A variational approach to the inversion of truncated Fourier operators

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Abstract. Truncated Fourier operators play an important role in many inverse problems of Signal and Image science. A variational approach to the regularization of their pseudo-inverses is considered. A particular regularization parameter, which can be interpreted in terms of *resolution level*, appears to play the essential role. This paper presents results on the behavior of the regularized solution as this parameter tends to zero. Notably, reasonably mild conditions are shown to ensure strong convergence of the regularized solution to the pseudo-inverse of the data.

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1 Introduction

The problem of Fourier Synthesis is of central importance in many applications pertaining to signal and image processing. At a rather abstract level, it can be formulated as follows:

> Recover a function from a partial and approximate knowledge of its Fourier transform.

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Among the many applications which belong to this general class of problems, let us mention aperture synthesis (in astronomy and earth observation) [5], deconvolution problems (see *e.g.* [6]), spectral analysis of signals (see *e.g.* [3]) and tomography [8, 9]. Notice that Shannon's interpolation formula may also be regarded as an explicit solution of a particular problem of Fourier synthesis.

In [6, 5], Lannes *et al.* stated and analyzed the problem in more specific terms, namely:

> Let V and W be subsets of $\mathbb{R}^d$. Assume that V is bounded and that W has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on W.

Here, $L^2(\Omega)$ denotes the space of square integrable complex valued functions having their (essential) support in the set $\Omega \subseteq \mathbb{R}^d$. (For every $p \in [1, \infty]$, $L^p(\Omega)$ is defined likewise and $\|\cdot\|_{L^p(\Omega)}$ denotes the corresponding L^p-norm.) In the case where W (resp. W^c) is bounded, the problem is referred to as that of *Fourier extrapolation* (resp. *Fourier interpolation*). It has been shown [6] that the problem of Fourier extrapolation is ill-posed, whereas the problem of Fourier interpolation is well-posed in the least square sense.

In practice, of course, the experimental data provide some knowledge of the Fourier transform on a bounded domain. In [6, 5], an original regularization principle for problems of Fourier extrapolation was designed. In essence, this regularization principle consists in reformulating the problem in terms of Fourier interpolation. It amounts to replacing the original problem of recovering the unknown object f_0 by that of recovering a *limited resolution* version of it, namely, $\phi * f_0$, where ϕ is some convolution kernel (or *point spread function*). Well-known results from the approximation of L^p-functions by mollification suggest that ϕ should be regarded as a member of the one-parameter family $\{\phi_\beta | \beta > 0\}$ defined by

$$\phi_\beta(x) = \frac{1}{\beta^d}\,\phi\left(\frac{x}{\beta}\right), \tag{1}$$

where $\phi \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1$. The reconstructed object may then be defined as the solution of the optimization problem

$$(\mathcal{P}_{\alpha,\beta}) \quad \left| \begin{array}{rl} \text{minimize} & \dfrac{1}{2}\left\| \hat{\phi}_\beta g - \mathbb{1}_W \hat{f} \right\|^2_{L^2(W)} + \dfrac{\alpha}{2}\left\| (1-\hat{\phi}_\beta)\hat{f} \right\|^2_{L^2(\mathbb{R}^d)} \\ \text{s.t.} & f \in L^2(V), \end{array} \right.$$

in which $\mathbb{1}_W$ denotes the characteristic function of W, and $\hat{f} = Uf$ denotes the image of f by the Fourier operator.

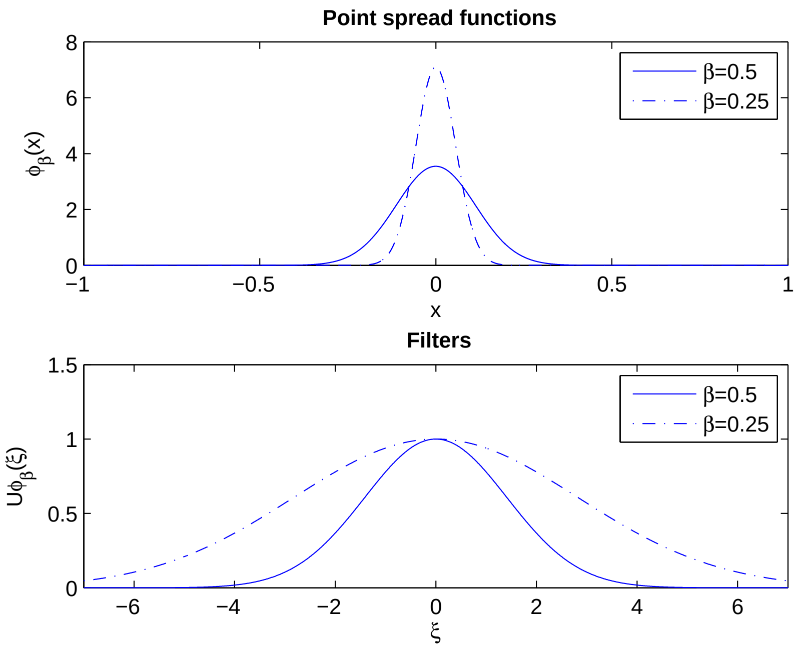

Figure 1: One-dimensional examples of functions ϕ_β (point spread functions) and $\hat{\phi}_\beta$ (filters): Gaussian case.

