+\tau\,z)|\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}z \\
&= C(d,\alpha,\epsilon)\,r^{2-\alpha}.
\end{aligned}$$

Moreover, by Definitions (2.1) and (4.1),

$$\mathcal{L}_r^\alpha[\phi^{\epsilon,\nu}(\cdot,t,y,s)](x) = \theta_\nu(t-s)\,\mathcal{L}_r^\alpha[\rho_\epsilon](x-y).$$

By Fubini and the convolution like structure of the integral, it follows that

$$\int_{Q_T^2} |\psi(v(x,t)) - \psi(u(y,s))| \, \mathcal{L}_r^\alpha[\phi^{\epsilon,\nu}(\cdot,y,t,s)](x)\,\mathrm{d}w \\ \leq \left(\|\psi(v)\|_{L^1(Q_T)} + \|\psi(u)\|_{L^1(Q_T)}\right) C(d,\alpha,\epsilon)\, r^{2-\alpha},$$

since $\int \theta_\nu = 1$. In a similar way we can estimate the φ-integral and conclude that

$$\mathcal{E}_2 \leq C_\epsilon\, r^{2-\alpha}. \tag{6.4}$$

From now on C_ϵ will denote various constants depending among other things on ϵ, but not on r,ν. For later use we note that $\mathcal{E}_2 \to 0$ as $r,\nu \downarrow 0$ and ϵ is fixed.

4. *Estimate of $\mathcal{E}_3$ – the linear case.* We consider the case $\varphi' \equiv a$ and $\psi' \equiv b$ for $a, b > 0$. In this case

$$\mathcal{E}_3 = G_d(\alpha) \int_{Q_T^2} \int_{|z|>r} \operatorname{sgn}(v-u)\, \phi^{\epsilon,\nu} \\ \cdot \frac{a\,(v(x+z,t)-v) - b\,(u(y+z,s)-u)}{|z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}w. \tag{6.5}$$

By the change of variables $z \mapsto b^{\frac{1}{\alpha}}\, z$, we see that

$$b\,\mathcal{L}^{\alpha,r}[v(\cdot,t)](x) = G_d(\alpha) \int_{|z|>r} \frac{v(x+z,t)-v(x,t)}{|b^{-\frac{1}{\alpha}}\, z|^{d+\alpha}}\, b^{-\frac{d}{\alpha}}\,\mathrm{d}z \\ = G_d(\alpha) \int_{|z|>b^{-\frac{1}{\alpha}} r} \frac{v(x+b^{\frac{1}{\alpha}}\, z,t)-v(x,t)}{|z|^{d+\alpha}}\,\mathrm{d}z,$$

and similarly that

$$a\,\mathcal{L}^{\alpha,r}[u(\cdot,s)](y) = G_d(\alpha) \int_{|z|>a^{-\frac{1}{\alpha}} r} \frac{u(y+a^{\frac{1}{\alpha}}\, z,s)-u(y,s)}{|z|^{d+\alpha}}\,\mathrm{d}z.$$

It follows that

$$\mathcal{E}_3 = G_d(\alpha) \int_{Q_T^2} \int_{(a\vee b)^{-\frac{1}{\alpha}} r<|z|<(a\wedge b)^{-\frac{1}{\alpha}} r} \cdots \frac{\mathrm{d}z}{|z|^{d+\alpha}} \\ + G_d(\alpha) \int_{Q_T^2} \int_{|z|>(a\wedge b)^{-\frac{1}{\alpha}} r} \operatorname{sgn}(v-u) \\ \cdot \frac{\left(v(x+b^{\frac{1}{\alpha}}\, z,t) - u(y+a^{\frac{1}{\alpha}}\, z,s)\right) - (v-u)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w \\ =: \mathcal{E}_{3,1} + \mathcal{E}_{3,2}, \tag{6.6}$$

where $\mathcal{E}_{3,1}$ contains only the u-terms if $a \geq b$, or only the v-terms in the other case. In the u-case, e.g.,

$$\mathcal{E}_{3,1} = G_d(\alpha) \int_{Q_T^2} \int_{a^{-\frac{1}{\alpha}} r<|z|<b^{-\frac{1}{\alpha}} r} \operatorname{sgn}(u-v)\, \frac{u(y+a^{\frac{1}{\alpha}}\, z,s)-u}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w.$$

The estimates for $\mathcal{E}_{3,1}$ are similar in both cases, and we only detail the u-case. As in the proof of Lemma 6.1, we use that

$$\operatorname{sgn}(u(y,s)-v(x,t))\,\left(u(y+a^{\frac{1}{\alpha}}\, z,s)-u(y,s)\right) \\ \leq \left|u(y+a^{\frac{1}{\alpha}}\, z,s)-v(x,t)\right| - |u(y,s)-v(x,t)|\,,$$

to deduce that

$$\begin{aligned}\mathcal{E}_{3,1} &\leq G_d(\alpha)\int_{Q_T^2}\int_{a^{-\frac{1}{\alpha}}r<|z|<b^{-\frac{1}{\alpha}}r}\frac{\left|u(y+a^{\frac{1}{\alpha}}\,z,s)-v(x,t)\right|-|u-v|}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w\\ &=G_d(\alpha)\int_{Q_T^2}|u-v|\\ &\quad\cdot\int_{a^{-\frac{1}{\alpha}}r<|z|<b^{-\frac{1}{\alpha}}r}\underbrace{\left(\phi^{\epsilon,\nu}(x,t,y+a^{\frac{1}{\alpha}}\,z,s)-\phi^{\epsilon,\nu}\right)}_{=\theta_\nu(t-s)\left(\rho_\epsilon(x-y-a^{\frac{1}{\alpha}}\,z)-\rho_\epsilon(x-y)\right)}|z|^{-d-\alpha}\,\mathrm{d}z\,\mathrm{d}w.\end{aligned}$$

We continue as in the derivation of (6.4), and use a Taylor expansion with integral remainder of ρ_ϵ. Since the first order term contains the factor

$$\int_{a^{-\frac{1}{\alpha}}r<|z|<b^{-\frac{1}{\alpha}}r}\frac{z}{|z|^{d+\alpha}}\,\mathrm{d}z=0,$$

we find an estimate similar to (6.4), namely

$$\mathcal{E}_{3,1}\leq C_\epsilon\left(\|u\|_{L^1(Q_T)}+\|v\|_{L^1(Q_T)}\right)r^{2-\alpha}. \tag{6.7}$$

We emphasize that C_ϵ can be chosen to be independent of a and b by (6.1) (more precisely $C_\epsilon=C(d,\alpha,\epsilon,\underline{\Lambda},\overline{\Lambda})$; this will be important in the next step.

5. *Estimate of* $\mathcal{E}_{3,2}$. Note that a,b are arbitrary reals such that (6.1) holds, i.e. $\overline{\Lambda}\geq a,b\geq\underline{\Lambda}$, and let $r_2\geq r_1>0$. Since $\underline{\Lambda}>0$ and r will be sent to zero, we assume without loss of generality that $r_1>\underline{\Lambda}^{-\frac{1}{\alpha}}\,r$. In particular, $r_1>(a\wedge b)^{-\frac{1}{\alpha}}\,r$. Then

$$\begin{aligned}\mathcal{E}_{3,2}&=\sum_{i=1}^{3}G_d(\alpha)\int_{Q_T^2}\int_{|z|\in I_i}\mathrm{sgn}(v-u)\\ &\qquad\cdot\frac{\left(v(x+b^{\frac{1}{\alpha}}\,z,t)-u(y+a^{\frac{1}{\alpha}}\,z,s)\right)-(v-u)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w\\ &=:\sum_{i=1}^{3}\mathcal{E}_{3,2,i},\end{aligned} \tag{6.8}$$

where $I_1=(r_2,+\infty)$, $I_2=(r_1,r_2)$ and $I_3=((a\wedge b)^{-\frac{1}{\alpha}}\,r,r_1)$.

By adding and subtracting $\mathrm{sgn}(v-u)\,u(y+b^{\frac{1}{\alpha}}\,z,s)$ and using Lemma 6.1 with $c=\tilde{c}=b^{\frac{1}{\alpha}}$ and $\gamma=\tilde{\gamma}=1$, we find that

$$\mathcal{E}_{3,2,i}\leq G_d(\alpha)\int_{Q_T^2}\int_{|z|\in I_i}\mathrm{sgn}(v-u)\,\frac{u(y+b^{\frac{1}{\alpha}}\,z,s)-u(y+a^{\frac{1}{\alpha}}\,z,s)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w.$$

By the BV-regularity of u, we then immediately deduce that

$$\mathcal{E}_{3,2,2}\leq G_d(\alpha)\,|u|_{L^1(0,T;BV)}\,|a^{\frac{1}{\alpha}}-b^{\frac{1}{\alpha}}|\int_{r_1<|z|<r_2}\frac{|z|\,\mathrm{d}z}{|z|^{d+\alpha}}.$$

Moreover, going back to the original variables $a^{\frac{1}{\alpha}}\,z\mapsto z$ and $b^{\frac{1}{\alpha}}\,z\mapsto z$, we find that

$$\begin{aligned}&\int_{Q_T^2}\int_{|z|>r_2}\mathrm{sgn}(v-u)\frac{u(y+a^{\frac{1}{\alpha}}\,z,s)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w\\ &=a\int_{Q_T^2}\int_{|z|>a^{\frac{1}{\alpha}}\,r_2}\mathrm{sgn}(v-u)\frac{u(y+z,s)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w,\end{aligned}$$

and a similar formula for the b-term. Hence we find that

$$\mathcal{E}_{3,2,1} \leq G_d(\alpha)\,(b-a) \int_{Q_T^2} \int_{|z|>(a\vee b)^{\frac{1}{\alpha}}\, r_2} \operatorname{sgn}(v-u)\, \frac{u(y+z,s)}{|z|^{d+\alpha}}\, \phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w$$
$$+ G_d(\alpha)\, \operatorname{sgn}(a-b)\,(a\wedge b) \int_{Q_T^2} \int_{(a\wedge b)^{\frac{1}{\alpha}}\, r_2<|z|<(a\vee b)^{\frac{1}{\alpha}}\, r_2} \dots,$$

where the integrands are the same. Since $\phi^{\epsilon,\nu}$ is an approximate unit,

$$\mathcal{E}_{3,2,1} \leq C(d)\, \frac{G_d(\alpha)}{\alpha}\, \|u\|_{L^1(Q_T)}\, \frac{|a-b|}{a\vee b}\, r_2^{-\alpha},$$

where $C(d) = 2\, S_d$.

It remains to estimate $\mathcal{E}_{3,2,3}$ in (6.8). By Lemma 6.1, with $c = a^{\frac{1}{\alpha}}$ and $\tilde{c} = b^{\frac{1}{\alpha}}$,

$$\mathcal{E}_{3,2,3} \leq G_d(\alpha) \int_{Q_T^2} \int_{(a\wedge b)^{-\frac{1}{\alpha}}\, r<|z|<r_1} |v-u|\, \theta_\nu(t-s) \cdot \{\rho_\epsilon(x-y+h(z)) - \rho_\epsilon(x-y)\}\, |z|^{-d-\alpha}\,\mathrm{d}z\,\mathrm{d}w \tag{6.9}$$

with $h(z) := (b^{\frac{1}{\alpha}} - a^{\frac{1}{\alpha}})\, z$. After a Taylor expansion of ρ_ϵ with integral remainder, we find that

$$\mathcal{E}_{3,2,3} \leq G_d(\alpha) \int_{Q_T^2} \int_{(a\wedge b)^{-\frac{1}{\alpha}}\, r<|z|<r_1} \int_0^1 (1-\tau)\, |v-u|\, \theta_\nu(t-s)\, |z|^{-d-\alpha}$$
$$\cdot \nabla^2 \rho_\epsilon\,(x-y+\tau\, h(z)) \cdot h(z)^2\, \mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}w.$$

Remember that the integral of the first order term in z is zero by symmetry. By a standard argument, $|v-u|$ is BV in y as composition of a BV with a Lipschitz function (cf. e.g. [11]). Hence, by an integration by parts with respect to y,

$$\mathcal{E}_{3,2,3} \leq G_d(\alpha) \int_0^T \int_{Q_T} \int_{(a\wedge b)^{-\frac{1}{\alpha}}\, r<|z|<r_1} \int_0^1 (1-\tau)\, \theta_\nu(t-s)\, |z|^{-d-\alpha}$$
$$\cdot \left\{ \int_{\mathbb{R}^d} \nabla \rho_\epsilon\,(x-y+\tau\, h(z)) \cdot h(z)\, h(z) \cdot \mathrm{d}\nabla_y |v(x,t) - u(\cdot,s)|(y) \right\}$$
$$\mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.$$

We use the notation $\mathrm{d}\nabla_y|v(x,t)-u(\cdot,s)|(y)$ in case $\nabla_y|v-u|$ is a measure. Then $|\nabla_y|v-u|| \leq |\nabla u|$ in the sense of measures since y is the space variable of u. It follows that

$$\mathcal{E}_{3,2,3} \leq G_d(\alpha) \int_0^T \int_{Q_T} \int_{|z|<r_1} \int_0^1 (1-\tau)\, \theta_\nu(t-s)\, |z|^{-d-\alpha}\, |h(z)|^2$$
$$\cdot \left\{ \int_{\mathbb{R}^d} |\nabla \rho_\epsilon\,(x-y+\tau\, h(z))|\, \mathrm{d}|\nabla u(\cdot,s)|(y) \right\} \mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s.$$

By Fubini[4] we integrate with respect to (x,t) before (y,s), and then we use that $h(z) = (b^{\frac{1}{\alpha}} - a^{\frac{1}{\alpha}})\, z$ and $\int |\nabla\rho_\epsilon| = \frac{1}{\epsilon}\int|\nabla\rho| = \frac{C(d)}{\epsilon}$ (by (4.1)), to see that

$$\begin{aligned} \mathcal{E}_{3,2,3} &\leq G_d(\alpha) \int_0^T \int_{|z|<r_1} \int_0^1 (1-\tau)\, |z|^{-d-\alpha} \\ &\qquad \cdot |h(z)|^2\, |u(\cdot,s)|_{BV}\, \mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}s \int_{Q_T} \theta_\nu |\nabla\rho_\epsilon|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(d)\, \frac{G_d(\alpha)}{2-\alpha}\, |u|_{L^1(0,T;BV)}\, (a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}})^2\, \frac{r_1^{2-\alpha}}{\epsilon}. \end{aligned} \tag{6.10}$$

[4]applied for fixed s, so that $\mathrm{d}|\nabla u(\cdot,s)|(y)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t$ is a tensor product of σ-finite measures!

6. *Estimate of $\mathcal{E}_3$ – conclusion in the linear case.* By the estimates of **4** and **5**, (6.6), (6.8), etc., we can then conclude that

$$
\begin{aligned}
\mathcal{E}_3 &\le \mathcal{E}_{3,1} + \mathcal{E}_{3,2,1} + \mathcal{E}_{3,2,2} + \mathcal{E}_{3,2,3} \\
&\le C_\epsilon \left(\|u\|_{L^1(Q_T)} + \|v\|_{L^1(Q_T)}\right) r^{2-\alpha} \\
&\quad + C(d)\, G_d(\alpha) \left\{ \frac{1}{\alpha} \|u\|_{L^1(Q_T)} \frac{|a-b|}{a \vee b} r_2^{-\alpha} \right. \\
&\quad + |u|_{L^1(0,T;BV)} \, |a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}| \int_{r_1<|z|<r_2} \frac{|z| \, \mathrm{d}z}{|z|^{d+\alpha}} \\
&\quad \left. + \frac{1}{2-\alpha} |u|_{L^1(0,T;BV)} \, (a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}})^2 \frac{r_1^{2-\alpha}}{\epsilon} \right\},
\end{aligned}
\tag{6.11}
$$

for arbitrary $r_2 \ge r_1 > \underline{\Lambda}^{-\frac{1}{\alpha}} r$. Note that the $\frac{1}{a \vee b}$-term has to be handled with care since it could be large in the general case when φ' and ψ' can be degenerate.

We conclude the estimate of $\mathcal{E}_3$ by choosing the values of constants r_1 and r_2. In the critical case where $\alpha = 1$, we take $r_1 = T \wedge 1$ and $r_2 = 1 \vee \frac{\|u_0\|_{L^1}}{(a \vee b)\,|u_0|_{BV}}$. Notice that if $|u_0|_{BV} = 0$, then $u_0 \equiv 0$ as constant integrable function, and (3.4) reduces to (2.6). In the sequel, we thus assume without loss of generality that $|u_0|_{BV} \neq 0$. Note then that $+\infty > r_2 \ge r_1 = T \wedge 1 > \underline{\Lambda}^{-\frac{1}{\alpha}} r$ for r small enough ($r \downarrow 0$ in the end). By easy computation and Lemma B.1 of the Appendix,

$$
\begin{aligned}
|a-b| \int_{r_1<|z|<r_2} \frac{|z| \, \mathrm{d}z}{|z|^{d+1}} &= C\, |a-b| \, (\ln r_2 - \ln r_1) \\
\le C\, |a-b| &\left\{ |\ln T| + \mathbf{1}_{\frac{\|u_0\|_{L^1}}{|u_0|_{BV}} > 1} \ln \frac{\|u_0\|_{L^1}}{|u_0|_{BV}} + (-\ln(a \vee b))^+ \right\} \\
\le C &\left\{ \Big(1 + |\ln T| + |u_0|_{BV}^{-1} \, \mathrm{E}_1(u_0)\Big) |a-b| + |a \ln a - b \ln b| \right\},
\end{aligned}
$$

where $C = C(d)$ and where $\mathrm{E}_1(u_0)$ is defined in (3.3). We finally deduce from (6.11) that, when $\alpha = 1$,

$$
\begin{aligned}
\mathcal{E}_3 &\le C_\epsilon \left(\|u\|_{L^1(Q_T)} + \|v\|_{L^1(Q_T)}\right) r \\
&\quad + C(d) \left\{ \|u\|_{L^1(Q_T)} \frac{|u_0|_{BV}}{\|u_0\|_{L^1}} |a-b| \right. \\
&\quad + \left(1 + |\ln T| + |u_0|_{BV}^{-1} \, \mathrm{E}_1(u_0)\right) |u|_{L^1(0,T;BV)} \, |a-b| \\
&\quad + |u|_{L^1(0,T;BV)} \, |a \ln a - b \ln b| \\
&\quad \left. + T\, |u|_{L^1(0,T;BV)} \, (a-b)^2 \frac{1}{\epsilon} \right\},
\end{aligned}
\tag{6.12}
$$

for all $T \wedge 1 > \underline{\Lambda}^{-1} r$. To divide by $\|u_0\|_{L^1}$, we have assumed without loss of generality that we are not in the case where $\|u_0\|_{L^1} = 0$, for which (3.4) also reduces to (2.6).