The *regularized data* $\hat{\phi}_\beta g$ are expected to correspond to $\phi_\beta * f_0$. The regularization term $\|(1-\hat{\phi}_\beta)\hat{f}\|^2/2$ can be interpreted as the *energy of f in the high frequency domain* (the inverse of β being interpreted as a *cutoff frequency*), whereas the fit term acts in the *low frequency domain*. Both terms are designed so as to be as little in conflict as possible, which suggests that the choice of a particular α is not as crucial as in the case of other regularization principles. Figure 1 gives one-dimensional examples of ϕ_β and $\hat{\phi}_\beta$.
In the pioneering works [6, 5], convergence analysis was not investigated. More precisely, one may (and one should) wonder about the destination of the reconstructed object when α and/or β converge to zero. This is the main purpose of this article. Let us emphasize that β appears here as the essential regularization parameter and that, consequently, the above regularization scheme is in fact quite different from Tikhonov's approach (in which α is the important parameter).

The paper is organized as follows. In Section 2, we review basic facts about Fourier synthesis. Ill-posedness of the Fourier extrapolation problem is outlined, as well as well-posedness of the Fourier interpolation problem. In Section 3, we consider the behavior of the solution of $(\mathcal{P}_{\alpha,\beta})$ as Tikhonov's parameter α goes to zero. This reveals difficulties which tend to confirm that β is the fundamental parameter of this regularization principle. In Section 4, we prove that the solution of $(\mathcal{P}_{\alpha,\beta})$ converges, as β tends to zero, to the Moore-Penrose inverse of the data g, under some regularity condition of the latter as well as conditions on the underlying point spread function. In the last section, the previous assumptions are somewhat relaxed, providing a regularization scheme which is morphologically less restrictive.

NOTATION: We shall denote by T_Ω the operator

$$\begin{array}{rccl} T_\Omega\colon & L^2(V) & \longrightarrow & L^2(\Omega) \\ & f & \longmapsto & T_\Omega f := \mathbb{1}_\Omega \hat{f} = \mathbb{1}_\Omega U f. \end{array}$$

Operators of this form will be referred to as *truncated Fourier operators*. The set of all continuous functions on $\mathbb{R}^d$ which vanish at infinity will be denoted by $C_0(\mathbb{R}^d)$. The closed ball of $\mathbb{R}^d$ centered at the origin of radius r will be denoted by B_r.

STANDING ASSUMPTION: Throughout, V and W are bounded subsets of $\mathbb{R}^d$ with non-empty interiors[1].

2 Ill-posedness and regularization

In this section, we review a few fundamental facts about Fourier synthesis. These results originate from [6]. For the sake of completeness, however, most statements will be proved. In fact, our setting as well as some of the proofs are somewhat different from those given in [6].

2.1 Fourier extrapolation

Proposition 1. *The linear operator T_W is compact and injective. The Hermitian operator $T_W^\star T_W$ is diagonalizable in a Hilbert basis $\{f_k\}_{k\in\mathbb{N}^*}$, and the eigenvalues are the values taken by a sequence $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ which*

[1] Assuming that $\operatorname{int} V \neq \emptyset$ and that W is bounded implies that $\ker T_W^\star = \{0\}$. For details, see Remark 3. Most of the results of this paper would still be true without the assumption $\operatorname{int} V \neq \emptyset$, but such an assumption is entirely natural and will simplify our developments.

converges to zero. The inverse $T_W^{-1}\colon \operatorname{ran} T_W \to L^2(V)$ *is unbounded and so is the pseudo-inverse* T_W^+*. Moreover,* $\operatorname{ran} T_W^{-1}$ *is not closed.*

Proof. Notice first that, for all $f \in L^2(V)$ and all $\xi \in \mathbb{R}^d$,

$$(T_W f)(\xi) = \int_{\mathbb{R}^d} e^{-2i\pi\langle x,\xi\rangle} \mathbb{1}_V(x)\mathbb{1}_W(\xi) f(x)\,\mathrm{d}x.$$

Since V and W are bounded, the kernel

$$\alpha(x,\xi) := e^{-2i\pi\langle x,\xi\rangle}\mathbb{1}_V(x)\mathbb{1}_W(\xi)$$

belongs to $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, which shows that T_W is Hilbert-Schmidt (thus compact). Now, recall that the Fourier transforms of compactly supported functions are entire functions, which implies that they are completely determined by their values on the set W (since we assume that W has nonempty interior). Consequently, T_W is injective. The proposition then results from Theorem 14 (see the appendix). ∎

Proposition 2. *The largest eigenvalue* λ_1 *of* $T_W^\star T_W$ *is strictly less than 1.*

Proof. An easy computation shows that, for all $g \in L^2(W)$, $T_W^\star g = \mathbb{1}_V U^{-1} g$, so that $T_W^\star T_W = \mathbb{1}_V U^{-1}\mathbb{1}_W U$ and

$$\begin{aligned}\lambda_1^2 = \|T_W^\star T_W f_1\|^2_{L^2(V)} &= \int_V \left|U^{-1}\mathbb{1}_W U f_1\right|^2 \\ &\le \int_{\mathbb{R}^d} \left|U^{-1}\mathbb{1}_W U f_1\right|^2 = \int_W \left|U f_1\right|^2,\end{aligned}$$

where the last equality results from Plancherel's Theorem; the last integral is strictly less than 1, for otherwise Uf_1 would vanish on the complement of W, thus on $\mathbb{R}^d$ (since it is analytic), in contradiction with the fact that $f_1 \neq 0$. ∎

We end this subsection by a remark on the domain of the Moore-Penrose pseudo-inverse T_W^+ of T_W (see also Footnote 1 on Page 4).

Remark 3. It is well known that $\mathcal{D}(T_W^+)$, the domain of T_W^+, satisfies

$$\mathcal{D}(T_W^+) = \operatorname{ran} T_W + \ker T_W^\star = \operatorname{ran} T_W + (\operatorname{ran} T_W)^\perp.$$

In our context, $T_W^\star$ is injective, again as a consequence of the analyticity of Fourier transforms of compactly supported function. It follows that $\mathcal{D}(T_W^+) = \operatorname{ran} T_W = \mathcal{D}(T_W^{-1})$, and that T_W^{-1} and T_W^+ coincide. Nevertheless, we shall use T_W^+ instead of T_W^{-1} throughout, in order to emphasize that our developments remain true in a more general framework. ∎

2.2 Fourier regularization

Let us first introduce the problem of *Fourier interpolation*, which consists in *inverting* the truncated Fourier operator T_Ω, in which Ω is assumed to have a bounded complement (and a nonempty interior).

Proposition 4. *Let us assume that $\Omega \subseteq \mathbb{R}^d$ is such that Ω^c is bounded. Then,*

(i) T_Ω *is bounded and injective;*

(ii) $\operatorname{ran} T_\Omega$ *is closed;*

(iii) $T_\Omega^{-1} \colon \operatorname{ran} T_\Omega \to L^2(V)$ *is bounded.*

PROOF. It is readily seen that $T_\Omega^\star T_\Omega = I - T_{\Omega^c}^\star T_{\Omega^c}$, in which I denotes the identity of $L^2(V)$. From the properties of the Fourier extrapolation problem, we deduce that $T_\Omega^\star T_\Omega$ can be diagonalized, and that its eigenvalues are the values of a sequence $0 < \mu_1 \leq \mu_2 \leq \ldots < 1$ which converges to 1. As a matter of fact, $\mu_k = 1 - \lambda_k(T_{\Omega^c}^\star T_{\Omega^c})$ for all k, and Proposition 2 ensures that $0 < \mu_1$. Consequently, both

$$T_\Omega^\star T_\Omega \colon L^2(V) \to \operatorname{ran}(T_\Omega^\star T_\Omega) \quad \text{and} \quad T_\Omega \colon L^2(V) \to \operatorname{ran} T_\Omega$$

have continuous inverses, and $\operatorname{ran} T_\Omega$ is closed. ∎

As pointed out in [6], this suggests to regularize the Fourier extrapolation problem by reformulating it in terms of Fourier interpolation. In a rough version of this approach, the reconstructed object is defined as the minimizer of a functional of the form

$$\frac{1}{2}\| g - T_W f \|^2_{L^2(W)} + \frac{1}{2}\| \mathbb{1}_{W_\beta} U f \|^2_{L^2(W_\beta)},$$

where W_β is the complement of the ball $B_{1/\beta}$ centered at the origin of radius $1/\beta$. The parameter β is chosen small enough to ensure that $W_\beta \cap W = \emptyset$. Clearly, this amounts to interpolate the Fourier transform on $(W_\beta \cup W)^c$ from its knowledge on $W_\beta \cup W$. The assigned value on W_β is zero, which is reminiscent of the well-known *zero filling* techniques. We stress, however, that a certain amount of interpolation (which depends on the value of β) is actually performed here, and that this formulation is variational in essence. Notice that the standard Tikhonov regularizer, namely,

$$\frac{1}{2} \| f \|^2_{L^2(\mathbb{R}^d)} = \frac{1}{2} \| U f \|^2_{L^2(\mathbb{R}^d)},$$

constrains Uf everywhere in $\mathbb{R}^d$, which cannot be optimal for the inverse problem under consideration.
It is important to realize that the new objective is no longer the reconstruction of the original object f_0, but that of a lower resolution version of it, namely, $\phi_\beta * f_0$, where the Fourier transform of ϕ_β is the characteristic function of the ball $B_{1/\beta}$. Since ϕ_β is a (radial) sinc function, it seems more appropriate to introduce some *apodization*. For obvious morphological reasons, the *point spread function* ϕ_β should be (essentially) positive, isotropic and radially decreasing. We are then led to define the reconstructed object as the solution of Problem $(\mathcal{P}_{\alpha,\beta})$ above[2]. Throughout, the *apodized point spread function* will be of the form given in Equation (1), where ϕ is assume to be a non-trivial integrable function.