When $\alpha > 1$, we simply choose $r_2 = +\infty$ in (6.11) and we get

$$
\begin{aligned}
\mathcal{E}_3 &\le C_\epsilon \left(\|u\|_{L^1(Q_T)} + \|v\|_{L^1(Q_T)}\right) r^{2-\alpha} \\
&\quad + C(d)\, G_d(\alpha) \left\{ \frac{1}{\alpha - 1} |u|_{L^1(0,T;BV)} \, |a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}| \, r_1^{1-\alpha} \right. \\
&\quad \left. + \frac{1}{2-\alpha} |u|_{L^1(0,T;BV)} \, (a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}})^2 \frac{r_1^{2-\alpha}}{\epsilon} \right\},
\end{aligned}
\tag{6.13}
$$

for all $r_1 > \underline{\Lambda}^{-\frac{1}{\alpha}} r$.

7. *Estimate of $\mathcal{E}_3$ - the general case via linearization.* The idea is now to reduce to the linear case in step **4** by freezing the "diffusion coefficients" $\varphi'(\xi)$ and $\psi'(\xi)$. To do so, we introduce the function

$$\chi_a^b(\xi) := \operatorname{sgn}(b-a)\,\mathbf{1}_{(a\wedge b, a\vee b)}(\xi), \tag{6.14}$$

for $\xi, a, b \in \mathbb{R}$. By (6.2), we then find that

$$\begin{aligned}\mathcal{E}_3 &= G_d(\alpha)\int_{Q_T^2}\int_{|z|>r}\operatorname{sgn}(v-u)\\ &\quad\cdot\frac{\int_{v(x,t)}^{v(x+z,t)}\psi'(\xi)\,\mathrm{d}\xi-\int_{u(y,s)}^{u(y+z,s)}\varphi'(\xi)\,\mathrm{d}\xi}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w\\ &= G_d(\alpha)\int_{Q_T^2}\int_{|z|>r}\int\operatorname{sgn}(v-u)\\ &\quad\cdot\frac{\chi_{v(x,t)}^{v(x+z,t)}(\xi)\,\psi'(\xi)-\chi_{u(y,s)}^{u(y+z,s)}(\xi)\,\varphi'(\xi)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}\xi\,\mathrm{d}z\,\mathrm{d}w.\end{aligned} \tag{6.15}$$

Let us notice that this integral is well-defined, since e.g. $\int|\chi_a^b(\xi)|\,\mathrm{d}\xi = |b-a|$ and, φ' and ψ' are assumed bounded by (6.1).

For each $\delta>0$, we define a regularized version of $\mathcal{E}_3$ as

$$\begin{aligned}\mathcal{E}_3(\delta) &:= G_d(\alpha)\int_{Q_T^2}\int_{|z|>r}\int\int\operatorname{sgn}(v-u)\\ &\quad\cdot\frac{\chi_{v(x,t)}^{v(x+z,t)}(\zeta)\,\psi'(\xi)-\chi_{u(y,s)}^{u(y+z,s)}(\zeta)\,\varphi'(\xi)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\omega_\delta(\xi-\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\xi\,\mathrm{d}z\,\mathrm{d}w,\end{aligned} \tag{6.16}$$

where the approximate unit $\omega_\delta(\xi) := \frac{1}{\delta}\,\omega\left(\frac{\xi}{\delta}\right)$, and

$$\omega\in C_b^\infty\cap L^1(\mathbb{R}),\quad \omega>0,\quad \int\omega=1.$$

For each $\zeta,\xi\in\mathbb{R}$, let $\Omega_\xi(\zeta) := \int_{-\infty}^{\zeta}\omega_\delta(\xi-w)\,\mathrm{d}w-\int_{-\infty}^{0}\omega_\delta(\xi-w)\,\mathrm{d}w$, and note that

$$\int\chi_{v(x,t)}^{v(x+z,t)}(\zeta)\,\omega_\delta(\xi-\zeta)\,\mathrm{d}\zeta=\int_{v(x,t)}^{v(x+z,t)}\Omega_\xi'(\zeta)\,\mathrm{d}\zeta=\Omega_\xi(v(x+z,t))-\Omega_\xi(v(x,t)).$$

Moreover, $\operatorname{sgn}(v-u)=\operatorname{sgn}(\Omega_\xi(v)-\Omega_\xi(u))$ since $\Omega_\xi(\cdot)$ is increasing, and since $\Omega_\xi(\cdot)$ is smooth and vanishes at zero, $\Omega_\xi(u)$ and $\Omega_\xi(v)$ have similar boundedness, integrability, and regularity properties as u and v. It follows that

$$\begin{aligned}&\mathcal{E}_3(\delta)\\ &= G_d(\alpha)\int\int_{Q_T^2}\int_{|z|>r}\operatorname{sgn}(\Omega_\xi(v)-\Omega_\xi(u))\,\phi^{\epsilon,\nu}\\ &\quad\cdot\frac{\psi'(\xi)\,(\Omega_\xi(v(x+z,t))-\Omega_\xi(v))-\varphi'(\xi)\,(\Omega_\xi(u(y+z,s))-\Omega_\xi(u))}{|z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\xi.\end{aligned}$$

This integrand has similar form and properties as the one in (6.5) for fixed ξ!

We continue in the critical case when $\alpha = 1$. We argue as in step **4** with $a = \varphi'(\xi)$ and $b = \psi'(\xi)$. By (6.12) we get that for all $T \wedge 1 > \underline{\Lambda}^{-1}\, r$,

$$
\begin{aligned}
\mathcal{E}_3(\delta) \leq & \int C_\epsilon \left(\|\Omega_\xi(u)\|_{L^1(Q_T)} + \|\Omega_\xi(v)\|_{L^1(Q_T)} \right) r \,\mathrm{d}\xi \\
& + C(d) \int \Big\{ \|\Omega_\xi(u)\|_{L^1(Q_T)} \frac{|u_0|_{BV}}{\|u_0\|_{L^1}} \, |\varphi'(\xi) - \psi'(\xi)| \\
& + \left(1 + |\ln T| + |u_0|_{BV}^{-1} \ \mathrm{E}_1(u_0)\right) |\Omega_\xi(u)|_{L^1(0,T;BV)} \, |\varphi'(\xi) - \psi'(\xi)| \\
& + |\Omega_\xi(u)|_{L^1(0,T;BV)} \, |\varphi'(\xi) \, \ln \varphi'(\xi) - \psi'(\xi) \, \ln \psi'(\xi)| \\
& + T \, |\Omega_\xi(u)|_{L^1(0,T;BV)} \, (\varphi'(\xi) - \psi'(\xi))^2 \, \frac{1}{\epsilon} \Big\} \,\mathrm{d}\xi \\
\leq & \; C_\epsilon \, r \int \|\Omega_\xi(u)\|_{L^1(Q_T)} + \|\Omega_\xi(v)\|_{L^1(Q_T)} \,\mathrm{d}\xi \\
& + C(d) \Big\{ A \, \frac{|u_0|_{BV}}{\|u_0\|_{L^1}} \, \|\varphi' - \psi'\|_\infty \\
& + \left(1 + |\ln T| + |u_0|_{BV}^{-1} \ \mathrm{E}_1(u_0)\right) B \, \|\varphi' - \psi'\|_\infty \\
& + B \, \|\varphi' \, \ln \varphi' - \psi' \, \ln \psi'\|_\infty \\
& + T \, B \, \|\varphi' - \psi'\|_\infty^2 \, \frac{1}{\epsilon} \Big\},
\end{aligned}
$$

with $A = \int \|\Omega_\xi(u)\|_{L^1(Q_T)} \,\mathrm{d}\xi$, $B = \int |\Omega_\xi(u)|_{L^1(0,T;BV)} \,\mathrm{d}\xi$, and

$$
\|\varphi' - \psi'\|_\infty = \operatorname*{ess\,sup}_{I(u_0)} |\varphi' - \psi'|.
$$

The supremum above can be taken only on $I(u_0)$, since φ' and ψ' are assumed to vanish outside this interval by (6.1). Note also that $C_\epsilon = C(d, \alpha, \epsilon, \underline{\Lambda}, \overline{\Lambda})$ can be chosen independent of $\varphi'(\xi)$ and $\psi'(\xi)$ as discussed below (6.7). A standard argument, see Appendix A, then reveals that

$$
\int \|\Omega_\xi(u)\|_{L^1(Q_T)} \,\mathrm{d}\xi = \|u\|_{L^1(Q_T)}, \tag{6.17}
$$

$$
\int |\Omega_\xi(u)|_{L^1(0,T;BV)} \,\mathrm{d}\xi = |u|_{L^1(0,T;BV)}, \tag{6.18}
$$

and hence that $A \leq T \, \|u_0\|_{L^1}$ and $B \leq T \, |u_0|_{BV}$ by (2.6).

By standard computations given in Appendix A,

$$
\lim_{\delta \downarrow 0} \mathcal{E}_3(\delta) = \mathcal{E}_3, \tag{6.19}
$$

and it follows after going to the limit in the estimate above, that

$$
\begin{aligned}
\mathcal{E}_3 \leq & \; C_\epsilon \, r \\
& + C(d) \Big\{ T \ \mathrm{E}_1(u_0) \, \|\varphi' - \psi'\|_\infty \\
& + T \, (1 + |\ln T|) \, |u_0|_{BV} \, \|\varphi' - \psi'\|_\infty \\
& + T \, |u_0|_{BV} \, \|\varphi' \, \ln \varphi' - \psi' \, \ln \psi'\|_\infty \\
& + T^2 \, |u_0|_{BV} \, \|\varphi' - \psi'\|_\infty^2 \, \frac{1}{\epsilon} \Big\},
\end{aligned} \tag{6.20}
$$

for all $T \wedge 1 > \underline{\Lambda}^{-1}\, r$ when $\alpha = 1$.

When $\alpha > 1$, similar arguments using (6.13) show that for all $r_1 > \underline{\Lambda}^{-\frac{1}{\alpha}} r$,

$$\begin{aligned}\mathcal{E}_3 \leq C_\epsilon\, r^{2-\alpha} \\ + C(d) \Big\{ \frac{G_d(\alpha)}{\alpha-1}\, T\, |u_0|_{BV}\, \|(\varphi')^{\frac{1}{\alpha}} - (\psi')^{\frac{1}{\alpha}}\|_\infty\, r_1^{1-\alpha} \\ + \frac{G_d(\alpha)}{2-\alpha}\, T\, |u_0|_{BV}\, \|(\varphi')^{\frac{1}{\alpha}} - (\psi')^{\frac{1}{\alpha}}\|_\infty^2\, \frac{r_1^{2-\alpha}}{\epsilon} \Big\}.\end{aligned} \tag{6.21}$$

8. *Conclusion.* We have to insert the estimates of the three preceding steps into (6.2). Let us begin by the case where $\alpha = 1$. By (6.3), (6.4) and (6.20),

$$\begin{aligned}\|u(\cdot,T) - v(\cdot,T)\|_{L^1} \leq 2\,(m_u(\nu) \vee m_v(\nu)) + C_\epsilon\, r \\ + T\, |u_0|_{BV}\, \|f' - g'\|_\infty \\ + C(d) \Big\{ |u_0|_{BV}\, \epsilon \\ + T\ \mathrm{E}_1(u_0)\, \|\varphi' - \psi'\|_\infty \\ + T\,(1 + |\ln T|)\, |u_0|_{BV}\, \|\varphi' - \psi'\|_\infty \\ + T\, |u_0|_{BV}\, \|\varphi'\, \ln \varphi' - \psi'\, \ln \psi'\|_\infty \\ + T^2\, |u_0|_{BV}\, \|\varphi' - \psi'\|_\infty^2\, \frac{1}{\epsilon} \Big\},\end{aligned}$$

for all $r, \epsilon > 0$ and $T > \nu > 0$ such that $T \wedge 1 > \underline{\Lambda}^{-1} r$. We complete the proof by sending r and ν to zero, and taking $\epsilon = T\, \|\varphi' - \psi'\|_\infty$.

When $\alpha > 1$, we find using (6.21) that

$$\begin{aligned}\|u(\cdot,T) - v(\cdot,T)\|_{L^1} \leq 2\,(m_u(\nu) \vee m_v(\nu)) + C_\epsilon\, r^{2-\alpha} \\ + T\, |u_0|_{BV}\, \|f' - g'\|_\infty \\ + C(d) \Big\{ |u_0|_{BV}\, \epsilon \\ + \frac{G_d(\alpha)}{\alpha-1}\, T\, |u_0|_{BV}\, \|(\varphi')^{\frac{1}{\alpha}} - (\psi')^{\frac{1}{\alpha}}\|_\infty\, r_1^{1-\alpha} \\ + \frac{G_d(\alpha)}{2-\alpha}\, T\, |u_0|_{BV}\, \|(\varphi')^{\frac{1}{\alpha}} - (\psi')^{\frac{1}{\alpha}}\|_\infty^2\, \frac{r_1^{2-\alpha}}{\epsilon} \Big\},\end{aligned}$$

for all $r, \epsilon > 0$, $T > \nu > 0$ and $r_1 > \underline{\Lambda}^{-\frac{1}{\alpha}} r$. We conclude by choosing $\epsilon = T^{\frac{1}{\alpha}}\, \|(\varphi')^{\frac{1}{\alpha}} - (\psi')^{\frac{1}{\alpha}}\|_\infty$ and $r_1 = T^{\frac{1}{\alpha}}$. The proof of Theorem 3.1 is complete. □

Remark 6.2.
(1) From the proof, we find that $C \leq C(d) \left(1 + \frac{G_d(\alpha)}{\alpha-1} + \frac{G_d(\alpha)}{2-\alpha}\right)$ in (3.4) when $\alpha > 1$. By (2.2), $\lim_{\alpha \uparrow 2} C(d, \alpha)$ is finite and only depends on d.
(2) In particular, $C \leq C(d) \left(1 + \frac{G_d(\alpha)}{\alpha-1} + \frac{G_d(\alpha)}{2-\alpha}\right)$ when $\alpha > 1$ also in (3.6).

6.3. **Proof of Theorem 3.8.** Here no linearization procedure is needed since $\varphi = \psi$. The new difficulty comes from the fact that the two Lévy measures are different. A key idea is to change variables to work with only one measure.

Proof of Theorem 3.8. We argue as in the preceding proof with $u = u^\alpha$ and $v = u^\beta$, i.e. $(u_0, f, \varphi) = (v_0, g, \psi)$. To simplify references to similar computations, we still use the letters u and v for a while.

1. *Applying Kuznetsov, initial estimates.* As in step **1** in the proof of Theorem 3.1, we apply Lemma 4.1 and estimate the $\mathcal{L}_r$-terms. We obtain estimates similar to

(6.2), and (6.4), and conclude that for all $\alpha,\beta\in(0,2)$, $r,\epsilon>0$ and $T>\nu>0$,

$$\begin{aligned}&\|u(\cdot,T)-v(\cdot,T)\|_{L^1}\\ &\leq C(d)\,|u_0|_{BV}\,\epsilon+2\,(m_u(\nu)\vee m_v(\nu))+C_\epsilon\,(r^{2-\alpha}+r^{2-\beta})\\ &\quad+\underbrace{\int_{Q_T^2}\operatorname{sgn}(v-u)\,\left(\mathcal{L}^{\beta,r}[\varphi(v(\cdot,t))](x)-\mathcal{L}^{\alpha,r}[\varphi(u(\cdot,s))](y)\right)\phi^{\epsilon,\nu}\,\mathrm{d}w}_{=:\mathcal{E}_3}.\end{aligned}\tag{6.22}$$

The new $r^{2-\beta}$-term comes from the new $\mathcal{L}_r^\beta$-term in the estimate corresponding to $\mathcal{E}_2$. Note that the terms in $\mathcal{E}_3$ only involve one function φ, but different α,β. Most of the remaining proof consists in estimating $\mathcal{E}_3$.