We shall prove in Proposition 6 below that Problem $(\mathcal{P}_{\alpha,\beta})$ is well-posed. A detailed account of the behavior of the solution of $(\mathcal{P}_{\alpha,\beta})$ as $\alpha \downarrow 0$ (and β fixed) will be given in the next section. Although the well-posedness of Problem $(\mathcal{P}_{\alpha,\beta})$ is a rather immediate consequence of Proposition 4, we adopt here a different viewpoint. This approach will bring us back to Tikhonov's regularization theory, as reviewed in the appendix. The key point lies in Lemma 5 below. Let $\beta > 0$ be fixed and let, for every $f_1, f_2 \in L^2(V)$,

$$\langle f_1, f_2 \rangle_\beta := \int_{\mathbb{R}^d} |1 - \hat{\phi}_\beta|^2 U f_1 \overline{U f_2}. \tag{2}$$

Lemma 5. *The sesquilinear mapping $\langle \cdot, \cdot \rangle_\beta$ is an inner product which turns $L^2(V)$ into a Hilbert space. The corresponding norm $\| \cdot \|_\beta$ is equivalent to the original L^2-norm.*

PROOF. The first part of the lemma is an easy exercise, and we only prove the equivalence of the norms. Since ϕ_β is assumed to be integrable, $\hat{\phi}_\beta$ is continuous and vanishes at infinity. Therefore, there exists $R > 0$ such that

$$\inf_{\|\xi\| \geq R} |1 - \hat{\phi}_\beta(\xi)| \geq \frac{1}{2}.$$

[2]Notice that the support of $\phi_\beta * f_0$ is clearly expected to be larger that V. In practice, it may therefore be suitable to replace V by a larger set V' in $(\mathcal{P}_{\alpha,\beta})$. However, it is clear that the injectivity of T_W is crucial, so that chosing (the closure of) $\operatorname{supp} \phi_\beta + V$ may not be possible (for example, a Gaussian ϕ would lead to the choice $V' = \mathbb{R}^d$). Moreover, our analysis would remain true with any bounded $V' \supset V$ in place of V, and since $L^2(V)$ is then obviously contained in $L^2(V')$, we shall merely ignore this distinction.

By Proposition 4, $T_{B_R^c}\colon L^2(V) \to \operatorname{ran} T_{B_R^c}$ is bi-continuous, so that, for all $f \in L^2(V)$,

$$\| f \|_\beta \geq \frac{1}{2} \| T_{B_R^c} f \|_{L^2(B_R^c)} \geq \frac{1}{2\|T_{B_R^c}^{-1}\|} \| f \|_{L^2(V)} .$$

On the other hand, Plancherel's Equality implies that, for all $f \in L^2(V)$,

$$\| f \|_\beta \leq \sup_{\xi \in \mathbb{R}^d} \left| 1 - \hat{\phi}_\beta(\xi) \right| \, \|\hat{f}\|_{L^2(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d} \left| 1 - \hat{\phi}_\beta(\xi) \right| \, \| f \|_{L^2(V)} . \blacksquare$$

Proposition 6. *Let $\alpha, \beta > 0$ be fixed. Then $(\mathcal{P}_{\alpha,\beta})$ has a unique solution $f_{\alpha,\beta}$, which depends continuously on $g \in L^2(W)$.*

Proof. Clearly, Problem $(\mathcal{P}_{\alpha,\beta})$ can then be rewritten as:

$$(\mathcal{P}'_{\alpha,\beta}) \quad \left| \begin{array}{rl} \text{minimize} & \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2_{L^2(W)} + \frac{\alpha}{2} \| f \|^2_\beta \\ \text{s.t.} & f \in L^2(V). \end{array} \right.$$

From the classical theory of Tikhonov's regularization (see the appendix, Proposition 17) the unique solution of $(\mathcal{P}'_{\alpha,\beta})$ is given by

$$f_{\alpha,\beta} = (T_W^\# T_W + \alpha I)^{-1} T_W^\# (\hat{\phi}_\beta g),$$

in which $T_W^\#$ denotes the adjoint of T_W with respect to the new inner product $\langle \cdot, \cdot \rangle_\beta$. The conclusion then follows from Lemma 5 and the continuity of the multiplication $g \mapsto \hat{\phi}_\beta g$ in $L^2(W)$. $\blacksquare$

3 Tikhonov-like regularization

In this section, we fix $\beta > 0$ and we investigate the behavior of the solution of $(\mathcal{P}_{\alpha,\beta})$ as $\alpha \downarrow 0$. This leads us to consider the following *limit problem*:

$$(\mathcal{P}_{0,\beta}) \quad \left| \begin{array}{rl} \text{minimize} & \frac{1}{2} \left\| \hat{\phi}_\beta g - T_W f \right\|^2 \\ \text{s.t.} & f \in L^2(V). \end{array} \right.$$

One may indeed wonder whether the solution of $(\mathcal{P}_{\alpha,\beta})$ converges to that of $(\mathcal{P}_{0,\beta})$ under the usual condition that $g \in \mathcal{D}(T_W^+)$. This simple question raises the following important conceptual difficulty: does Problem $(\mathcal{P}_{0,\beta})$ have any solution at all, that is to say, does the regularized datum $\hat{\phi}_\beta g$ belong to $\mathcal{D}(T_W^+)$? This condition, which we shall discuss later on (see Proposition 8 below), turns out to be necessary and sufficient for the norm of the reconstructed object $f_{\alpha,\beta}$ not to diverge to infinity as $\alpha \downarrow 0$.