2. *Change of variables and first estimate of $\mathcal{E}_3$.* We perform several changes of variables to move the differences between $\mathcal{L}^{\alpha,r}$ and $\mathcal{L}^{\beta,r}$ from the Lévy measure to the z-translations. This is similar in spirit to what we did in the preceding proof to obtain (6.6). First we let $\tilde z=|z|^{\gamma^{-1}-1}\,z$ ($\gamma>0$), and note that $\mathrm{d}\tilde z=\gamma^{-1}|z|^{d\,(\gamma^{-1}-1)}\,\mathrm{d}z$[5] so that $|z|^{-d-\beta}\,\mathrm{d}z=\gamma\,|\tilde z|^{-d-\beta\,\gamma}\,\mathrm{d}\tilde z$. Take $\gamma=\gamma_\beta:=\sqrt{\frac{\alpha}{\beta}}$, and check that $-d-\beta\,\gamma=-d-\sqrt{\alpha\,\beta}$ and

$$\mathcal{L}^{\beta,r}[\varphi(v(\cdot,t))](x)=G_d(\beta)\,\gamma_\beta\int_{|z|>r^{\gamma_\beta^{-1}}}\frac{\varphi\left(v(x+|z|^{\gamma_\beta-1}\,z,t)\right)-\varphi(v(x,t))}{|z|^{d+\sqrt{\alpha\,\beta}}}\,\mathrm{d}z.$$

Then we use the change of variable $z\mapsto(G_d(\beta)\gamma_\beta)^{\frac{1}{\sqrt{\alpha\beta}}}\,z$ and get that

$$\mathcal{L}^{\beta,r}[\varphi(v(\cdot,t))](x)=\int_{|z|>r_\beta}\frac{\varphi\left(v(x+c_\beta\,|z|^{\gamma_\beta-1}\,z,t)\right)-\varphi(v(x,t))}{|z|^{d+\sqrt{\alpha\,\beta}}}\,\mathrm{d}z,$$

where $c_\beta:=(G_d(\beta)\,\gamma_\beta)^{\frac{1}{\beta}}>0$ and $r_\beta:=(G_d(\beta)\,\gamma_\beta)^{-\frac{1}{\sqrt{\alpha\,\beta}}}\,r^{\gamma_\beta^{-1}}>0$. Similar computations for u show that

$$\mathcal{L}^{\alpha,r}[\varphi(u(\cdot,s))](y)=\int_{|z|>r_\alpha}\frac{\varphi(u\left(y+c_\alpha\,|z|^{\gamma_\alpha-1}\,z,s)\right)-\varphi(u(y,s))}{|z|^{d+\sqrt{\alpha\,\beta}}}\,\mathrm{d}z,$$

where $\gamma_\alpha:=\sqrt{\frac{\beta}{\alpha}}$, $\quad c_\alpha:=(G_d(\alpha)\,\gamma_\alpha)^{\frac{1}{\alpha}}$ and $r_\alpha:=(G_d(\alpha)\,\gamma_\alpha)^{-\frac{1}{\sqrt{\alpha\,\beta}}}\,r^{\gamma_\alpha^{-1}}$. Hence

$$\begin{aligned}\mathcal{E}_3&=\int_{Q_T^2}\int_{r_\alpha\wedge r_\beta<|z|<r_\alpha\vee r_\beta}\cdots\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}\\ &\quad+\int_{Q_T^2}\int_{|z|>r_\alpha\vee r_\beta}\operatorname{sgn}(v-u)\\ &\quad\cdot\frac{\left\{\varphi\left(v(x+c_\beta\,|z|^{\gamma_\beta-1}\,z,t)\right)-\varphi\left(u(y+c_\alpha\,|z|^{\gamma_\alpha-1}\,z,s)\right)\right\}-\{\varphi(v)-\varphi(u)\}}{|z|^{d+\sqrt{\alpha\,\beta}}}\\ &\quad\cdot\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w\\ &=:\mathcal{E}_{3,1}+\mathcal{E}_{3,2},\end{aligned}$$

where the integrand of $\mathcal{E}_{3,1}$ only contains either u-terms or v-terms. As in the preceding proof, cf. (6.6) and (6.7), we find that

$$\mathcal{E}_{3,1}\leq C_\epsilon\,o_r(1),\tag{6.23}$$

[5]Indeed, $\mathrm{d}\tilde z=F(z)\,\mathrm{d}z$ for $F(z)=|\det(D(|z|^{\gamma^{-1}-1}\,z))|$ and

$$D\left(|z|^{\gamma^{-1}-1}\,z\right)=(\gamma^{-1}-1)\,|z|^{\gamma^{-1}-3}\,z\otimes z+|z|^{\gamma^{-1}-1}\,\mathrm{Id}.$$

Hence F is positive, $F(\lambda\,z)=|\lambda|^{d\,(\gamma^{-1}-1)}\,F(z)$ for all $\lambda\in\mathbb{R}$, and radial since

$$F(R\,e)=\left|\det\left((\gamma^{-1}-1)\,R\,e\,(R\,e)^t+R\,R^t\right)\right|=\left|\det\left(R\,((\gamma^{-1}-1)\,e\,e^t+\mathrm{Id})\,R^t\right)\right|=\gamma^{-1},$$

for all orthogonal matrices $R\in\mathbb{R}^{d\times d}$ and column vectors e of the canonical basis.

where $o_r(1) = \max_{\sigma=\alpha,\beta}(r_\alpha \vee r_\beta)^{2\gamma_\sigma - \sqrt{\alpha\beta}} \to 0$ as $r \downarrow 0$ and α, β are fixed.

Most of the remaining proof consists in estimating $\mathcal{E}_{3,2}$. Before continuing, let us list the following properties that will be needed: for any $d \in \mathbb{N}$ and $\lambda \in (0,2)$,

$$\begin{cases} \lim_{\alpha,\beta\to\lambda} \gamma_\alpha = \lim_{\alpha,\beta\to\lambda} \gamma_\beta = 1, \\ \lim_{\alpha,\beta\to\lambda} c_\alpha = \lim_{\alpha,\beta\to\lambda} c_\beta = G_d(\lambda)^{\frac{1}{\lambda}} > 0, \\ \lim_{\alpha,\beta\to\lambda} \frac{|\gamma_\alpha-\gamma_\beta|}{|\alpha-\beta|} = \frac{1}{\lambda}, \\ \limsup_{\alpha,\beta\to\lambda} \frac{|c_\alpha-c_\beta|}{|\alpha-\beta|} < +\infty. \end{cases} \tag{6.24}$$

In particular, the limsup is a constant of the form $C = C(d,\lambda)$ (note also that this limsup is in fact a limit but this is will not be needed). These properties are immediate consequences of (2.2).

3. *First estimate of $\mathcal{E}_{3,2}$.* We introduce parameters $r_2 \geq r_1 > 0$. Notice that $r_1 > r_\alpha \vee r_\beta$ for sufficiently small r ($r \downarrow 0$ in the next step). Let us define

$$\begin{aligned} \mathcal{E}_{3,2} = \sum_{i=1}^{3} \mathcal{E}_{3,2,i} := \sum_{i=1}^{3} \int_{Q_T^2} \int_{|z|\in I_i} \operatorname{sgn}(v-u) \\ \cdot \frac{\{\varphi\left(v(x + c_\beta\,|z|^{\gamma_\beta-1}\,z,t)\right) - \varphi\left(u(y + c_\alpha\,|z|^{\gamma_\alpha-1}\,z,s)\right)\} - \{\varphi(v)-\varphi(u)\}}{|z|^{d+\sqrt{\alpha\beta}}} \\ \cdot \phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w \end{aligned}$$

for $I_1 = (r_2, +\infty)$, $I_2 = (r_1, r_2)$ and $I_3 = (r_\alpha \vee r_\beta, r_1)$. An application of Lemma 6.1 with $c = \tilde{c} = c_\beta$ and $\gamma = \tilde{\gamma} = \gamma_\beta$, shows that

$$\begin{aligned} \mathcal{E}_{3,2,i} \leq \int_{Q_T^2} \int_{|z|\in I_i} \operatorname{sgn}(v-u)\,\phi^{\epsilon,\nu} \\ \cdot \frac{\varphi\left(u(y + c_\beta\,|z|^{\gamma_\beta-1}\,z,s)\right) - \varphi\left(u(y + c_\alpha\,|z|^{\gamma_\alpha-1}\,z,s)\right)}{|z|^{d+\sqrt{\alpha\beta}}}\,\mathrm{d}z\,\mathrm{d}w. \end{aligned} \tag{6.25}$$

We now estimate these terms.

Let us begin with $\mathcal{E}_{3,2,1}$. Going back to the original variables, $c_\alpha\,|z|^{\gamma_\alpha-1}z \mapsto z$,

$$\begin{aligned} &\int_{Q_T^2} \int_{|z|>r_2} \operatorname{sgn}(v-u) \frac{\varphi\left(u(y + c_\alpha\,|z|^{\gamma_\alpha-1}\,z,s)\right)}{|z|^{d+\sqrt{\alpha\beta}}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w \\ &= G_d(\alpha) \int_{Q_T^2} \int_{|z|>c_\alpha\,r_2^{\gamma_\alpha}} \operatorname{sgn}(v-u) \frac{\varphi\left(u(y+z,s)\right)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}\,\mathrm{d}z\,\mathrm{d}w. \end{aligned}$$

Let us continue by assuming that $c_\alpha\,r_2^{\gamma_\alpha} \geq c_\beta\,r_2^{\gamma_\beta}$. By the above identity and a similar one for the β-term, we then find that

$$\begin{aligned} &\mathcal{E}_{3,2,1} \\ &\leq \int_{Q_T^2} \int_{|z|>c_\alpha\,r_2^{\gamma_\alpha}} \operatorname{sgn}(v-u)\,\varphi(u(y+z,s))\,\phi^{\epsilon,\nu} \left(\frac{G_d(\beta)}{|z|^{d+\beta}} - \frac{G_d(\alpha)}{|z|^{d+\alpha}}\right) \mathrm{d}z\,\mathrm{d}w \\ &\quad + G_d(\beta) \int_{Q_T^2} \int_{c_\beta\,r_2^{\gamma_\beta}<|z|<c_\alpha\,r_2^{\gamma_\alpha}} \operatorname{sgn}(v-u)\,\varphi(u(y+z,s))\,\phi^{\epsilon,\nu}\,\frac{\mathrm{d}z\,\mathrm{d}w}{|z|^{d+\beta}}. \end{aligned}$$

By (1.5) and (2.6), $\|\varphi(u)\|_{L^1(Q_T)} \leq M\,\|u_0\|_{L^1}$ for $M = T\,\operatorname{ess\,sup}_{I(u_0)}|\varphi'|$, and then by Fubini,

$$\begin{aligned} &\mathcal{E}_{3,2,1} \\ &\leq M\,\|u_0\|_{L^1} \left\{ \int_{|z|>c_\alpha\,r_2^{\gamma_\alpha}} \left|\frac{G_d(\beta)}{|z|^{d+\beta}} - \frac{G_d(\alpha)}{|z|^{d+\alpha}}\right| \mathrm{d}z + G_d(\beta) \int_{c_\beta\,r_2^{\gamma_\beta}<|z|<c_\alpha\,r_2^{\gamma_\alpha}} \frac{\mathrm{d}z}{|z|^{d+\beta}} \right\}. \end{aligned}$$

Doing the same reasoning when $c_\alpha\, r_2^{\gamma_\alpha} < c_\beta\, r_2^{\gamma_\beta}$ and taking the maximum, we finally get

$$\begin{aligned}(6.26)\qquad \mathcal{E}_{3,2,1} \leq M\,\|u_0\|_{L^1} \int_{|z|>(c_\alpha\, r_2^{\gamma_\alpha})\wedge(c_\beta\, r_2^{\gamma_\beta})} \left|\frac{G_d(\beta)}{|z|^{d+\beta}} - \frac{G_d(\alpha)}{|z|^{d+\alpha}}\right| \mathrm{d}z \\ + C(d)\, M\,\|u_0\|_{L^1} \max_{\sigma=\alpha,\beta} \int_{|z|\in \mathrm{co}\{c_\alpha\, r_2^{\gamma_\alpha}, c_\beta\, r_2^{\gamma_\beta}\}} \frac{\mathrm{d}z}{|z|^{d+\sigma}},\end{aligned}$$

where $C(d) = \max_{[0,2]} G_d$ is finite by (2.2) and from now on $\mathrm{co}\{a,b\}$ designs the interval $(a\wedge b, a\vee b)$.

Next, by (1.5) and (2.6), $|\varphi(u)|_{L^1(0,T;BV)} \leq M\,|u_0|_{BV}$. Hence by integrating first with respect to y in (6.25), we find that

$$(6.27)\qquad \mathcal{E}_{3,2,2} \leq M\,|u_0|_{BV} \int_{r_1<|z|<r_2} |c_\alpha\,|z|^{\gamma_\alpha} - c_\beta\,|z|^{\gamma_\beta}|\,\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}.$$

Finally, by Lemma 6.1

$$\begin{aligned}\mathcal{E}_{3,2,3} \leq \int_{Q_T^2}\int_{r_\alpha\vee r_\beta<|z|<r_1} |\varphi(v)-\varphi(u)|\,\theta_\nu(t-s) \\ \cdot\{\rho_\epsilon(x-y+h(z)) - \rho_\epsilon(x-y)\}\,\frac{\mathrm{d}z\,\mathrm{d}w}{|z|^{d+\sqrt{\alpha\,\beta}}},\end{aligned}$$

with $h(z) := (c_\beta\,|z|^{\gamma_\beta-1} - c_\alpha\,|z|^{\gamma_\alpha-1})\,z$. This estimate is similar to (6.9), but with a new displacement, new functions $\varphi(u)$ and $\varphi(v)$, and the new power $\sqrt{\alpha\,\beta}$. By arguing as before, we find that

$$\mathcal{E}_{3,2,3} \leq \frac{C(d)}{\epsilon}\int_0^T\int_{|z|<r_1}\int_0^1 (1-\tau)\,|z|^{-d-\sqrt{\alpha\,\beta}}\,|h(z)|^2\,|\varphi(u(\cdot,s))|_{BV}\,\mathrm{d}\tau\,\mathrm{d}z\,\mathrm{d}s,$$

instead of (6.10). Since $|\varphi(u)|_{L^1(0,T;BV)} \leq M\,|u_0|_{BV}$, we get that

$$(6.28)\qquad \mathcal{E}_{3,2,3} \leq C(d)\,M\,|u_0|_{BV}\,\frac{1}{\epsilon}\int_{|z|<r_1} |c_\beta\,|z|^{\gamma_\beta} - c_\alpha\,|z|^{\gamma_\alpha}|^2\,\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}.$$

4. *The general estimate.* Let us resume the preceding estimates. By (6.22), (6.23), (6.26), (6.27), (6.28) and the fact that $\mathcal{E}_3 = \mathcal{E}_{3,1} + \mathcal{E}_{3,2,1} + \mathcal{E}_{3,2,2} + \mathcal{E}_{3,2,3}$, we have proved that for all $\alpha,\beta\in(0,2)$, $\epsilon>0$, $T>\nu>0$, $r_2\geq r_1>0$ and $r>0$ small enough,

$$\begin{aligned}&\|u^\alpha(\cdot,T) - u^\beta(\cdot,T)\|_{L^1} \\ &\leq 2\,(m_u(\nu)\vee m_v(\nu)) + C_\epsilon\,(r^{2-\alpha} + r^{2-\beta} + o_r(1)) \\ &\quad + C(d)\,|u_0|_{BV}\,\epsilon \\ &\quad + M\,\|u_0\|_{L^1}\int_{|z|>(c_\alpha\, r_2^{\gamma_\alpha})\wedge(c_\beta\, r_2^{\gamma_\beta})} \left|\frac{G_d(\beta)}{|z|^{d+\beta}} - \frac{G_d(\alpha)}{|z|^{d+\alpha}}\right|\mathrm{d}z \\ &\quad + C(d)\,M\,\|u_0\|_{L^1}\max_{\sigma=\alpha,\beta}\int_{|z|\in\mathrm{co}\{c_\alpha\, r_2^{\gamma_\alpha}, c_\beta\, r_2^{\gamma_\beta}\}} \frac{\mathrm{d}z}{|z|^{d+\sigma}} \\ &\quad + M\,|u_0|_{BV}\int_{r_1<|z|<r_2} |c_\alpha\,|z|^{\gamma_\alpha} - c_\beta\,|z|^{\gamma_\beta}|\,\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}} \\ &\quad + C(d)\,M\,|u_0|_{BV}\,\frac{1}{\epsilon}\int_{|z|<r_1} |c_\beta\,|z|^{\gamma_\beta} - c_\alpha\,|z|^{\gamma_\alpha}|^2\,\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}.\end{aligned}$$

Now, we pass to the limit as $r,\nu\downarrow 0$, thanks to (6.23). Next, we replace the L^1-norm at time T by the $C([0,T];L^1)$-norm, which can be done without loss of generality since $t\,\|\varphi'\|_\infty \leq T\,\|\varphi'\|_\infty = M$, for all $t\leq T$. Finally, we replace ϵ by $\epsilon\,|\alpha-\beta|$, which can also be done since ϵ is arbitrary. We deduce that for all $\alpha, \beta \in (0,2)$, $\epsilon > 0$, and $r_2 \geq r_1 > 0$,

$$\begin{aligned}
&\|u^\alpha - u^\beta\|_{C([0,T];L^1)} \\
&\leq C(d)\,|u_0|_{BV}\,\epsilon\,|\alpha-\beta| \\
&\quad + M\,\|u_0\|_{L^1} \underbrace{\int_{|z|>(c_\alpha\, r_2^{\gamma_\alpha})\wedge(c_\beta\, r_2^{\gamma_\beta})} \left|\frac{G_d(\beta)}{|z|^{d+\beta}} - \frac{G_d(\alpha)}{|z|^{d+\alpha}}\right| \mathrm{d}z}_{=:J_1} \\
&\quad + C(d)\,M\,\|u_0\|_{L^1} \underbrace{\max_{\sigma=\alpha,\beta} \int_{|z|\in \mathrm{co}\{c_\alpha\, r_2^{\gamma_\alpha},\, c_\beta\, r_2^{\gamma_\beta}\}} \frac{\mathrm{d}z}{|z|^{d+\sigma}}}_{=:J_2} \\
&\quad + M\,|u_0|_{BV} \underbrace{\int_{r_1<|z|<r_2} |c_\alpha\,|z|^{\gamma_\alpha} - c_\beta\,|z|^{\gamma_\beta}|\,\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}}_{=:J_3} \\
&\quad + \frac{C(d)\,M\,|u_0|_{BV}}{\epsilon} \underbrace{\frac{1}{|\alpha-\beta|}\int_{|z|<r_1} |c_\beta\,|z|^{\gamma_\beta} - c_\alpha\,|z|^{\gamma_\alpha}|^2\,\frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}}_{=:J_4}.
\end{aligned} \tag{6.29}$$

The rest of proof consists in estimating $\limsup_{\alpha,\beta\to\lambda} \frac{J_i}{|\alpha-\beta|}$ $(i=1,\ldots,4)$. We will use the letter C to denote various constants $C = C(d,\lambda)$.

5. *The case* $\lambda \in (1,2)$. We first let $r_2 \to +\infty$ so that $(c_\alpha\, r_2^{\gamma_\alpha}) \wedge (c_\beta\, r_2^{\gamma_\beta}) \to +\infty$, since all these coefficients are positive (cf. step **2**). We get at the limit

$$J_1 = J_2 = 0 \tag{6.30}$$

and $J_3 = \int_{|z|>r_1} \left|c_\alpha\,|z|^{-d-\sigma_\alpha} - c_\beta\,|z|^{-d-\sigma_\beta}\right| \mathrm{d}z$, with $\sigma_\alpha := \sqrt{\alpha\,\beta} - \gamma_\alpha$ and $\sigma_\beta := \sqrt{\alpha\,\beta} - \gamma_\beta$.

Let us estimate J_3. We recognize a term of the same form than in (5.2) with the new "locally Lipschitz" coefficients c_α, c_β and powers $\sigma_\alpha, \sigma_\beta$. Arguing as before,

$$J_3 \leq |c_\alpha - c_\beta| \max_{\sigma=\sigma_\alpha,\sigma_\beta} \int_{|z|>r_1} \frac{\mathrm{d}z}{|z|^{d+\sigma}} + (c_\alpha \vee c_\beta) \underbrace{\int_{|z|>r_1} \left||z|^{-d-\sigma_\alpha} - |z|^{-d-\sigma_\beta}\right| \mathrm{d}z}_{=:\tilde{J}_3},$$

where $\tilde{J}_3 \leq S_d \left|\frac{r_1^{-\sigma_\alpha}}{\sigma_\alpha} - \frac{r_1^{-\sigma_\beta}}{\sigma_\beta}\right| + 2\,S_d \left|\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_\beta}\right| \mathbf{1}_{r_1<1}$. By (6.24),

$$\limsup_{\alpha,\beta\to\lambda} \frac{J_3}{|\alpha-\beta|} \leq C \underbrace{(r_1^{1-\lambda} + \mathbf{1}_{r_1<1})}_{\leq C\, r_1^{1-\lambda} \text{ if } \lambda>1} + C \underbrace{\limsup_{\alpha,\beta\to\lambda} \frac{1}{|\alpha-\beta|} \left|\frac{r_1^{-\sigma_\alpha}}{\sigma_\alpha} - \frac{r_1^{-\sigma_\beta}}{\sigma_\beta}\right|}_{=:\tilde{\tilde{J}}_3},$$

where a Taylor expansion with integral remainder shows that

$$\tilde{\tilde{J}}_3 = \limsup_{\alpha,\beta\to\lambda} \frac{|\sigma_\alpha - \sigma_\beta|}{|\alpha-\beta|} \left|\int_0^1 \frac{\sigma_\tau\, r_1^{-\sigma_\tau}\, \ln r_1 + r_1^{-\sigma_\tau}}{\sigma_\tau^2}\, \mathrm{d}\tau\right| \leq C\, r_1^{1-\lambda}\,(1 + |\ln r_1|),$$

with $\sigma_\tau := \tau\,\sigma_\alpha + (1-\tau)\,\sigma_\beta$. We deduce the following estimate:

$$\limsup_{\alpha,\beta\to\lambda} \frac{J_3}{|\alpha-\beta|} \leq C\, r_1^{1-\lambda}\,(1 + |\ln r_1|). \tag{6.31}$$

Let us notice that this estimate fails when $\lambda = 1$, because $\sigma_\alpha, \sigma_\beta \to \lambda - 1 = 0$ as $\alpha, \beta \to 1$.

Let us now estimate J_4. By adding and subtracting terms,

$$J_4 \leq \frac{1}{2} \sum_{\pm} \frac{|c_\alpha \mp c_\beta|^2}{|\alpha-\beta|} \underbrace{\int_{|z|<r_1} ||z|^{\gamma_\alpha} \pm |z|^{\gamma_\beta}|^2 \, \frac{\mathrm{d}z}{|z|^{d+\sqrt{\alpha\,\beta}}}}_{=:J_{4,\pm}}.$$

By expanding the squares and integrating,

$$J_{4,\pm} = S_d \left(\frac{r_1^{2\,\gamma_\alpha - \sqrt{\alpha\,\beta}}}{2\,\gamma_\alpha - \sqrt{\alpha\,\beta}} + \frac{r_1^{2\,\gamma_\beta - \sqrt{\alpha\,\beta}}}{2\,\gamma_\beta - \sqrt{\alpha\,\beta}} \pm 2\, \frac{r_1^{\gamma_\alpha+\gamma_\beta-\sqrt{\alpha\,\beta}}}{\gamma_\alpha+\gamma_\beta-\sqrt{\alpha\,\beta}} \right).$$

By (6.24), the limit of $J_{4,+}$ is easy to compute and we get

$$\limsup_{\alpha,\beta\to 1} \frac{J_4}{|\alpha-\beta|} \leq C\, r_1^{2-\lambda} + C \underbrace{\limsup_{\alpha,\beta\to 1} \frac{J_{4,-}}{(\alpha-\beta)^2}}_{=:\tilde{J}_{4,-}}.$$

We estimate $\tilde{J}_{4,-}$ by multiplying and dividing by $(\gamma_\alpha - \gamma_\beta)^2$ and changing the variables by $a := \gamma_\alpha - \frac{\sqrt{\alpha\,\beta}}{2}$ and $b := \gamma_\beta - \frac{\sqrt{\alpha\,\beta}}{2}$. We get

$$\begin{aligned}\tilde{J}_{4,-} \leq & \limsup_{\alpha,\beta\to 1} \frac{(\gamma_\alpha-\gamma_\beta)^2}{|\alpha-\beta|^2} \\ & \cdot \limsup_{a,b\to c} \frac{1}{|a-b|^2} \left(\frac{r_1^{2\,a}}{2\,a} + \frac{r_1^{2\,b}}{2\,b} - \frac{2\,r_1^{a+b}}{a+b} \right),\end{aligned}$$

where $c := 1 - \frac{\lambda}{2} > 0$ is the limit of a, b as $\alpha, \beta \to \lambda$. By (6.24) and the estimation of the last limit in Lemma B.2(ii) in appendix,

$$\tilde{J}_{4,-} \leq C\, r_1^{2-\lambda}\,(1+\ln^2 r_1).$$

We conclude that

$$\limsup_{\alpha,\beta\to 1} \frac{J_4}{|\alpha-\beta|} \leq C\, r_1^{2-\lambda}\,(1+\ln^2 r_1). \tag{6.32}$$

Note that this estimate works even if $\lambda = 1$.

We are now ready to conclude the proof and show (3.8) when $\lambda \in (1,2)$. Recall that we estimate $\mathrm{Lip}_\alpha(u;\lambda)$ using (6.29) with $r_2 = +\infty$. The limsups of the terms on the right-hand side are estimated by (6.30), (6.31) and (6.32). We get for all $\epsilon > 0$ and $r_1 > 0$,

$$\mathrm{Lip}_\alpha(u;\lambda) \leq C\,|u_0|_{BV} \left\{ \epsilon + M \left(r_1^{1-\lambda}\,(1+|\ln r_1|) + \frac{r_1^{2-\lambda}}{\epsilon}\,(1+\ln^2 r_1) \right) \right\}.$$

We complete the proof by taking $\epsilon = M^{\frac{1}{\lambda}}\,(1+|\ln M|)$ and $r_1 = M^{\frac{1}{\lambda}}$.

6. *The case* $\lambda = 1$. We have to estimate again J_i in (6.29) ($i = 1, \ldots, 4$). This time, we do not let $r_2 \to +\infty$.

For J_1, we recognize again a term of the form (5.2) and we argue in the same way to estimate it. The only difference is that the fixed cutting parameter $\tilde{r}$ is replaced by a moving one $(c_\alpha\, r_2^{\gamma_\alpha}) \wedge (c_\beta\, r_2^{\gamma_\beta})$. But, by (6.24) it follows that $\lim_{\alpha,\beta\to 1}(c_\alpha\, r_2^{\gamma_\alpha}) \wedge (c_\beta\, r_2^{\gamma_\beta}) = G_d(1)\, r_2$ with $G_d(1) > 0$, and we leave it to the reader to verify that this is sufficient to extend the proof of (5.3) to the current case. Now, this estimate becomes

$$\limsup_{\alpha,\beta\to 1} \frac{J_1}{|\alpha-\beta|} \leq C\,(G_d(1)\,r_2)^{-1}\,(1+|\ln(G_d(1)\,r_2)|) \leq C\, r_2^{-1}\,(1+|\ln r_2|). \tag{6.33}$$

For J_2, we use that

$$J_2 = S_d \max_{\sigma=\alpha,\beta} \frac{1}{\sigma} \left|(c_\alpha\, r_2^{\gamma_\alpha})^{-\sigma} - (c_\beta\, r_2^{\gamma_\beta})^{-\sigma}\right|$$
$$= S_d \max_{\sigma=\alpha,\beta} |c_\alpha\, r_2^{\gamma_\alpha} - c_\beta\, r_2^{\gamma_\beta}| \int_0^1 \left(\tau\, c_\alpha\, r_2^{\gamma_\alpha} + (1-\tau)\, c_\beta\, r_2^{\gamma_\beta}\right)^{-\sigma-1} \mathrm{d}\tau.$$

By (6.24) and a simple passage to the limit under the integral sign,

$$\limsup_{\alpha,\beta\to 1} \frac{J_2}{|\alpha-\beta|} \leq C\, r_2^{-2} \limsup_{\alpha,\beta\to 1} \frac{|c_\alpha\, r_2^{\gamma_\alpha} - c_\beta\, r_2^{\gamma_\beta}|}{|\alpha-\beta|}.$$

To estimate the last limit, we write

$$|c_\alpha\, r_2^{\gamma_\alpha} - c_\beta\, r_2^{\gamma_\beta}| \leq |c_\alpha - c_\beta|\, (r_2^{\gamma_\alpha} \vee r_2^{\gamma_\beta}) + (c_\alpha \vee c_\beta)\, |r_2^{\gamma_\alpha} - r_2^{\gamma_\beta}|,$$

where $|r_2^{\gamma_\alpha} - r_2^{\gamma_\beta}| = |\gamma_\alpha - \gamma_\beta|\, |\ln r_2| \int_0^1 r_2^{\tau\,\gamma_\alpha + (1-\tau)\,\gamma_\beta}\, \mathrm{d}\tau$. Hence, again by (6.24),

$$\limsup_{\alpha,\beta\to 1} \frac{J_2}{|\alpha-\beta|} \leq C\, r_2^{-1}\, (1 + |\ln r_2|). \tag{6.34}$$

We have to do again the estimate of J_3, since the preceding one (6.31) fails.

$$J_3 \leq |c_\alpha - c_\beta| \max_{\sigma=\alpha,\beta} \int_{r_1<|z|<r_2} |z|^{\gamma_\sigma} \frac{dz}{|z|^{d+\sqrt{\alpha\,\beta}}}$$
$$+ (c_\alpha \vee c_\beta) \underbrace{\int_{r_1<|z|<r_2} \left||z|^{\gamma_\alpha} - |z|^{\gamma_\beta}\right| \frac{dz}{|z|^{d+\sqrt{\alpha\,\beta}}}}_{=:\tilde{J}_3},$$

so that by (6.24) and a simple passage to the limit under the integral sign,

$$\limsup_{\alpha,\beta\to 1} \frac{J_3}{|\alpha-\beta|} \leq C\, (|\ln r_1| + |\ln r_2|) + C \limsup_{\alpha,\beta\to 1} \frac{\tilde{J}_3}{|\alpha-\beta|}. \tag{6.35}$$

To estimate $\tilde{J}_3$, we first assume that $\alpha, \beta \neq 1$, so that $\gamma_\alpha - \sqrt{\alpha\,\beta} = (1-\alpha)\,\gamma_\alpha \neq 0$ and $\gamma_\beta - \sqrt{\alpha\,\beta} \neq 0$. Hence, $\tilde{J}_3 = S_d\,(\int_{r_1}^1 \cdots + \int_1^{r_2} \dots)$ in polar coordinates, and

$$\tilde{J}_3 \leq S_d \sum_{i=1,2} \left| \frac{r_i^{\gamma_\alpha - \sqrt{\alpha\,\beta}} - 1}{\gamma_\alpha - \sqrt{\alpha\,\beta}} - \frac{r_i^{\gamma_\beta - \sqrt{\alpha\,\beta}} - 1}{\gamma_\beta - \sqrt{\alpha\,\beta}} \right|.$$

By Lemma B.2(i) in the appendix,

$$\tilde{J}_3 \leq 2\, S_d\, |\gamma_\alpha - \gamma_\beta| \max_{i=1,2} \max_{\sigma=\alpha,\beta} (1 \vee r_i^{\gamma_\sigma - \sqrt{\alpha\,\beta}})\, \ln^2 r_i.$$

By sending α or $\beta \to 1$, we see that this inequality holds also when α or $\beta = 1$. Hence, by (6.24) and (6.35),

$$\limsup_{\alpha,\beta\to 1} \frac{J_3}{|\alpha-\beta|} \leq C\, (|\ln r_1| \vee \ln^2 r_1 + |\ln r_2| \vee \ln^2 r_2). \tag{6.36}$$

Finally, for J_4, we use (6.32) which is still valid and we are ready to show (3.8) in the critical case. By (6.29), (6.33), (6.34), (6.36) and (6.32), we have for all $\epsilon > 0$, and $r_2 \geq r_1 > 0$,

$$\begin{aligned} \mathrm{Lip}_\alpha(u;1) \leq{}& C\, |u_0|_{BV}\, \epsilon \\ &+ C\, M\, \|u_0\|_{L^1}\, r_2^{-1}\, (1 + |\ln r_2|) \\ &+ C\, M\, |u_0|_{BV} (|\ln r_1| \vee \ln^2 r_1 + |\ln r_2| \vee \ln^2 r_2) \\ &+ C\, M\, |u_0|_{BV}\, \frac{r_1}{\epsilon}\, (1 + \ln^2 r_1). \end{aligned}$$

We complete the proof by taking $\epsilon = M\,(1+|\ln M|)$, $r_1 = M \wedge 1$, $r_2 = 1 \vee \frac{\|u_0\|_{L^1}}{|u_0|_{BV}}$, and noting that $\|u_0\|_{L^1} \leq |u_0|_{BV}$ if $r_2 = 1$. □

7. Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3. Let us first recall the notions of entropy solutions of (1.6) and (1.7) introduced in [44, 16]. For (1.7), we use an equivalent definition introduced in [40].

Definition 7.1 (Entropy solutions). *Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$ and (1.4)–(1.5) hold. Let $u \in L^\infty(Q_T) \cap L^\infty(0,T;L^1)$.*

(1) *u is an entropy solution of (1.6) if, for all $k \in \mathbb{R}$ and all nonnegative $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$,*

$$\int_{Q_T} \Big(|u-k|\,\partial_t\phi + q_f(u,k)\cdot\nabla\phi - \operatorname{sgn}(u-k)\,\varphi(u)\,\phi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} |u_0(x)-k|\,\phi(x,0)\,\mathrm{d}x \geq 0.$$

(2) *u is an entropy solution of (1.7) if,*
   (a) *$\varphi(u) \in L^2(0,T;H^1)$,*
   (b) *and for all $k \in \mathbb{R}$ and all nonnegative $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$,*

$$\int_{Q_T} \Big(|u-k|\,\partial_t\phi + q_f(u,k)\cdot\nabla\phi + |\varphi(u)-\varphi(k)|\,\Delta\phi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} |u_0(x)-k|\,\phi(x,0)\,\mathrm{d}x \geq 0.$$

To prove Theorem 3.3, we need to establish some technical lemmas. Let us begin by a compactness result.

Lemma 7.1. *Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, (1.4)–(1.5) hold, and for each $\alpha \in (0,2)$, let u^α be the unique entropy solution to (1.1). Then, there exist $u, w \in L^\infty(Q_T) \cap C([0,T];L^1)$ such that $u = \lim_{\alpha\downarrow 0} u^\alpha$ and $w = \lim_{\alpha\uparrow 2} u^\alpha$, up to subsequences, in $C([0,T];L^1_{\mathrm{loc}})$ and almost everywhere in Q_T.*

Proof. We only do the proof for w, the proof for u being similar. Let us consider a sequence $\alpha_m \uparrow 2$ and let us define $E := \{u^{\alpha_m}\}_m$. We will show that E is relatively compact in $C([0,T];L^1_{\mathrm{loc}})$. First we take a sequence $\{u_0^n\}_n \subset L^\infty \cap L^1 \cap BV(\mathbb{R}^d)$ that converges to u_0 in $L^1(\mathbb{R}^d)$, and let E_n denote the family $\{u_n^{\alpha_m}\}_m$ of entropy solutions to (1.1) with $\alpha = \alpha_m$ and u_0^n as initial data. We begin by showing that E_n is relatively compact in $C([0,T];L^1_{\mathrm{loc}})$.

The family E_n is equicontinuous in $C([0,T];L^1)$ by Corollary 3.6, and Remark 6.2(2). For each $t \in [0,T]$, $\{u_n^{\alpha_m}(\cdot,t)\}_m$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by the $L^1 \cap BV$-bound (2.6) and Helly's theorem. By the Arzela-Ascoli theorem, E_n is relatively compact in $C([0,T];L^1_{\mathrm{loc}})$ for any $n \in \mathbb{N}$.

The relative compactness of E, and thus the existence of $w \in C([0,T];L^1_{\mathrm{loc}})$, is now a consequence of the L^1-contraction principle since

$$\sup_{m\in\mathbb{N}} \|u^{\alpha_m} - u_n^{\alpha_m}\|_{C([0,T];L^1)} \leq \|u_0 - u_0^n\|_{L^1} \to 0 \quad \text{as } n \to +\infty. \tag{7.1}$$

Taking a subsequence if necessary, we can assume that u^{α_m} converges to w in $C([0,T];L^1_{\mathrm{loc}})$ and almost everywhere in Q_T. In particular, by the a priori estimate (2.6), we infer that $w \in L^\infty(Q_T)$. To prove that $w \in C([0,T];L^1)$, we observe that E is equicontinuous in $C([0,T];L^1)$ by the triangle inequality, the convergence estimate (7.1), and the equicontinuity of E_n. Hence, for any $R > 0$, $m \in \mathbb{N}$, and $t, s \in [0, T]$,

$$\begin{aligned}
&\|(w(\cdot,t) - w(\cdot,s))\, \mathbf{1}_{|x|<R}\|_{L^1} \\
&\leq \|u^{\alpha_m}(\cdot,t) - u^{\alpha_m}(\cdot,s)\|_{L^1} \\
&\quad + \|(w(\cdot,t) - u^{\alpha_m}(\cdot,t))\, \mathbf{1}_{|x|<R}\|_{L^1} + \|(u^{\alpha_m}(\cdot,s) - w(\cdot,s))\, \mathbf{1}_{|x|<R}\|_{L^1} \\
&\leq o(1) + 2\, \|(w - u^{\alpha_m})\, \mathbf{1}_{|x|<R}\|_{C([0,T];L^1)},
\end{aligned}$$

where $o(1) \to 0$ as $|t - s| \to 0$ uniformly in R and m. We then conclude that

$$\|(w(\cdot,t) - w(\cdot,s))\|_{L^1} \leq o(1) \quad \text{as } |t - s| \to 0$$

by first sending $m \to +\infty$ and then $R \to +\infty$ using Fatou's lemma. □

Let us now verify that these limits satisfy the entropy inequalities of the preceding definition.