Theorem 7. *Let $\beta > 0$ be fixed and let $g \in \mathcal{D}(T_W^+)$.*

(i) *If $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$, then the unique solution $f_{\alpha,\beta} \in L^2(V)$ of $(\mathcal{P}_{\alpha,\beta})$ converges strongly in $L^2(V)$, as $\alpha \downarrow 0$, to the unique solution of $(\mathcal{P}_{0,\beta})$, namely, $T_W^+(\hat{\phi}_\beta g)$.*

(ii) *If $\hat{\phi}_\beta g \notin \mathcal{D}(T_W^+)$, then $\| f_{\alpha,\beta}\|_{L^2(V)}$ tends to infinity as $\alpha \downarrow 0$.*

Proof.

i. On rewriting $(\mathcal{P}_{\alpha,\beta})$ as $(\mathcal{P}'_{\alpha,\beta})$, this point is an immediate consequence of the Tikhonov classical theory (see the appendix, Theorem 18) and Lemma 5.

ii. Assume that $\hat{\phi}_\beta g \notin \mathcal{D}(T_W^+)$, that is to say, that $(\mathcal{P}_{0,\beta})$ has no solution. Suppose, in order to obtain a contradiction, that $\| f_{\alpha,\beta}\|$ does not tend to infinity as $\alpha \downarrow 0$. There then exists a positive sequence $(\alpha_n)_{n\in\mathbb{N}^*}$ converging to 0 which is such that the sequence $(f_n)_{\in\mathbb{N}^*}$ defined by

$$f_n := f_{\alpha_n,\beta}$$

is bounded in $L^2(V)$. By the Weak Compactness Theorem, taking a subsequence if necessary, we can assume that $(f_n)_{n\in\mathbb{N}^*}$ converges weakly to some f' in $L^2(V)$. Writing that f_n is the solution of $(\mathcal{P}_{\alpha_n,\beta})$, we find that, for every $f \in L^2(V)$,

$$\frac{1}{2}\left\| \hat{\phi}_\beta g - T_W f_n \right\|^2_{L^2(W)} + \frac{\alpha_n}{2}\|(1-\hat{\phi}_\beta)\hat{f}_n\|^2_{L^2(\mathbb{R}^d)} \leq \frac{1}{2}\left\| \hat{\phi}_\beta g - T_W f \right\|^2_{L^2(W)} + \frac{\alpha_n}{2}\|(1-\hat{\phi}_\beta)\hat{f}\|^2_{L^2(\mathbb{R}^d)}. \quad (3)$$

It is obvious that $\hat{\phi}_\beta g - T_W f_n$ converges weakly to $\hat{\phi}_\beta g - T_W f'$ as $n \to \infty$. Hence, we have

$$\left\| \hat{\phi}_\beta g - T_W f' \right\|_{L^2(W)} \leq \liminf_{n\to\infty} \left\| \hat{\phi}_\beta g - T_W f_n \right\|_{L^2(W)}.$$

Letting $n \to \infty$ in (3) now yields the inequality:

$$\frac{1}{2}\left\| \hat{\phi}_\beta g - T_W f' \right\|^2_{L^2(W)} \leq \frac{1}{2}\left\| \hat{\phi}_\beta g - T_W f \right\|^2_{L^2(W)};$$

notice indeed that

$$\frac{\alpha_n}{2}\|(1-\hat{\phi}_\beta)\hat{f}_n\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\alpha_n}{2}\sup_{\xi\in\mathbb{R}^d}\left|1-\hat{\phi}_\beta(\xi)\right| \|f_n\|^2_{L^2(V)} \to 0,$$

since $\alpha_n \to 0$ and $(f_n)_{n\in\mathbb{N}^*}$ is bounded in $L^2(V)$. We have then proved that f' is a solution of $(\mathcal{P}_{0,\beta})$, which is the desired contradiction. ∎

The next proposition will show that, in Theorem 7, the second alternative occurs quite often.

Proposition 8. *Let $\phi \in L^1(\mathbb{R}^d)$ be such that $\hat{\phi}$ is analytic. Then, for every $\beta > 0$ and $g \in \mathcal{D}(T_W^+)$, $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$ if and only if* $\operatorname{supp}(\phi_\beta * T_W^+ g) \subseteq V$.

Proof. By the analytic continuation theorem, $\tilde{g} := UT_W^+ g$ is the unique analytic extension of g on $\mathbb{R}^d$. If $\operatorname{supp}(\phi_\beta * T_W^+ g) \subseteq V$, then $\phi_\beta * T_W^+ g \in L^2(V)$ and $U(\phi_\beta * T_W^+ g) = \hat{\phi}_\beta \tilde{g}$. It is then obvious that $A(\phi_\beta * T_W^+ g) = \hat{\phi}_\beta g$.
Assume now that $\operatorname{supp}(\phi_\beta * T_W^+ g)$ is strictly larger than V. Suppose, in order to obtain a contradiction, that $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$. In our context, $\mathcal{D}(T_W^+) = \operatorname{ran} T_W$ and there exists $f \in L^2(V)$ such that $\mathbb{1}_W Uf = \hat{\phi}_\beta g$. As the product of two analytic functions, $\hat{\phi}_\beta \tilde{g}$ is analytic, and it coincides with the entire function Uf on W. Since $\operatorname{int} W \neq \emptyset$, the analytic continuation theorem shows that $Uf = \hat{\phi}_\beta \tilde{g}$. Taking inverse Fourier transforms yields the equality $f = \phi_\beta * (U^{-1}\tilde{g})$ (in $L^2(\mathbb{R}^d)$). Then,

$$f = \phi_\beta * (U^{-1}UT_W^+ g) = \phi_\beta * T_W^+ g \quad (\text{in } L^2(\mathbb{R}^d)).$$