Lemma 7.2. *Under the assumptions of Lemma 7.1, u and w satisfy the entropy inequalities of Definition 7.1*(1) *and* (2b) *respectively.*

In the proof we need the following lemma:

Lemma 7.3. *A function $u \in L^\infty(Q_T) \cap L^\infty(0,T;L^1)$ is an entropy solution of* (1.1) *(cf. Definition 2.1) if and only if for all convex $\eta \in C^1(\mathbb{R})$, all $r > 0$ and all nonnegative $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$,*

$$\begin{aligned}
&\int_{Q_T} \Big(\eta(u)\, \partial_t \phi + q_f^\eta(u) \cdot \nabla \phi\Big)\, \mathrm{d}x\, \mathrm{d}t \\
&+ \int_{Q_T} \Big(q_\varphi^\eta(u)\, \mathcal{L}_r^\alpha[\phi] + \eta'(u)\, \mathcal{L}^{\alpha,r}[\varphi(u)]\, \phi\Big)\, \mathrm{d}x\, \mathrm{d}t \qquad (7.2)\\
&+ \int_{\mathbb{R}^d} \eta(u_0(x))\, \phi(x,0)\, \mathrm{d}x \geq 0,
\end{aligned}$$

where $q_g^\eta(u) := \int_0^u \eta'(\tau)\, g'(\tau)\, \mathrm{d}\tau$ (for $g = f, \varphi$).

This result is well-known for (local) conservation laws, see e.g. [36, p. 27]. Because of the presence of the discontinuous sign function in the Kruzhkov formulation (2.5), any proof will be more technical than in the local case and we therefore provide one in Appendix C.

Proof of Lemma 7.2. We begin with the proof for w which is easier.

1. *Entropy inequalities for w.* Using the definition of $\mathcal{L}_r^\alpha$ and $\mathcal{L}^{\alpha,r}$ in (2.1), we send $r \to +\infty$ in the entropy inequality (2.5) and find that

$$\begin{aligned}
&\int_{Q_T} \Big(|u^\alpha - k|\, \partial_t \phi + q_f(u^\alpha, k) \cdot \nabla \phi - |\varphi(u^\alpha) - \varphi(k)|\, (-\triangle)^{\frac{\alpha}{2}} \phi\Big)\, \mathrm{d}x\, \mathrm{d}t \qquad (7.3)\\
&+ \int_{\mathbb{R}^d} |u_0(x) - k|\, \phi(x,0)\, \mathrm{d}x \geq 0.
\end{aligned}$$

Since $(-\triangle)^{\frac{\alpha}{2}} \phi = \mathcal{F}^{-1}\left(|2\,\pi \cdot|^\alpha\, \mathcal{F}\phi\right)$ and $-\triangle \phi = \mathcal{F}^{-1}\left(|2\,\pi \cdot|^2\, \mathcal{F}\phi\right)$, by Plancherel

$$-(-\triangle)^{\frac{\alpha}{2}} \phi \to \triangle\, \phi \quad \text{in } L^2(Q_T) \text{ as } \alpha \uparrow 2. \qquad (7.4)$$

To get the entropy inequalities of Definition 7.1(2b), we must pass to the limit in (7.3). This is straightforward for the local terms due to Lemma 7.1 and (2.6).

For the nonlocal term, we first observe that

$$
\begin{aligned}
&-\int_{Q_T} |\varphi(u^\alpha)-\varphi(k)|\,(-\triangle)^{\frac{\alpha}{2}}\phi\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{Q_T} \underbrace{\Big\{|\varphi(u^\alpha)-\varphi(k)|-|\varphi(k)|\Big\}}_{=:q(u^\alpha)}\Big\{\triangle\phi-\triangle\phi-(-\triangle)^{\frac{\alpha}{2}}\phi\Big\}\,\mathrm{d}x\,\mathrm{d}t\\
&\leq \int_{Q_T} q(u^\alpha)\,\triangle\phi\,\mathrm{d}x\,\mathrm{d}t+\|q(u^\alpha)\|_{L^2(Q_T)}\|\triangle\phi+(-\triangle)^{\frac{\alpha}{2}}\phi\|_{L^2(Q_T)}.
\end{aligned}
$$

By (7.4), the second term tends to zero since $\|q(u^\alpha)\|_{L^2(Q_T)}$ is bounded independently of α. The boundedness follows from (2.6) and an (L^1,L^∞)-interpolation argument since $q\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $q(0)=0$. By the $C([0,T];L^1_{\mathrm{loc}})$-convergence of u^α (up to a subsequence), the first term converges as $\alpha\uparrow 2$ to

$$
\int_{Q_T} |\varphi(w)-\varphi(k)|\,\triangle\phi\,\mathrm{d}x\,\mathrm{d}t.
$$

This completes the proof for w.

2. *Entropy inequalities for u.* Let us fix $r>0$ for the duration of this proof and start from the entropy inequalities (7.2), written for convex and C^1-entropies η.

There is again no difficulty to pass to the limit as $\alpha\downarrow 0$ in the local terms of (7.2). For the first nonlocal term, we use that $\mathcal{L}^\alpha_r[\phi]\to 0$ uniformly on Q_T. This is readily seen from (2.1) and (2.2). Let us also notice that q^η_φ, defined just below (7.2), satisfies $q^\eta_\varphi\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $q^\eta_\varphi(0)=0$. Hence

$$
\int_{Q_T} q^\eta_\varphi(u^\alpha)\,\mathcal{L}^\alpha_r[\phi]\,\mathrm{d}x\,\mathrm{d}t\to 0,
$$

since $q^\eta_\varphi(u^\alpha)$ is bounded in $C([0,T];L^1)$. For the remaining nonlocal term, we split the integral and get

$$
\begin{aligned}
&\int_{Q_T} \eta'(u^\alpha)\,\mathcal{L}^{\alpha,r}[\varphi(u^\alpha)]\,\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\leq -\underbrace{G_d(\alpha)\int_{|z|>r}\frac{\mathrm{d}z}{|z|^{d+\alpha}}}_{=:I}\int_{Q_T}\eta'(u^\alpha)\,\varphi(u^\alpha)\,\phi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C\,\underbrace{\frac{G_d(\alpha)}{r^{d+\alpha}}}_{=:J}\,\|\varphi(u^\alpha)\|_{C([0,T];L^1)}\,\|\phi\|_{L^1(Q_T)},
\end{aligned}
$$

where C is an L^∞-bound on $\eta'(u^\alpha)$. Notice that for all fixed r, $\lim_{\alpha\downarrow 0} I=1$ and $\lim_{\alpha\downarrow 0} J=0$ by (2.2). Since η' is continuous, we can pass to the limit as $\alpha\downarrow 0$ in the inequality above, thanks to (2.6), the almost everywhere convergence of u^α (up to a subsequence), and the dominated convergence theorem.

The limit in (7.2) then implies that

$$
\int_{Q_T}\Big(\eta(u)\,\partial_t\phi+q^\eta_f(u)\cdot\nabla\phi-\eta'(u)\,\varphi(u)\,\phi\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^d}\eta(u_0(x))\,\phi(x,0)\,\mathrm{d}x\geq 0,
$$

for all convex C^1-entropies η and fluxes $q^\eta_f(u)=\int_0^u\eta'(\tau)\,f'(\tau)\,\mathrm{d}\tau$. It is then classical to get the desired Kruzhkov entropy inequalities of Definition 7.1(1) from these inequalities, see e.g. the if part of the proof in Appendix C. □

To prove that w satisfies (2a) of Definition 7.1, we need to derive an $H^{\frac{\alpha}{2}}$-estimate on u^α. In the sequel, $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ denotes the fractional Sobolev space of $u\in L^2(\mathbb{R}^d)$

such that $\iint_{\mathbb{R}^{2d}} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y < +\infty$. The $H^{\frac{\alpha}{2}}$-semi-norm can be defined in both the following equivalent ways:

$$|u|^2_{H^{\frac{\alpha}{2}}} := \frac{G_d(\alpha)}{2}\iint_{\mathbb{R}^{2d}} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d} |2\,\pi\,\xi|^\alpha\,|\mathcal{F}u|^2\,\mathrm{d}\xi. \tag{7.5}$$

The $H^{\frac{\alpha}{2}}$-norm is defined as $\|u\|^2_{H^{\frac{\alpha}{2}}} := \|u\|^2_{L^2} + |u|^2_{H^{\frac{\alpha}{2}}}$. The equality in (7.5) is standard, cf. e.g. [1]. In the sequel, the knowledge of the precise constants will be important to get estimates uniform in $\alpha \uparrow 2$. For the sake of completeness, we therefore provide a short computation of them in Appendix C.

Lemma 7.4. *Let $\alpha \in (0,2)$, $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, (1.4)–(1.5) hold, and u^α be the unique entropy solution to (1.1). Then*

$$\int_{\mathbb{R}^d} \Phi(u^\alpha(x,T))\,\mathrm{d}x + |\varphi(u^\alpha)|^2_{L^2(0,T;H^{\frac{\alpha}{2}})} \leq \int_{\mathbb{R}^d} \Phi(u_0(x))\,\mathrm{d}x,$$

where $\Phi(u) := \int_0^u \varphi(\tau)\,\mathrm{d}\tau$ for all $u \in \mathbb{R}$.

Remark 7.5. Note that Φ is nonnegative, convex, and 0 at 0.

Proof. We can take $\eta = \Phi$ in (7.2), since it is C^1 and convex by (1.5). Using also Lemma 7.3 and the continuity of u^α in time with values in $L^1(\mathbb{R}^d)$, as in Remark 2.1, we find that for all $\phi \in C_c^\infty(\mathbb{R}^{d+1})$,

$$\begin{aligned}
&\int_{Q_T} \Big(\Phi(u^\alpha)\,\partial_t\phi + q_f^\Phi(u^\alpha)\cdot\nabla\phi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{Q_T} \Big(q_\varphi^\Phi(u^\alpha)\,\mathcal{L}_r^\alpha[\phi] + \varphi(u^\alpha)\,\mathcal{L}^{\alpha,r}[\varphi(u^\alpha)]\,\phi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{\mathbb{R}^d} \Phi(u_0(x))\,\phi(x,0)\,\mathrm{d}x \geq \int_{\mathbb{R}^d} \Phi(u^\alpha(x,T))\,\phi(x,T)\,\mathrm{d}x.
\end{aligned} \tag{7.6}$$

Then take $\phi(x,t) = \gamma_R(x)$, where $R > 0$ and γ_R is an approximation of $\mathbf{1}_{|x|<R}$ such that $\gamma_R \in C_c^\infty(\mathbb{R}^d)$, $\{\gamma_R\}_{R>0}$ is bounded in $W^{2,\infty}(\mathbb{R}^d)$, $\gamma_R \to 1$ in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ as $R \to +\infty$. It is obvious that the ∇- and $\mathcal{L}_r^\alpha$-terms in (7.6) vanish as $R \to +\infty$, since $q_g^\Phi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $q_g^\Phi(0) = 0$ for $g = f, \varphi$. For the $\mathcal{L}^{\alpha,r}$-term, a standard computation shows that for all $u, v \in L^2(\mathbb{R}^d)$ and $r > 0$,

(7.7)

$$\begin{aligned}
&-\int_{\mathbb{R}^d} u\,\mathcal{L}^{\alpha,r}[v]\,\mathrm{d}x \\
&= -G_d(\alpha)\iint_{|z|>r} u(x)\,\frac{v(x+z)-v(x)}{|z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}x \\
&= \frac{G_d(\alpha)}{2}\Bigg\{\iint_{|x-y|>r} u(x)\,v(x)\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+\alpha}} + \iint_{|x-y|>r} u(y)\,v(y)\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+\alpha}} \\
&\quad -\iint_{|x-y|>r} u(x)\,v(y)\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+\alpha}} - \iint_{|x-y|>r} u(y)\,v(x)\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+\alpha}}\Bigg\} \\
&= \frac{G_d(\alpha)}{2}\iint_{|x-y|>r} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Hence, by the dominated convergence theorem,

$$\begin{aligned}
&\int_{Q_T} \varphi(u^\alpha)\,\mathcal{L}^{\alpha,r}[\varphi(u^\alpha)]\,\gamma_R\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{G_d(\alpha)}{2}\int_0^T \iint_{|x-y|>r} (\varphi(u^\alpha(x,t)) - \varphi(u^\alpha(y,t)))\\
&\quad \cdot (\varphi(u^\alpha(x,t))\,\gamma_R(x) - \varphi(u^\alpha(y,t))\,\gamma_R(y))\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+\alpha}}\,\mathrm{d}t\\
&\to \frac{G_d(\alpha)}{2}\int_0^T \iint_{|x-y|>r} \frac{(\varphi(u^\alpha(x,t)) - \varphi(u^\alpha(y,t)))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t
\end{aligned}$$

as $R \to +\infty$. Going to the limit in (7.6), we then find that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi(u^\alpha(x,T))\,\mathrm{d}x\\
&+ \frac{G_d(\alpha)}{2}\int_0^T \iint_{|x-y|>r} \frac{(\varphi(u^\alpha(x,t)) - \varphi(u^\alpha(y,t)))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\le \int_{\mathbb{R}^d} \Phi(u_0(x))\,\mathrm{d}x.
\end{aligned}$$

The proof is complete by sending $r \downarrow 0$ and using the monotone convergence theorem. □

From this energy type of estimate, we have the following result:

Lemma 7.6. *Under the assumptions of Lemma 7.1,* $\varphi(w) \in L^2(0,T;H^1)$.

Proof. Recall first that by (2.6) and a (L^1, L^∞)-interpolation argument, $\{u^\alpha\}_{\alpha\in(0,2)}$ is bounded in $L^2(0,T;L^2)$. Using in addition the preceding lemma, we find a constant C such that for all $\alpha \in (0,2)$,

$$\|\varphi(u^\alpha)\|_{L^2(0,T;H^{\frac{\alpha}{2}})} \le C.$$

Using the Fourier formula in (7.5),

$$\int_{Q_T} (1 + |2\,\pi\,\xi|^\alpha)\,|\mathcal{F}\varphi(u^\alpha)|^2\,\mathrm{d}\xi\,\mathrm{d}t \le C$$

(recall that $\mathcal{F}$ is the Fourier transform in space). Now we use the following inequalities: for all $1 \le \beta \le \alpha$ and all $\xi \in \mathbb{R}^d$,

$$(1 + |2\,\pi\,\xi|^\beta) \le (1 + |2\,\pi\,\xi|)^\beta \le (1 + |2\,\pi\,\xi|)^\alpha \le 2^{\alpha-1}\,(1 + |2\,\pi\,\xi|^\alpha).$$

We deduce that

$$\int_{Q_T} (1 + |2\,\pi\,\xi|^\beta)\,|\mathcal{F}\varphi(u^\alpha)|^2\,\mathrm{d}\xi\,\mathrm{d}t \le 2^{\alpha-1}\,C.$$

Going back to the integral formula in (7.5),

$$\|\varphi(u^\alpha)\|^2_{L^2(0,T;L^2)} + \frac{G_d(\beta)}{2}\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\varphi(u^\alpha)(x,t) - \varphi(u^\alpha)(y,t))^2}{|x-y|^{d+\beta}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le 2^{\alpha-1}\,C.$$

By Fatou's lemma, applied for $\alpha \uparrow 2$ with fixed β,

$$\|\varphi(w)\|^2_{L^2(0,T;L^2)} + \frac{G_d(\beta)}{2}\int_0^T \iint_{\mathbb{R}^{2d}} \frac{(\varphi(w)(x,t) - \varphi(w)(y,t))^2}{|x-y|^{d+\beta}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le 2\,C.$$

Finally, Fatou's lemma applied to the Fourier formula shows that

$$\begin{aligned} 2\,C &\geq \liminf_{\beta\uparrow 2} \int_{Q_T} (1+|2\,\pi\,\xi|^\beta)\,|\mathcal{F}\varphi(w)|^2\,\mathrm{d}\xi\,\mathrm{d}t \\ &\geq \int_{Q_T} (1+|2\,\pi\,\xi|^2)\,|\mathcal{F}\varphi(w)|^2\,\mathrm{d}\xi\,\mathrm{d}t. \end{aligned}$$

The proof is complete. □

We end by the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $u, w \in L^\infty(Q_T) \cap C([0,T];L^1)$ be defined in Lemma 7.1. By previous lemmas, they are entropy solutions of (1.6) and (1.7), respectively. By uniqueness (cf. [44, 16, 40]), the whole sequences converge and the proof is complete. □

8. Optimal example

In this last section, we exhibit an example of an equation for which Theorems 3.1 and 3.8 are optimal. Note that the modulus in f is the same than in [27, 48]. This modulus is optimal for linear fluxes, i.e. for equations of the form $\partial_t u + F \cdot \nabla u = 0$ where $F \in \mathbb{R}^d$. This is readily seen by the formula $u(x,t) = u_0(x - t\,F)$. Here, we focus on the new fractional diffusion term. The proofs work for $\alpha = 2$ and our example is also optimal for the results in [24]. Let us finally mention that this example is motivated by Remark 2.1 of [33] and similar remarks in [37, 35, 2].

Let us consider, for every $\alpha \in [0,2]$ and $\gamma, a > 0$,

$$\begin{cases} \partial_t u + a\,(-\triangle)^{\frac{\alpha}{2}}\,u = 0, \\ u(x,0) = \gamma\,\mathbf{1}_Q(\gamma^{-1}\,x), \end{cases} \tag{8.1}$$

where $Q := [-1,1]^d$. This is (1.1) with u_0 as above, $f \equiv 0$ and $\varphi' \equiv a$. Notice that

$$\begin{cases} \|u_0\|_{L^1} = 2^d\,\gamma^{d+1}, \\ |u_0|_{BV} = d\,2^d\,\gamma^d, \\ \mathrm{E}_i(u_0) = d\,2^d\,\gamma^d\left(1+\left(\ln\frac{\gamma}{d}\right)^i\right)\mathbf{1}_{\gamma>d}, \quad (i=1,2), \end{cases} \tag{8.2}$$

where $\mathrm{E}_i(u_0)$ is defined in (3.3).

8.1. **Optimality of Theorem 3.1.** Let us fix $\alpha \in [0,2]$ and let us use the notation $u =: u_a$. Given $T > 0$ and other parameters $b, c > 0$, we define

$$\omega_{a-b} := \begin{cases} |a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}|, & \alpha > 1, \\ |a\,\ln a - b\,\ln b|, & \alpha = 1, \\ |a-b|, & \alpha < 1, \end{cases}$$

$$\sigma_T := \begin{cases} T^{\frac{1}{\alpha}}, & \alpha > 1, \\ T\,|\ln T|, & \alpha = 1, \\ T, & \alpha < 1, \end{cases}$$

$$\sigma_\gamma := \begin{cases} \gamma^d, & \alpha > 1, \\ \gamma^d\,\ln\gamma, & \alpha = 1, \\ \gamma^{d+1-\alpha}, & \alpha < 1. \end{cases}$$

We also introduce the best Lipschitz constant of $a \mapsto u_a$ at $a = c$:

$$\mathrm{Lip}_\varphi(u;c) := \limsup_{a,b\to c} \frac{\|u_a - u_b\|_{C([0,T];L^1)}}{|a-b|}.$$

Theorem 3.1 and (8.2) imply that the function $a \geq 0 \mapsto u_a \in C([0,T];L^1)$ is continuous at $a = 0$ and locally Lipschitz continuous for $a > 0$ with for all $c > 0$,

$$\begin{aligned} \|u_a - u_b\|_{C([0,T];L^1)} &= O(\omega_{a-b}) \quad \text{as } a, b \downarrow 0, \\ \mathrm{Lip}_\varphi(u;c) &= O(\sigma_T) \quad \text{as } T \downarrow 0, \\ \mathrm{Lip}_\varphi(u;c) &= O(\sigma_\gamma) \quad \text{as } \gamma \to +\infty, \end{aligned}$$

while all the respective remaining parameters are fixed. The result below states that these estimates are optimal.

Proposition 8.1. *Let $\alpha \in [0,2]$ and $c > 0$.*

(i) *For all $T, \gamma > 0$, $\liminf_{a,b\downarrow 0} \frac{\|u_a - u_b\|_{C([0,T];L^1)}}{\omega_{a-b}} > 0$.*
(ii) *For all $\gamma > 0$, $\liminf_{T\downarrow 0} \frac{\mathrm{Lip}_\varphi(u;c)}{\sigma_T} > 0$.*
(iii) *For all $T > 0$, $\liminf_{\gamma\to+\infty} \frac{\mathrm{Lip}_\varphi(u;c)}{\sigma_\gamma} > 0$.*

Remark 8.2. This result shows that the modulus of continuity in $\varphi - \psi$ derived in (3.5) is optimal for linear diffusion functions. It also shows that the T- and u_0-dependencies of this modulus are optimal in the limits $T \downarrow 0$ or $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}} \to +\infty$ (recall that $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}} \sim \gamma$ by (8.2)).

8.2. **Optimality of Theorem 3.8.** Let us now use the notation $u =: u^\alpha$ to emphasize the dependence on α. Given $\lambda \in (0,2)$, we define

$$\tilde{\sigma}_M := \begin{cases} M^{\frac{1}{\lambda}} \, |\ln M|, & \lambda > 1, \\ M \, \ln^2 M, & \lambda = 1, \\ M, & \lambda < 1, \end{cases}$$

$$\tilde{\sigma}_\gamma := \begin{cases} \gamma^d, & \lambda > 1, \\ \gamma^d \, \ln^2 \gamma, & \lambda = 1, \\ \gamma^{d+1-\lambda} \, \ln \gamma, & \lambda < 1, \end{cases}$$

where $M := T\,a$. We also consider the best Lipschitz constant of $\alpha \mapsto u^\alpha$ at $\alpha = \lambda$ defined in (3.7). Then, Theorem 3.8 and (8.2) imply that for all $\lambda \in (0,2)$,

$$\begin{aligned} \mathrm{Lip}_\alpha(u;\lambda) &= O(\tilde{\sigma}_M) \quad \text{as } M \downarrow 0, \\ \mathrm{Lip}_\alpha(u;\lambda) &= O(\tilde{\sigma}_\gamma) \quad \text{as } \gamma \to +\infty, \end{aligned}$$

while all the respective remaining parameters are fixed. The result below states that these estimates are optimal.

Proposition 8.3. *Let $T, a > 0$, $M = T\,a$, and $\lambda \in (0,2)$. There exist $M_0, \gamma_0 > 0$ such that:*

(i) *For all $\gamma_0 \geq \gamma > 0$, $\liminf_{M\downarrow 0} \frac{\mathrm{Lip}_\alpha(u;\lambda)}{\tilde{\sigma}_M} > 0$.*
(ii) *For all $M_0 \geq M > 0$, $\liminf_{\gamma\to+\infty} \frac{\mathrm{Lip}_\alpha(u;\lambda)}{\tilde{\sigma}_\gamma} > 0$.*

Remark 8.4. This result shows that the M- and u_0-dependencies in (3.8) are optimal at the limits $M = T\,\|\varphi'\|_\infty \downarrow 0$ or $\frac{\|u_0\|_{L^1}}{|u_0|_{BV}} \to +\infty$.

8.3. **Proofs.**

Proof of Proposition 8.1. Let us prove each items in order.

1. *Item* (i). Let us first assume that $T = \gamma = 1$. The general case will follow from a rescaling argument given at the end of the proof. Let us define

$$\mathcal{E}_Q := \int_Q u_a(x,1)\,\mathrm{d}x - \int_Q u_b(x,1)\,\mathrm{d}x. \tag{8.3}$$

Since $\|u_a - u_b\|_{C([0,1];L^1)} \geq \|u_a(\cdot,1) - u_b(\cdot,1)\|_{L^1} \geq |\mathcal{E}_Q|$, it suffices to show that $\liminf_{a,b\downarrow 0} \frac{|\mathcal{E}_Q|}{\omega_{a-b}} > 0$. It is well-known that $u_a(x,t) = \mathcal{F}^{-1}(e^{-t\,a\,|2\,\pi\cdot|^\alpha}) * \mathbf{1}_Q(x)$. A short computation shows that

$$
\begin{aligned}
\mathcal{E}_Q &= \int \mathcal{F}^{-1}(e^{-a\,|2\,\pi\cdot|^\alpha} - e^{-b\,|2\,\pi\cdot|^\alpha})\,(\mathbf{1}_Q * \mathbf{1}_Q)\,\mathrm{d}x \\
&= \int (e^{-a\,|2\,\pi\,\xi|^\alpha} - e^{-b\,|2\,\pi\,\xi|^\alpha})\,(\mathcal{F}\mathbf{1}_Q)^2\,\mathrm{d}\xi \\
&= \frac{2^d}{\pi^d}\int (e^{-a\,|\xi|^\alpha} - e^{-b\,|\xi|^\alpha})\prod_{i=1}^d \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi \\
&= \frac{2^d}{\pi^d}\int\int_0^1 (b-a)\,|\xi|^\alpha\,e^{-(\tau\,a+(1-\tau)\,b)\,|\xi|^\alpha}\prod_{i=1}^d \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\tau\,\mathrm{d}\xi,
\end{aligned}
\tag{8.4}
$$

where $\xi =: (\xi_1,\dots,\xi_d)$ and $\operatorname{sinc}(\xi_i) := \frac{\sin\xi_i}{\xi_i}$. To get the third line, we have used the formula $\mathcal{F}\mathbf{1}_Q(\xi) = \prod_{i=1}^d \frac{\sin(2\,\pi\,\xi_i)}{\pi\,\xi_i}$ and the change of variable $2\,\pi\,\xi \mapsto \xi$. We now give separate arguments for the cases $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$.

a. *The case* $\alpha < 1$. This is obvious since $0 < \int |\xi|^\alpha \prod_{i=1}^d \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi < +\infty$.

b. *The case* $\alpha > 1$. Note that $|\xi|^\alpha \leq d^{\alpha-1}\sum_{i=1}^d |\xi_i|^\alpha$. Hence, by (8.4),

$$
|\mathcal{E}_Q| \geq I_{a,b}\int\int_0^1 |a-b|\,|\xi_1|^\alpha\,e^{-d^{\alpha-1}\,(\tau\,a+(1-\tau)\,b)\,|\xi_1|^\alpha}\,\operatorname{sinc}^2(\xi_1)\,\mathrm{d}\tau\,\mathrm{d}\xi_1 \tag{8.5}
$$

where

$$
I_{a,b} = \frac{2^d}{\pi^d}\prod_{i=2}^d \int e^{-d^{\alpha-1}\,(a\vee b)\,|\xi_i|^\alpha}\,\operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi_i.
$$

Since $e^{-d^{\alpha-1}\,(a\vee b)\,|\xi_i|^\alpha} \to 1$ as $a,b\downarrow 0$,

$$
I_{a,b} \geq C_0 := \frac{2^{d-1}}{\pi^d}\prod_{i=2}^d \int \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi_i > 0, \tag{8.6}
$$

for all $a,b > 0$ sufficiently small. Hence, assuming e.g. that $a > b$, we get

$$
\begin{aligned}
|\mathcal{E}_Q| \geq{}& C_0 \underbrace{\int a\,|\xi_1|^{\alpha-2}\,e^{-d^{\alpha-1}\,a\,|\xi_1|^\alpha}\,\sin^2(\xi_1)\,\mathrm{d}\xi_1}_{=:I_a} \\
&- C_0\int b\,|\xi_1|^{\alpha-2}\,e^{-d^{\alpha-1}\,b\,|\xi_1|^\alpha}\,\sin^2(\xi_1)\,\mathrm{d}\xi_1.
\end{aligned}
\tag{8.7}
$$

Before continuing, notice that this estimate is valid for $\alpha = 1$; this is will be useful later. Let us continue the case $\alpha > 1$ by changing variables,

$$
I_a = a^{\frac{1}{\alpha}}\int |\xi_1|^{\alpha-2}\,e^{-d^{\alpha-1}\,|\xi_1|^\alpha}\,\sin^2(a^{-\frac{1}{\alpha}}\,\xi_1)\,\mathrm{d}\xi_1.
$$

Doing the same for the b-integral and adding and subtracting term,

$$
\begin{aligned}
|\mathcal{E}_Q| \geq{}& C_0\,(a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}})\int |\xi_1|^{\alpha-2}\,e^{-d^{\alpha-1}\,|\xi_1|^\alpha}\,\sin^2(a^{-\frac{1}{\alpha}}\,\xi_1)\,\mathrm{d}\xi_1 \\
&+ C_0\,b^{\frac{1}{\alpha}}\int |\xi_1|^{\alpha-2}\,e^{-d^{\alpha-1}\,|\xi_1|^\alpha}\left\{\sin^2(a^{-\frac{1}{\alpha}}\,\xi_1) - \sin^2(b^{-\frac{1}{\alpha}}\,\xi_1)\right\}\mathrm{d}\xi_1 \\
=:{}& C_0\,(a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}})\,I_1 + C_0\,b^{\frac{1}{\alpha}}\,I_2.
\end{aligned}
\tag{8.8}
$$

By a Taylor expansion and an integration by parts,

$$
\begin{aligned}
|I_2| &\leq (a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}) \left| \int_0^1 \int a_{\alpha,\tau}^{-2} \underbrace{|\xi_1|^{\alpha-2}\, \xi_1\, e^{-d^{\alpha-1}\,|\xi_1|^\alpha}}_{=:f(\xi_1)} \right. \\
&\quad \left. \cdot \underbrace{2 \sin\left(a_{\alpha,\tau}^{-1}\,\xi_1\right) \cos\left(a_{\alpha,\tau}^{-1}\,\xi_1\right)}_{=\sin\left(2\,a_{\alpha,\tau}^{-1}\,\xi_1\right)} \, \mathrm{d}\xi_1\, \mathrm{d}\tau \right| \\
&\leq \frac{1}{2}\, b^{-\frac{1}{\alpha}}\, (a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}}) \left| \int_0^1 \int f'(\xi_1)\, \cos\left(2\, a_{\alpha,\tau}^{-1}\,\xi_1\right) \mathrm{d}\xi_1\, \mathrm{d}\tau \right|,
\end{aligned}
$$

where $a_{\alpha,\tau} := \tau\, a^{\frac{1}{\alpha}} + (1-\tau)\, b^{\frac{1}{\alpha}}$ and f' is integrable when $\alpha > 1$. We deduce that $C_0\, b^{\frac{1}{\alpha}}\, I_2 = (a^{\frac{1}{\alpha}} - b^{\frac{1}{\alpha}})\, o(1)$ as $a, b \downarrow 0$, since for fixed τ, $\cos\left(2\, a_{\alpha,\tau}^{-1}\, \cdot\right)$ converges to its zero mean value in L^∞-weak-$\star$. By a similar argument $\sin^2(a^{-\frac{1}{\alpha}}\, \cdot)$ also weakly-$\star$ converges to its positive mean value m and hence

$$
\lim_{a,b\downarrow 0} I_1 = m \int |\xi_1|^{\alpha-2}\, e^{-d^{\alpha-1}\,|\xi_1|^\alpha}\, \mathrm{d}\xi_1 > 0.
$$

We thus conclude the result from (8.8).

c. *The case* $\alpha = 1$. We restart from (8.7) assuming again that $a > b$, a, b small. This time we cut I_a into three pieces.

$$
I_a = \int_{1<|\xi_1|<a^{-1}} \cdots + \int_{|\xi_1|<1} \cdots + \int_{|\xi_1|>a^{-1}} \cdots .
$$

We do the same for the b-integral and we get

$$
\begin{aligned}
|\mathcal{E}_Q| \geq{}& C_0 \int_{1<|\xi_1|<a^{-1}} a\, |\xi_1|^{-1}\, e^{-a\,|\xi_1|}\, \sin^2(\xi_1)\, \mathrm{d}\xi_1 \\
&- C_0 \int_{1<|\xi_1|<b^{-1}} b\, |\xi_1|^{-1}\, e^{-b\,|\xi_1|}\, \sin^2(\xi_1)\, \mathrm{d}\xi_1 \\
&+ C_0 \left(\int_{|\xi_1|<1} \cdots - \int_{|\xi_1|<1} \cdots \right) + C_0 \left(\int_{|\xi_1|>a^{-1}} \cdots - \int_{|\xi_1|>b^{-1}} \cdots \right).
\end{aligned}
$$

The last two terms are $O(a-b) = (b\, \ln b - a\, \ln a)\, o(1)$ as $a, b \downarrow 0$. To show this, we follow line by line the arguments of **a** and **b** respectively, noting that all integrals are well-defined because of the new domains of integration. Let now I denote the remaining term. Recalling that $a > b$,

$$
\begin{aligned}
I ={}& C_0 \int_{1<|\xi_1|<a^{-1}} |\xi_1|^{-1}\, (a\, e^{-a\,|\xi_1|} - b\, e^{-b\,|\xi_1|})\, \sin^2(\xi_1)\, \mathrm{d}\xi_1 \\
&- C_0 \int_{a^{-1}<|\xi_1|<b^{-1}} b\, |\xi_1|^{-1}\, e^{-b\,|\xi_1|}\, \sin^2(\xi_1)\, \mathrm{d}\xi_1 \\
=:{}& I_1 + I_2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|I_2| &\leq C_0 \int_{a^{-1}<|\xi_1|<b^{-1}} b\, |\xi_1|^{-1}\, \mathrm{d}\xi_1 \\
&= 2\, C_0\, b\, (\ln a - \ln b) \leq 2\, C_0\, (a-b) = (b\, \ln b - a\, \ln a)\, o(1)
\end{aligned}
$$

as $a, b \downarrow 0$. Hence it remains to show that $\liminf_{a,b \overset{a>b>0}{\to} 0} \frac{I_1}{b\,\ln b - a\,\ln a} > 0$. Since

$$a\,e^{-a\,|\xi_1|} - b\,e^{-b\,|\xi_1|} = (a-b) \int_0^1 \{1 - (\tau\,a + (1-\tau)\,b)\,|\xi_1|\}\; e^{-(\tau\,a + (1-\tau)\,b)\,|\xi_1|}\,\mathrm{d}\tau$$
$$\geq \begin{cases} \frac{e^{-1}}{2}\,(a-b) & \text{for all } |\xi_1| \leq \frac{a^{-1}}{2}, \\ 0 & \text{for all } |\xi_1| \leq a^{-1}, \end{cases}$$

we find that

$$\begin{aligned} I_1 &\geq C_0\,\frac{e^{-1}}{2}\,(a-b) \int_{\frac{5\,\pi}{4} < |\xi_1| < \frac{a^{-1}}{2}} |\xi_1|^{-1}\, \sin^2(\xi_1)\,\mathrm{d}\xi_1 \\ &\geq \frac{C_0}{4}\,\frac{e^{-1}}{2}\,(a-b) \int_{\frac{5\,\pi}{4} < |\xi_1| < \frac{a^{-1}}{2}} |\xi_1|^{-1}\,\mathrm{d}\xi_1. \end{aligned}$$

To get the last line, we have used that since $\sin^2(\cdot) \geq \frac{1}{2}$ on $E := [\frac{\pi}{4}, \frac{3\,\pi}{4}] + \pi\,\mathbb{Z}$, with $\mathbb{R} \setminus E = E + \frac{\pi}{2}$,

$$\begin{aligned} \int_{\frac{5\,\pi}{4} < |\xi_1| < \frac{a^{-1}}{2}} |\xi_1|^{-1}\, \sin^2(\xi_1)\,\mathrm{d}\xi_1 &= \int_{\frac{5\,\pi}{4} < |\xi_1|} \underbrace{|\xi_1|^{-1}\, \mathbf{1}_{|\xi_1| < \frac{a^{-1}}{2}}}_{=:g(|\xi_1|)}\, \sin^2(\xi_1)\,\mathrm{d}\xi_1 \\ &\geq \frac{1}{2} \int_{E \cap \{\frac{5\,\pi}{4} < |\xi_1|\}} g(|\xi_1|)\,\mathrm{d}\xi_1 \\ &\geq \frac{1}{4} \int_{\frac{5\,\pi}{4} < |\xi_1|} g(|\xi_1|)\,\mathrm{d}\xi_1, \end{aligned} \tag{8.9}$$

by translation and since g is nonincreasing. It follows that

$$I_1 \geq \tilde{C}_0\,(b-a)\,\ln a + O(a-b) \geq \tilde{C}_0\,(b\,\ln b - a\,\ln a) + (b\,\ln b - a\,\ln a)\,o(1)$$

as $a, b \downarrow 0$, where $\tilde{C}_0 = \frac{C_0}{4\,e} > 0$. Here we have used that $b\,\ln a \geq b\,\ln b$, and since $b\ln b - a\ln a > 0$ for small $a > b > 0$, the proof of (i) is complete under the assumption that $T = \gamma = 1$.