This contradicts our working assumption on the support of $(\phi_\beta * T_W^+ g)$. ∎

At a first sight, the last proposition may be regarded as a serious drawback of the whole regularization methodology. Together with Theorem 7 (i), it says in particular that point spread functions with non-compact support (such as the Gaussian kernel shown in Figure 1) give rise to somewhat inconsistent regularization schemes. We believe instead that this only stresses the fact that α should not be considered as the fundamental regularization parameter. As it may already be clear to the reader, the parameter playing the essential

role in $(\mathcal{P}_{\alpha,\beta})$ is β. This motivates the next sections, in which we shall obtain results on the behavior of $f_{\alpha,\beta}$ as β goes to zero.
In the Tikhonov regularization theory (see the appendix), the spectral decomposition of $T^\star T + \alpha I$ appears as the main tool. Clearly, the fact that the eigenspaces do not change with α is one of the keys to the convergence theorem (Theorem 18). As we shall see, the results of the next sections do not rely on any spectral argument (at least, we where not able to exhibit an inner product leaving the eigenspaces invariant with β). This is why the techniques to be used pertain essentially to variational analysis.

4 Mollification

Theorem 9. *Let $\alpha > 0$ be fixed and $\phi \in L^1(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \phi(x)dx = 1$, that is to say $\hat{\phi}(0) = 1$. Assume that there exists $s > 0$ such that*

$$\left|1 - \hat{\phi}(\xi)\right| \sim_{\xi\to 0} \left\|\xi\right\|^s, \tag{4}$$

up to a positive multiplicative constant. Assume in addition that

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\phi}(\xi) \neq 1. \tag{5}$$

Let $g \in \mathcal{D}(T_W^+)$ be such that

$$\int_{\mathbb{R}^d} \left\|\xi\right\|^{2s} \left|\tilde{g}(\xi)\right|^2 \mathrm{d}\xi < \infty, \tag{6}$$

where $\tilde{g}$ is the unique analytic extension of g on $\mathbb{R}^d$. Then, the unique solution $f_{\alpha,\beta}$ of Problem $(\mathcal{P}_{\alpha,\beta})$ converges strongly to $T_W^+ g$ in $L^2(V)$.

The proof of Theorem 9 relies on two technical lemmas, which we establish now.

Lemma 10. *Let ϕ be as in the theorem, and let*

$$m_\beta := \min_{\|\xi\|=1} \left|1 - \hat{\phi}(\beta\xi)\right|^2 \quad \text{and} \quad M_\beta := \max_{\|\xi\|=1} \left|1 - \hat{\phi}(\beta\xi)\right|^2. \tag{7}$$

We then have the following properties:

(i) *For all $\beta > 0$, $0 < m_\beta \leq M_\beta \leq \left(1 + \|\phi\|_{L^1(\mathbb{R}^d)}\right)^2$;*

(ii) $\sup_{\beta>0}(M_\beta/m_\beta) < \infty$ *and M_β tends to zero as $\beta \downarrow 0$;*

(iii) *there exist positive constants* ν_0 *and* C_0 *such that, for every* $\beta \in (0,1]$ *and every* $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\nu_0 \left(\|\xi\|^{2s} \mathbb{1}_{B_{1/\beta}}(\xi) + \frac{1}{M_\beta} \mathbb{1}_{B^c_{1/\beta}}(\xi) \right) \le \frac{|1-\hat{\phi}(\beta\xi)|^2}{|1-\hat{\phi}(\beta\xi/\|\xi\|)|^2} \le C_0 \|\xi\|^{2s}.$$

PROOF.

(i) By the Riemann-Lebesgue Lemma, $\hat{\phi} \in C_0(\mathbb{R}^d)$ and $\|\hat{\phi}\|_{L^\infty(\mathbb{R}^d)} \le \|\phi\|_{L^1(\mathbb{R}^d)}$. We deduce, in particular, that $M_\beta \le (1 + \|\phi\|_{L^1(\mathbb{R}^d)})^2$. Moreover, (5) implies that $|1-\hat{\phi}|$ is positive on $\mathbb{R}^d \setminus \{0\}$ and it follows that $m_\beta > 0$, as the infimum of a continuous and positive function on a compact set.

(ii) The continuity and positivity, on $\mathbb{R}^d \setminus \{0\}$, of $|1-\hat{\phi}|$ also imply that the ratio M_β/m_β is continuous with respect to $\beta > 0$. Moreover, since $\hat{\phi} \in C_0(\mathbb{R}^d)$, we see that $|1-\hat{\phi}(\xi)|$ tends to 1 as $\|\xi\| \to \infty$. It follows that M_β/m_β tends to 1 as $\beta \to \infty$. On the other hand, Condition (4) implies that M_β/m_β also tends to 1 as $\beta \downarrow 0$. Since the ratio M_β/m_β has finite limits as $\beta \downarrow 0$ and as $\beta \to \infty$, it must be bounded above on $(0,\infty)$. Finally, Condition (4) clearly implies that M_β goes to zero as $\beta \downarrow 0$.

(iii) By (4), $|1-\hat{\phi}(\xi)|^2/\|\xi\|^{2s}$ tends to some positive constant C as $\xi \to 0$. In particular, there exists $r > 0$ such that

$$\forall \xi \in B_r, \qquad \frac{C}{2}\|\xi\|^{2s} \le |1-\hat{\phi}(\xi)|^2 \le 2C\|\xi\|^{2s}. \tag{8}$$

Let $r_0 := \min\{1, r\}$, and let

$$m := \min_{\{r_0 \le \|\xi\| \le 1\}} \left|1-\hat{\phi}(\xi)\right|^2 \quad \text{and} \quad M := \max_{\{r_0 \le \|\xi\| \le 1\}} \left|1-\hat{\phi}(\xi)\right|^2.$$

We have $0 < m \le M < \infty$ and, for all ξ such that $r_0 \le \|\xi\| \le 1$,

$$m\|\xi\|^{2s} \le m \le \left|1-\hat{\phi}(\xi)\right|^2 \le M \le \frac{M}{r_0^{2s}} \|\xi\|^{2s}. \tag{9}$$

Let $\nu_1 := \min\{m, C/2\}$ and $C_1 := \max\{M/r_0^{2s}, 2C\}$. Then, ν_1 and C_1 are positive and (8) and (9) imply that, for all ξ such that $|\xi| \le 1$,

$$\nu_1 \|\xi\|^{2s} \le \left|1-\hat{\phi}(\xi)\right|^2 \le C_1 \|\xi\|^{2s}. \tag{10}$$

Consequently, for every $\beta \in (0,1]$ and every $\xi \in B_{1/\beta} \setminus \{0\}$,

$$\nu_1 \beta^{2s} \|\xi\|^{2s} \leq |1 - \hat{\phi}(\beta\xi)|^2 \leq C_1 \beta^{2s} \|\xi\|^{2s},$$
$$\nu_1 \beta^{2s} \leq |1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2 \leq C_1 \beta^{2s}.$$

Since all terms of the last inequality are positive, we deduce that, for every $\beta \in (0,1]$ and every $\xi \in B_{1/\beta} \setminus \{0\}$,

$$\frac{\nu_1}{C_1} \|\xi\|^{2s} \leq \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} \leq \frac{C_1}{\nu_1} \|\xi\|^{2s}. \tag{11}$$

Let now

$$m' := \min_{\{\|\xi\| > 1\}} \left|1 - \hat{\phi}(\xi)\right|^2 \quad \text{and} \quad M' := \max_{\{\|\xi\| > 1\}} \left|1 - \hat{\phi}(\xi)\right|^2.$$

The fact that $\xi \mapsto |1 - \hat{\phi}(\xi)|$ is positive and continuous on $\mathbb{R}^d \setminus \{0\}$ and that it tends to 1 as $\|\xi\| \to \infty$ clearly implies that $0 < m' \leq M' < \infty$. Thus, for every $\beta \in (0,1]$ and every ξ such that $\|\xi\| \geq 1/\beta$,

$$\frac{m'}{M_\beta} \leq \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} \leq \frac{M'}{m_\beta}.$$

Moreover, (10) implies that $m_\beta \leq C_1 \beta^{2s}$, so that

$$\frac{M'}{m_\beta} \leq \frac{M'}{C_1 \beta^{2s}} \leq \frac{M'}{C_1} \|\xi\|^{2s}.$$

We have then proved that, for every $\beta \in (0,1]$ and every ξ such that $\|\xi\| \geq 1/\beta$,

$$\frac{m'}{M_\beta} \leq \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} \leq \frac{M'}{C_1} \|\xi\|^{2s}. \tag{12}$$

Finally, let $\nu_0 := \min\{\nu_1/C_1, m'\}$ and $C_0 := \max\{C_1/\nu_1, M'/C_1\}$. Then, ν_0 and C_0 are positive, and (11) and (12) yield the desired inequality. ∎

Lemma 11. *Let $\phi \in L^1(\mathbb{R}^d)$ and, for $\beta > 0$, define ϕ_β as in (1). Then, for every $\psi \in L^2(\mathbb{R}^d)$, $\hat{\phi}_\beta \psi$ converges strongly to $\hat{\phi}(0)\psi$ in $L^2(\mathbb{R}^d)$ as $\beta \downarrow 0$.*

Proof. Let $a := \int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = \hat{\phi}(0)$ and $f := U^{-1}\psi \in L^2(\mathbb{R}^d)$. Well-known results on the approximation of L^p-functions by mollification say that $\phi_\beta * f$ converges strongly to af as $\beta \downarrow 0$. Taking the Fourier transform (which is an isometry of $L^2(\mathbb{R}^d)$), we see that $U(\phi_\beta * f) = \hat{\phi}_\beta \psi$ converges strongly to $a\psi = \hat{\phi}(0)\psi$ in $L^2(\mathbb{R}^d)$. ∎

Proof of the theorem. The proof is divided in three steps. In Step 1, we derive an L^2-estimate of $f_{\alpha,\beta}$ which does not depend on β. In Step 2, we establish the weak convergence of $f_{\alpha,\beta}$ to $T_W^+ g$. Finally, in Step 3, we use a compactness argument to show that the convergence is indeed strong.