For general $T, \gamma > 0$ fixed, the result follows from rescaling. Let $w(x,t) := \gamma^{-1}\,u(\gamma\,x, T\,t)$ and note that

$$\begin{cases} w_t + T\,\gamma^{-\alpha}\,a\,(-\triangle)^{\frac{\alpha}{2}}\,w = 0, \\ w(x,0) = \mathbf{1}_Q(x). \end{cases}$$

Set $\mu := T\,\gamma^{-\alpha}$ and $w =: w_{\mu\,a}$ to emphasize the dependence on the new "nonlinearity" $\mu\,a$. Then by the results of the $T = \gamma = 1$ case above,

$$\liminf_{a,b \downarrow 0} \frac{\|w_{\mu\,a} - w_{\mu\,b}\|_{C([0,1];L^1)}}{\omega_{\mu\,a - \mu\,b}} > 0,$$

where $\omega_{\cdot - \cdot}$ is defined on page 35. By a simple change of variables,

$$\|u_a - u_b\|_{C([0,T];L^1)} = \gamma^{d+1}\,\|w_{\mu\,a} - w_{\mu\,b}\|_{C([0,1];L^1)},$$

and since $\omega_{\mu\,a - \mu\,b} \sim \omega_{a-b}$ as $a, b \downarrow 0$ (μ is fixed!), (i) holds for any $T, \gamma > 0$.

2. *Item* (ii). Let us adapt the preceding arguments. We only give the proof for the case $\gamma = 1$ and $c = 1$, noting that the general result then easily follows from the rescaling $w(x,t) = \gamma^{-1}\, u(\gamma\, x, \gamma^\alpha\, c^{-1}\, t)$. We have

$$\begin{aligned}\mathrm{Lip}_\varphi(u;1) &\geq \lim_{a,b\to 1}\left|\frac{\int_Q u_a(x,T)\,\mathrm{d}x - \int_Q u_b(x,T)\,\mathrm{d}x}{a-b}\right| \\ &= \frac{2^d}{\pi^d}\int T\,|\xi|^\alpha\, e^{-T\,|\xi|^\alpha}\prod_{i=1}^d \mathrm{sinc}^2(\xi_i)\,\mathrm{d}\xi, \qquad (8.10)\end{aligned}$$

thanks to (8.4) written for time T. At this stage, the case $\alpha < 1$ follows from a direct passage to the limit. For the other ones, we argue as in (8.5)–(8.6), and find that there exists $C_0 > 0$ such that for all sufficiently small T,

$$\mathrm{Lip}_\varphi(u;1) \geq C_0 \underbrace{\int T\,|\xi_1|^\alpha\, e^{-d^{\alpha-1}\,T\,|\xi_1|^\alpha}\ \mathrm{sinc}^2(\xi_1)\,\mathrm{d}\xi_1}_{=:I}.$$

It remains to prove that $\liminf_{T\downarrow 0}\frac{I}{\sigma_T} > 0$. The case $\alpha > 1$ follows, as before, from the change of variable $T^{\frac{1}{\alpha}}\,\xi_1 \mapsto \xi_1$ and the L^∞-weak-$\star$ convergence of $\sin^2(T^{-\frac{1}{\alpha}}\,\cdot)$. For the $\alpha = 1$ case, we again split I into three parts,

$$I = \int_{1<|\xi_1|<T^{-1}}\cdots + \int_{|\xi_1|<1}\cdots + \int_{|\xi_1|>T^{-1}}\cdots.$$

As in case (i), the two last terms are $O(T) = T\,|\ln T|\,o(1)$ as $T \downarrow 0$, and the remaining integral can be bounded below as in (8.9) by

$$\tilde{C}_0 \int_{\frac{5\pi}{4}<|\xi_1|<T^{-1}} T\,|\xi_1|^{-1}\,\mathrm{d}\xi_1 \geq \tilde{C}_0\, T\,|\ln T| + T\,|\ln T|\,o(1) \quad \text{as } T\downarrow 0,$$

where $\tilde{C}_0 > 0$ is another constant independent of T small enough. The proof is complete.

3. *Item* (iii). We assume that $T = c = 1$, and note that the general case follows from the rescaling $w(x,t) = u(T^{\frac{1}{\alpha}}\, c^{\frac{1}{\alpha}}\, x, T\, t)$. We start as in the preceding case, considering this time integrals on $\gamma\, Q$ in (8.3). Arguing as in (8.4) by replacing Q by $\gamma\, Q$, we find that

$$\begin{aligned}\mathcal{E}_{\gamma\, Q} &= \int_{\gamma\, Q} u_a(x,1)\,\mathrm{d}x - \int_{\gamma\, Q} u_b(x,1)\,\mathrm{d}x \\ &= \frac{2^d}{\pi^d}\,\gamma^{2\,d+1}\int\int_0^1 (b-a)|\xi|^\alpha\, e^{-(\tau\, a+(1-\tau)\, b)\,|\xi|^\alpha}\prod_{i=1}^d \mathrm{sinc}^2(\gamma\,\xi_i)\,\mathrm{d}\tau\,\mathrm{d}\xi,\end{aligned}$$

and hence

$$\mathrm{Lip}_\varphi(u;1) \geq \lim_{a,b\to 1}\left|\frac{\mathcal{E}_{\gamma\, Q}}{a-b}\right| = \frac{2^d}{\pi^d}\,\gamma^{2\,d+1}\int |\xi|^\alpha\, e^{-|\xi|^\alpha}\prod_{i=1}^d \mathrm{sinc}^2(\gamma\,\xi_i)\,\mathrm{d}\xi.$$

After changing variables $\gamma\,\xi \mapsto \xi$, we then get that

$$\mathrm{Lip}_\varphi(u;1) \geq \frac{2^d}{\pi^d}\,\gamma^{d+1}\int \gamma^{-\alpha}\,|\xi|^\alpha\, e^{-\gamma^{-\alpha}\,|\xi|^\alpha}\prod_{i=1}^d \mathrm{sinc}^2(\xi_i)\,\mathrm{d}\xi.$$

This is the same expression as in (8.10) with $\gamma^{-\alpha}$ in place of T. Note that

$$\gamma^{d+1}\,\sigma_T\Big|_{T=\gamma^{-\alpha}} = \sigma_\gamma$$

according to the definitions of σ_T and σ_γ on page 35, and hence by the proof of (ii) we have that $\liminf_{\gamma\to+\infty}\frac{\mathrm{Lip}_\varphi(u;1)}{\sigma_\gamma} > 0$. The proof of (iii) is complete. □

Remark 8.5. In the proof of Corollary 3.6, a rescaling in time transformed the continuous dependence estimate (3.4) into the time continuity estimate (3.6). Hence, we leave it to the reader to verify that the same rescaling allows us to prove that (8.1) is also an example for which Corollary 3.6 is optimal.

Proof of Proposition 8.3. We adapt the arguments of the proof of Proposition 8.1(i).

1. *Item* (i). To avoid confusion with the proof of (ii) below, we denote the fixed parameter γ by $\tilde{\gamma}$. We consider the new difference

$$\mathcal{E}_Q := \int_{\tilde{\gamma}\, Q} u^\alpha(x,T)\,\mathrm{d}x - \int_{\tilde{\gamma}\, Q} u^\beta(x,T)\,\mathrm{d}x$$

with moving powers $\alpha, \beta \in (0,2)$ and time T. We let $M = T\,a$ and argue as in (8.4) to see that

$$\begin{aligned}
&\mathcal{E}_Q \\
&= \frac{2^d}{\pi^d}\,\tilde{\gamma}^{2\,d+1} \int (e^{-M\,|\xi|^\alpha} - e^{-M\,|\xi|^\beta}) \prod_{i=1}^d \operatorname{sinc}^2(\tilde{\gamma}\,\xi_i)\,\mathrm{d}\xi \\
&= \frac{2^d}{\pi^d}\,\tilde{\gamma}^{2\,d+1} \int \int_0^1 (\beta-\alpha)\,(\ln|\xi|)\,M\,|\xi|^{\tau\,\alpha+(1-\tau)\,\beta} \\
&\quad \cdot e^{-M\,|\xi|^{\tau\,\alpha+(1-\tau)\,\beta}} \prod_{i=1}^d \operatorname{sinc}^2(\tilde{\gamma}\,\xi_i)\,\mathrm{d}\tau\,\mathrm{d}\xi,
\end{aligned}$$

so that

$$\operatorname{Lip}_\alpha(u;\lambda) \geq \frac{2^d}{\pi^d}\,\tilde{\gamma}^{2\,d+1} \Bigg| \underbrace{\int (\ln|\xi|)\,M\,|\xi|^\lambda\, e^{-M\,|\xi|^\lambda} \prod_{i=1}^d \operatorname{sinc}^2(\tilde{\gamma}\,\xi_i)\,\mathrm{d}\xi}_{=:I} \Bigg|. \tag{8.11}$$

To complete the proof, we must show that $\liminf_{M\downarrow 0} \frac{|I|}{\tilde{\sigma}_M} > 0$.

a. *The case* $\lambda < 1$. Now

$$\lim_{M\downarrow 0} \frac{I}{M} = \int (\ln|\xi|)\,|\xi|^\lambda \prod_{i=1}^d \operatorname{sinc}^2(\tilde{\gamma}\,\xi_i)\,\mathrm{d}\xi =: I_{\tilde{\gamma}}.$$

Since $\operatorname{sinc}(0) \neq 0$, $\lim_{\tilde{\gamma}\downarrow 0} I_{\tilde{\gamma}} = +\infty$, and we see that (i) holds for $\tilde{\gamma}$ small enough.

In the other two cases we split I in two, $I = \int_{|\xi|<1} \cdots + \int_{|\xi|>1} \cdots$. The first integral is of order $O(M) = \tilde{\sigma}_M\, o(1)$ as $M \downarrow 0$, by a direct passage to the limit. Arguing as in the preceding proof (cf. (8.5)–(8.6)), the last integral can be bounded from below by

$$\int_{|\xi_1|>1} (\ln|\xi_1|)\,M\,|\xi_1|^{\lambda-2}\, e^{-d^{\lambda-1}\,M\,|\xi_1|^\lambda}\, \sin^2(\tilde{\gamma}\,\xi_1)\,\mathrm{d}\xi_1 =: J,$$

up to some positive multiplicative constant C_0 independent of M small enough. Note that C_0 will also depend on $\tilde{\gamma} > 0$ which is constant in this proof. Hence it suffices to show that $\liminf_{M\downarrow 0} \frac{J}{\tilde{\sigma}_M} > 0$.

b. *The case* $\lambda > 1$. By the change of variables $M^{\frac{1}{\lambda}}\,\xi_1 \mapsto \xi_1$,

$$\begin{aligned}
J = {} & M^{\frac{1}{\lambda}} \int_{|\xi_1|>M^{\frac{1}{\lambda}}} (\ln|\xi_1|)\,|\xi_1|^{\lambda-2}\, e^{-d^{\lambda-1}\,|\xi_1|^\lambda}\, \sin^2(M^{-\frac{1}{\lambda}}\,\tilde{\gamma}\,\xi_1)\,\mathrm{d}\xi_1 \\
& - \lambda^{-1}\, M^{\frac{1}{\lambda}}\,(\ln M) \int_{|\xi_1|>M^{\frac{1}{\lambda}}} |\xi_1|^{\lambda-2}\, e^{-d^{\lambda-1}\,|\xi_1|^\lambda}\, \sin^2(M^{-\frac{1}{\lambda}}\,\tilde{\gamma}\,\xi_1)\,\mathrm{d}\xi_1.
\end{aligned}$$

It is clear that the first term is $O(M^{\frac{1}{\lambda}}) = M^{\frac{1}{\lambda}}\,|\ln M|\,o(1)$ as $M \downarrow 0$, and that the second one has the expected behavior due to L^∞-weak-$\star$ convergence arguments.

c. *The case* $\lambda = 1$. We write $J = \int_{|\xi_1|>M^{-1}} \cdots + \int_{1<|\xi_1|<M^{-1}} \cdots$. The first term is $O(M\,|\ln M|) = M\,(\ln^2 M)\,o(1)$ as $M \downarrow 0$ by the change of variables argument of the $\lambda > 1$ case. For the remaining term, we argue as in (8.9), using this time that $\sin^2(\tilde\gamma\,\cdot)$ is bounded below by $\frac{1}{2}$ on $\tilde\gamma^{-1}E$. Taking N so large that the new function g (defined below) is nonincreasing on $((4\,N+1)\,\frac{\pi}{4}\,\tilde\gamma^{-1}, +\infty)$, we get a lower bound of the form

$$\begin{aligned}
&\int_{1<|\xi_1|<M^{-1}} (\ln|\xi_1|)\,M\,|\xi_1|^{-1}\,e^{-M\,|\xi_1|}\sin^2(\tilde\gamma\,\xi_1)\,\mathrm{d}\xi_1\\
&\geq e^{-1}\,M \int_{(4\,N+1)\,\frac{\pi}{4}\,\tilde\gamma^{-1}<|\xi_1|} \underbrace{(\ln|\xi_1|)\,|\xi_1|^{-1}\,\mathbf{1}_{|\xi|<M^{-1}}}_{=:g(|\xi_1|)} \sin^2(\tilde\gamma\,\xi_1)\,\mathrm{d}\xi_1\\
&\geq \frac{e^{-1}}{4}\,M \int_{(4\,N+1)\,\frac{\pi}{4}\,\tilde\gamma^{-1}<|\xi_1|<M^{-1}} (\ln|\xi_1|)\,|\xi_1|^{-1}\,\mathrm{d}\xi_1\\
&= \frac{e^{-1}}{4}\,M\,\ln^2 M + M\,(\ln^2 M)\,o(1) \quad \text{as } M \downarrow 0.
\end{aligned}$$

The proof of (i) is now complete.

2. *Item* (ii). To avoid confusion with the preceding proof, we denote the fixed parameter $M = T\,a$ by $\tilde M$. Then, by (8.11),

$$\operatorname{Lip}_\alpha(u;\lambda) \geq \frac{2^d}{\pi^d}\,\tilde M \,\Big|\underbrace{\gamma^{2\,d+1}\int (\ln|\xi|)\,|\xi|^\lambda\,e^{-\tilde M\,|\xi|^\lambda}\prod_{i=1}^d \operatorname{sinc}^2(\gamma\,\xi_i)\,\mathrm{d}\xi}_{=:I}\Big| \tag{8.12}$$

and it suffices to show that $\liminf_{\gamma\to+\infty} \frac{|I|}{\tilde\sigma_\gamma} > 0$.

a. *The case* $\lambda > 1$. Since $\ln|\xi|$ has different signs inside and outside the unit ball, we split the integral I in two,

$$I = \int_{|\xi|<1} \cdots + \int_{|\xi|>1} \cdots =: I_1 + I_2.$$

By the inequality $|\ln|\xi||\,|\xi|^\lambda \leq d^{\lambda-1}\sum_{i=1}^d |\ln|\xi_i||\,|\xi_i|^\lambda$ for $|\xi| < 1$ and the change of variables $\gamma\,\xi_j \mapsto \xi_j$ for $j \neq i$, we find that

$$\begin{aligned}
|I_1| &\leq d^{\lambda-1}\,\gamma^{2\,d+1}\sum_{i=1}^d \int_{|\xi|<1} |\ln|\xi_i||\,|\xi_i|^\lambda \prod_{j=1}^d \operatorname{sinc}^2(\gamma\,\xi_j)\,\mathrm{d}\xi\\
&\leq d^{\lambda-1}\,\gamma^d \sum_{i=1}^d \left\{\int_{|\xi_i|<1} \frac{|\ln|\xi_i||\,|\xi_i|^\lambda}{\xi_i^2}\,\mathrm{d}\xi_i \prod_{j\neq i}\int \operatorname{sinc}^2(\xi_j)\,\mathrm{d}\xi_j\right\}.
\end{aligned}$$

Here we also have used that $\sin^2(\gamma\,\xi_i) \leq 1$. It follows that $\limsup_{\gamma\to+\infty}\frac{|I_1|}{\gamma^d} \leq C(d,\lambda)$, a constant that does not depend on $\tilde M$.

Let us now see that I_2 is the dominant term provided that the fixed parameter $\tilde M$ is chosen sufficiently small. We have

$$I_2 \geq \gamma^{2\,d+1}\int (\ln|\xi_1|)\,|\xi_1|^\lambda\,e^{-\tilde M\,|\xi|^\lambda}\prod_{i=1}^d \operatorname{sinc}^2(\gamma\,\xi_i)\,\mathrm{d}\xi,$$

and then, letting $\xi_\gamma := (\xi_1, \gamma^{-1}\xi_2, \dots, \gamma^{-1}\xi_d)$ and changing variables $\gamma\,\xi_i \mapsto \xi_i$ for $i \neq 1$, we find that

$$I_2 \geq \gamma^d \int \left\{ (\ln|\xi_1|)\,|\xi_1|^{\lambda-2} \int e^{-\tilde{M}\,|\xi_\gamma|^\lambda} \prod_{i=2}^d \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi_2 \dots \mathrm{d}\xi_d \right\} \sin^2(\gamma\,\xi_1)\,\mathrm{d}\xi_1.$$

By L^∞-weak-$\star$ convergence arguments, $\liminf_{\gamma\to+\infty} \frac{I_2}{\gamma^d} \geq \tilde{m}\, I_{\tilde{M}}$, where

$$\tilde{m} := \int_0^{2\pi} \sin^2(\xi_1)\,\mathrm{d}\xi_1 \prod_{i=2}^d \int \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi_i > 0$$

and $I_{\tilde{M}} := \int (\ln|\xi_1|)\,|\xi_1|^{\lambda-2}\, e^{-\tilde{M}\,|\xi_1|^\lambda}\,\mathrm{d}\xi_1$. Since $\lim_{\tilde{M}\downarrow 0} I_{\tilde{M}} = +\infty$, it suffices to fix $\tilde{M} > 0$ small to get (ii) in the $\lambda > 1$ case.

b. *The case* $\lambda < 1$. We restart from (8.12), change the variables $\gamma\,\xi \mapsto \xi$, and pass to the limit as $\gamma \to +\infty$. The result follows.

c. *The case* $\lambda = 1$. Let us rewrite I in (8.12) as

$$I = \underbrace{\gamma^{2d+1} \int_{\gamma^{-1}<|\xi|<1} \dots}_{=:-J_1} + \underbrace{\gamma^{2d+1} \int_{|\xi|>1} \dots}_{=:J_2} + \gamma^{2d+1} \int_{|\xi|<\gamma^{-1}} \dots.$$

By the arguments of the $\lambda < 1$ case, the last integral is of order $O(\gamma^d \ln\gamma) = \gamma^d\,(\ln^2\gamma)\,o(1)$ as $\gamma \to +\infty$. For J_2, we use that

$$\left| \int_{|\xi|>1} \dots \right| \leq \int_{\cup_{i=1}^d \left\{\xi : |\xi_i| > \frac{1}{\sqrt{d}}\right\}} |\dots| \leq \sum_{i=1}^d \int_{\xi : |\xi_i| > \frac{1}{\sqrt{d}}} |\dots| = d \int_{\xi : |\xi_1| > \frac{1}{\sqrt{d}}} |\dots|$$

by symmetry of the I-integrand (cf. (8.12)). We then bound $|\ln|\xi||\,|\xi|\,e^{-\tilde{M}|\xi|}$ by some constant C, change the variables $\gamma\,\xi_i \mapsto \xi_i$ for $i \neq 1$, get

$$|J_2| \leq d\,C\,\gamma^d \int_{|\xi_1|>\frac{1}{\sqrt{d}}} \xi_1^{-2}\,\mathrm{d}\xi_1 \prod_{i=2}^d \int \operatorname{sinc}^2(\xi_i)\,\mathrm{d}\xi_i$$

and conclude that $J_2 = O(\gamma^d) = \gamma^d\,(\ln^2\gamma)\,o(1)$ as $\gamma \to +\infty$.

Since $J_1 > 0$, it remains to show that $\liminf_{\gamma\to+\infty} \frac{J_1}{\gamma^d \ln^2\gamma} > 0$. It will be convenient to use the notation $\hat{\xi} := (\xi_2, \dots, \xi_d)$. By the change of variables $\gamma\,\xi \mapsto \xi$ and the inequality $|\xi|\,\xi_1^{-2} \geq |\xi|^{-1}$,

$$J_1 \geq -\gamma^d \int_{1<|\xi|<\gamma} \frac{\ln(\gamma^{-1}|\xi|)}{|\xi|}\, f(\hat{\xi})\, \sin^2(\xi_1)\,\mathrm{d}\xi,$$

where $f(\hat{\xi}) := e^{-\tilde{M}} \prod_{i=2}^d \operatorname{sinc}^2(\xi_i)$. Let us restrict to the domain of integration

$$A := \left\{ (\xi_1, \hat{\xi}) \text{ s.t. } \frac{5\pi}{4} < |\xi_1| < \frac{1}{2}\sqrt{\gamma^2 - |\hat{\xi}|^2} \text{ and } \epsilon^2 < |\hat{\xi}| < \epsilon \right\},$$

where $\epsilon > 0$ is fixed and so small that $A \subset \{1 < |\xi| < \gamma\}$ and $C_0 := \min_A f(\hat{\xi}) > 0$. Then,

$$J_1 \geq -C_0\,\gamma^d \int_{\epsilon^2<|\hat{\xi}|<\epsilon} \int_{\frac{5\pi}{4}<|\xi_1|<\frac{1}{2}\sqrt{\gamma^2-|\hat{\xi}|^2}} \frac{\ln(\gamma^{-1}|\xi|)}{|\xi|}\, \sin^2(\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\hat{\xi}.$$

Arguing as in (8.9),

$$J_1 \geq -\tilde{C}_0\,\gamma^d \int_{\epsilon^2<|\hat{\xi}|<\epsilon} \int_{\frac{5\pi}{4}<|\xi_1|<\frac{1}{2}\sqrt{\gamma^2-|\hat{\xi}|^2}} \frac{\ln(\gamma^{-1}|\xi|)}{|\xi|}\,\mathrm{d}\xi_1\,\mathrm{d}\hat{\xi},$$

where $\tilde{C}_0 = \frac{C_0}{4} > 0$. If γ is large enough, then for all $\xi \in A$, $\gamma^{-1}|\xi| \leq \gamma^{-1}(|\xi_1| + |\hat{\xi}|) < 1$. We can then use that

$$-\frac{\ln(\gamma^{-1}|\xi|)}{|\xi|} \geq -\frac{\ln\left(\gamma^{-1}(|\xi_1|+|\hat{\xi}|)\right)}{|\xi_1|+|\hat{\xi}|},$$

and by integrating the right-hand side, we get

$$\begin{aligned}
J_1 &\geq \tilde{C}_0\,\gamma^d \int_{\epsilon^2<|\hat{\xi}|<\epsilon} \ln^2\left(\gamma^{-1}\left(\frac{5\,\pi}{4}+|\hat{\xi}|\right)\right) \mathrm{d}\hat{\xi} \\
&\quad - \tilde{C}_0\,\gamma^d \int_{\epsilon^2<|\hat{\xi}|<\epsilon} \ln^2\left(\frac{1}{2}\sqrt{1-\gamma^{-2}\,|\hat{\xi}|^2}+\gamma^{-1}\,|\hat{\xi}|\right) \mathrm{d}\hat{\xi} \\
&\geq \tilde{\tilde{C}}_0\,\gamma^d\,\ln^2\gamma + \gamma^d\,(\ln^2\gamma)\,o(1) \quad \text{as } \gamma \to +\infty.
\end{aligned}$$

Here $\tilde{\tilde{C}}_0$ is another positive constant independent of γ large enough. The proof is complete. □

Appendix A. Proofs of (6.17), (6.18) and (6.19)

Proof of (6.17) *and* (6.18). Recall that $\Omega_\xi(\cdot)$ is defined on page 21 and χ_a^b in (6.14). For A, we use that

$$\begin{aligned}
\int \|\Omega_\xi(u)\|_{L^1(Q_T)}\,\mathrm{d}\xi &= \int\int_{Q_T}\left|\int \chi_0^{u(x,t)}(\zeta)\,\omega_\delta(\xi-\zeta)\,\mathrm{d}\zeta\right| \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi \\
&= \int\int_{Q_T}\left|\chi_0^{u(x,t)}(\zeta)\right|\int \omega_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\zeta = \|u\|_{L^1(Q_T)}.
\end{aligned}$$

For B, we consider $\{u_n\}_n \subset C([0,T];W^{1,1})$ converging to u in $C([0,T];L^1)$ and such that $\int_{Q_T}|\nabla u_n| \to |u|_{L^1(0,T;BV)}$. Then

$$\begin{aligned}
&\int\int_{Q_T} |\nabla\Omega_\xi(u_n)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi \\
&= \int\int_{Q_T} \omega_\delta(\xi-u_n(x,t))\,|\nabla u_n(x,t)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi = \int_{Q_T}|\nabla u_n|,
\end{aligned}$$

so that

$$\begin{aligned}
\int |\Omega_\xi(u)|_{L^1(0,T;BV)}\,\mathrm{d}\xi &\leq \int\int_0^T \left\{\liminf_{n\to+\infty}\int_{\mathbb{R}^d}|\nabla\Omega_\xi(u_n)|\,\mathrm{d}x\right\}\mathrm{d}t\,\mathrm{d}\xi \\
&\leq \lim_{n\to+\infty}\int\int_{Q_T}|\nabla\Omega_\xi(u_n)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi = |u|_{L^1(0,T;BV)},
\end{aligned}$$

due to the lower semi-continuity of the BV-semi-norm with respect to the L^1-norm and to Fatou's lemma.[6] □

[6] For the reverse inequality, we use that, at all fixed time and for all $\Phi \in C_c^1(\mathbb{R}^d,\mathbb{R}^d)$ such that $|\Phi| \leq 1$,

$$\begin{aligned}
\int |\Omega_\xi(u)|_{BV}\,\mathrm{d}\xi &\geq \int\int_{\mathbb{R}^d}\Omega_\xi(u)\,\mathrm{div}\Phi\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int\int_{\mathbb{R}^d}\int \chi_0^u(\zeta)\,\omega_\delta(\xi-\zeta)\,\mathrm{div}\Phi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^d} u\,\mathrm{div}\Phi\,\mathrm{d}x,
\end{aligned}$$

and next we take the supremum with respect to Φ.

Proof of (6.19). Recall that $\mathcal{E}_3$ and $\mathcal{E}_3(\delta)$ are defined in (6.2) and (6.16), respectively. See also the original assumption (3.2) of the theorem, and the simplifying assumption (6.1). First we define

$$F_v(x,t,y,s,z,\xi) := \operatorname{sgn}(v-u)\,\frac{\chi_{v(x,t)}^{v(x+z,t)}(\xi)}{|z|^{d+\alpha}}\,\phi^{\epsilon,\nu}.$$

Let us recall that F_v is integrable on $Q_T^2 \times \{|z|>r\} \times \mathbb{R}$ since $\int |\chi_a^b(\xi)|\,\mathrm{d}\xi = |b-a|$. Hence, by Fubini the function

$$G_v(\xi) := \int_{Q_T^2} \int_{|z|>r} F_v(x,t,y,s,z,\xi)\,\mathrm{d}z\,\mathrm{d}w$$

is integrable with respect to $\xi \in \mathbb{R}$. But, by (6.15), (6.16),

$$\mathcal{E}_3 = G_d(\alpha) \int G_v(\xi)\,\psi'(\xi) - G_u(\xi)\,\varphi'(\xi)\,\mathrm{d}\xi$$

and $\mathcal{E}_3(\delta) = G_d(\alpha) \int \psi'(\xi)\, G_v * \omega_\delta(\xi) - \varphi'(\xi)\, G_u * \omega_\delta(\xi)\,\mathrm{d}\xi$, where $*$ is the convolution product in $\mathbb{R}$. Since ω_δ is an approximate unit, the convolution products inside the integral respectively converge to G_v and G_u in $L^1(\mathbb{R})$ as $\delta \downarrow 0$. Using in addition that φ' and ψ' are bounded by (6.1), $\lim_{\delta\downarrow 0} \mathcal{E}_3(\delta) = \mathcal{E}_3$. □

Appendix B. Some technical lemmas

Lemma B.1. *For all $a, b > 0$, $|a-b|\,(-\ln(a \vee b))^+ \le |a-b| + |a\,\ln a - b\,\ln b|$.*

Proof. We assume without loss of generality that $a \vee b = a$ and that $a \le e^{-1}$ (the result is trivial otherwise). Then

$$|a\,\ln a - b\,\ln b| = -\int_b^a (1+\ln\tau)\,\mathrm{d}\tau = -|a-b| - \int_b^a (\ln\tau)\,\mathrm{d}\tau,$$

since $1+\ln\tau$ is negative, and hence

$$|a-b| + |a\,\ln a - b\,\ln b| \ge -|a-b|\,\ln a,$$

since the logarithm is nondecreasing. This completes the proof. □

Lemma B.2. *For all $x > 0$, $a, b \ne 0$ and $c > 0$,*

(i) $\left|\frac{x^a-1}{a} - \frac{x^b-1}{b}\right| \le |a-b|\,(1 \vee x^a \vee x^b)\,\ln^2 x$,

(ii) $\lim_{a,b\to c}\left\{\frac{1}{(a-b)^2}\left|\frac{x^{2a}}{2a} + \frac{x^{2b}}{2b} - \frac{2x^{a+b}}{a+b}\right|\right\} \le C\,x^{2c}\,(1+\ln^2 x)$,

where $C = C(c)$.

Proof. (i) Let $f(a) = \frac{x^a-1}{a}$. Observe that $(\ln x)\int_0^1 x^{\tau a}\,\mathrm{d}\tau = f(a)$ by a Taylor expansion of x^a at $a=0$. It then follows that by differentiating under the integral sign that $f'(a) = (\ln^2 x)\int_0^1 \tau\,x^{\tau a}\,\mathrm{d}\tau$ and

$$\begin{aligned} f(a) - f(b) &= (a-b)\int_0^1 f'(\tau\,a + (1-\tau)\,b)\,\mathrm{d}\tau \\ &= (a-b)\,(\ln^2 x)\int_0^1\int_0^1 \tilde\tau\,x^{\tilde\tau\,(\tau\,a+(1-\tau)\,b)}\,\mathrm{d}\tilde\tau\,\mathrm{d}\tau. \end{aligned}$$

Since $x^{\tilde\tau\,(\tau\,a+(1-\tau)\,b)} \le 1 \vee x^{a\vee b} \vee x^{a\wedge b}$, we find that

$$|f(a) - f(b)| \le |a-b|\,(\ln^2 x)\,(1 \vee x^a \vee x^b),$$

and the proof of (i) is complete.

(ii) Note that

$$\frac{x^{2\,a}}{2\,a}+\frac{x^{2\,b}}{2\,b}-\frac{2\,x^{a+b}}{a+b}=\frac{b\,(a+b)\,x^{2\,a}+a\,(a+b)\,x^{2\,b}-4\,a\,b\,x^{a+b}}{2\,a\,b\,(a+b)}$$
$$=\frac{(x^a-x^b)^2}{2\,(a+b)}+\frac{(b\,x^a-a\,x^b)^2}{2\,a\,b\,(a+b)}.$$

The first term satisfies $|x^a - x^b| \leq |a-b|\,(x^a \vee x^b)\,|\ln x|$. Moreover, by adding and subtracting terms and the inequality $\frac{(A+B)^2}{2} \leq A^2 + B^2$, we find that the second term is bounded by

$$(b\,x^a - a\,x^b)^2 \leq \frac{1}{2}\,(a-b)^2\,(x^a+x^b)^2 + \frac{1}{2}\,(a+b)^2\,(x^a-x^b)^2.$$

The proof now follows from these two inequalities. □

Appendix C. Proofs of Lemma 7.3 and (7.5)

Proof of Lemma 7.3. The if part follows by approximating the Kruzhkov entropy $u \mapsto |u-k|$ by smooth convex entropies $u \mapsto \eta_n(u) := \sqrt{(u-k)^2 + n^{-2}} - n^{-1}$. The functions $\eta_n(\cdot)$ and $\eta_n'(\cdot)$ are locally uniformly bounded and converge pointwise to $|\cdot - k|$ and the everywhere representative of its weak derivative given by (2.3). Hence, if a function $u = u(x,t)$ is bounded and such that $\eta_n(u)$ satisfies (7.2), we can use the dominated convergence theorem to pass to the limit and find that $|u-k|$ satisfies (2.5).

To prove the only if part, we note that we may approximate (locally uniformly) any convex entropy $\eta \in C^1(\mathbb{R})$ by a family of piecewise linear functions $\tilde{\eta}_n$ of the form

$$u \mapsto \tilde{\eta}(u) = a + b\,u + \sum_{i=1}^{m} c_i\,|u-k_i| \tag{C.1}$$

where $a, b, k_i \in \mathbb{R}$, $c_i \geq 0$, and $m \in \mathbb{N}$. See e.g. [36, p. 27] for a proof. We need to refine this construction to ensure everywhere convergence of the derivatives $\tilde{\eta}_n'$. Consider the everywhere defined representative of $\tilde{\eta}'$ defined by

$$u \mapsto \tilde{\eta}'(u) = b + \sum_{i=1}^{m} c_i\ \mathrm{sgn}(u-k_i), \tag{C.2}$$

where the sign function is everywhere defined by (2.3). Since η' is continuous, it can be approximated uniformly on compact sets by piecewise constant functions of the form (C.2). Take such a sequence $\{\tilde{\eta}_n'\}_n$ that converges locally uniformly on $\mathbb{R}$ and redefine $\{\tilde{\eta}_n\}_n$ to be the primitives such that $\tilde{\eta}_n(0) = \eta(0)$, i.e. functions of the form (C.1). It follows that both $\tilde{\eta}_n$ and $\tilde{\eta}_n'$ converge locally uniformly towards η and η'.

Consider next the entropy solution $u = u(x,t)$ of (1.1) and note that the left-hand side of the entropy inequality (7.2) is linear with respect to η, that (7.2) holds with $\eta(u) = a + b\,u$ since u is a weak (distributional) solution of (1.1), and that (7.2) holds with $\eta(u) = c_i\,|u-k_i|$ by the Kruzhkov inequality (2.5) since $c_i \geq 0$. The reader may then check that (7.2) also holds with $\eta = \tilde{\eta}$ and the everywhere representative of $\tilde{\eta}'$ given by (C.2).

Since u is bounded, we may use the dominated convergence theorem to pass to the limit in (7.2) with $\eta = \tilde{\eta}_n$ to find that (7.2) holds also for the η in the limit. The proof is complete. □

Proof of (7.5). Combining (1.2) and (2.1),

$$\begin{aligned}\int_{\mathbb{R}^d} |2\,\pi\,\xi|^\alpha\,|\mathcal{F}u|^2\,\mathrm{d}\xi &= \int_{\mathbb{R}^d} u\,\mathcal{F}^{-1}\left(|2\,\pi\,\cdot|^\alpha\,\mathcal{F}u\right)\mathrm{d}x\\ &= \int_{\mathbb{R}^d} u\,(-\triangle)^{\frac{\alpha}{2}}\,u\,\mathrm{d}x\\ &= -\lim_{r\downarrow 0}\int_{\mathbb{R}^d} u\,\mathcal{L}^{\alpha,r}[u]\,\mathrm{d}x\\ &= \frac{G_d(\alpha)}{2}\iint_{\mathbb{R}^{2\,d}} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

thanks to (7.7) with $v = u$ to get the last line. □

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