Step 1: L^2-estimate. For all $f \in L^2(V)$,

$$\frac{1}{2}\left\|\hat{\phi}_\beta g - T_W f_{\alpha,\beta}\right\|^2_{L^2(W)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{1}{2}\left\|\hat{\phi}_\beta g - T_W f\right\|^2_{L^2(W)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}\right\|^2_{L^2(\mathbb{R}^d)}.$$

Let us take $f = T_W^+ g$. Then,

$$\frac{1}{2}\left\|\hat{\phi}_\beta g - T_W f_{\alpha,\beta}\right\|^2_{L^2(W)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{1}{2}\left\|\hat{\phi}_\beta g - T_W T_W^+ g\right\|^2_{L^2(W)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)UT_W^+ g\right\|^2_{L^2(\mathbb{R}^d)}. \tag{13}$$

Let us recall that $UT_W^+ g$ is the unique analytic extension of g on $\mathbb{R}^d$, that is to say $UT_W^+ g = \tilde{g}$. Since $T_W T_W^+ g = g$, Inequality (13) can be written as follows:

$$\frac{1}{2}\left\|\hat{\phi}_\beta g - T_W f_{\alpha,\beta}\right\|^2_{L^2(W)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{1}{2}\left\|(1-\hat{\phi}_\beta)g\right\|^2_{L^2(W)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\tilde{g}\right\|^2_{L^2(\mathbb{R}^d)}.$$

In particular,

$$\frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{1}{2}\left\|(1-\hat{\phi}_\beta)\tilde{g}\right\|^2_{L^2(\mathbb{R}^d)} + \frac{\alpha}{2}\left\|(1-\hat{\phi}_\beta)\tilde{g}\right\|^2_{L^2(\mathbb{R}^d)}.$$

and, dividing by $\alpha/2$, it follows that

$$\left\|(1-\hat{\phi}_\beta)\hat{f}_{\alpha,\beta}\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{1+\alpha}{\alpha}\left\|(1-\hat{\phi}_\beta)\tilde{g}\right\|^2_{L^2(\mathbb{R}^d)}. \tag{14}$$

For every $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$1 - \hat{\phi}_\beta(\xi) = 1 - \hat{\phi}(\beta\xi) = \Big(1 - \hat{\phi}(\beta\xi/\|\xi\|)\Big) \frac{1 - \hat{\phi}(\beta\xi)}{1 - \hat{\phi}(\beta\xi/\|\xi\|)}.$$

Recall indeed that Condition (5) ensures that we do not divide by 0. We then see that (14) can be written as

$$\int_{\mathbb{R}^d} \Big|1 - \hat{\phi}(\beta\xi/\|\xi\|)\Big|^2 \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} |\hat{f}_{\alpha,\beta}(\xi)|^2 \,\mathrm{d}\xi \\ \leq \frac{1+\alpha}{\alpha} \int_{\mathbb{R}^d} |1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2 \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} |\tilde{g}(\xi)|^2 \,\mathrm{d}\xi.$$

Defining m_β and M_β as in (7), we deduce that

$$m_\beta \int_{\mathbb{R}^d} \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} |\hat{f}_{\alpha,\beta}(\xi)|^2 \,\mathrm{d}\xi \\ \leq \frac{1+\alpha}{\alpha} M_\beta \int_{\mathbb{R}^d} \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} |\tilde{g}(\xi)|^2 \,\mathrm{d}\xi.$$

By Lemma 10 (i), we can divide this inequality by $m_\beta > 0$ and obtain

$$\int_{\mathbb{R}^d} \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} |\hat{f}_{\alpha,\beta}(\xi)|^2 \,\mathrm{d}\xi \\ \leq \frac{1+\alpha}{\alpha} \frac{M_\beta}{m_\beta} \int_{\mathbb{R}^d} \frac{|1 - \hat{\phi}(\beta\xi)|^2}{|1 - \hat{\phi}(\beta\xi/\|\xi\|)|^2} |\tilde{g}(\xi)|^2 \,\mathrm{d}\xi.$$

By Lemma 10 (iii), it is now easy to deduce from the above inequality that if $\beta \in (0, 1]$, then

$$\nu_0 \int_{\mathbb{R}^d} \left(\mathbb{1}_{B_{1/\beta}}(\xi) \|\xi\|^{2s} + \frac{1}{M_\beta} \mathbb{1}_{B^c_{1/\beta}}(\xi) \right) \Big| \hat{f}_{\alpha,\beta}(\xi) \Big|^2 \,\mathrm{d}\xi \\ \leq \frac{1+\alpha}{\alpha} \frac{M_\beta}{m_\beta} C_0 \int_{\mathbb{R}^d} \big\|\xi\big\|^{2s} \big|\tilde{g}(\xi)\big|^2 \,\mathrm{d}\xi \leq \frac{1+\alpha}{\alpha} C_0 C_1,$$

where

$$C_1 := \left(\sup_{\beta > 0} \frac{M_\beta}{m_\beta} \right) \int_{\mathbb{R}^d} \big\|\xi\big\|^{2s} \big|\tilde{g}(\xi)\big|^2 \,\mathrm{d}\xi.$$

Notice that Lemma 10 (ii) and Assumption (6) imply that C_1 is a finite (positive) real number. At this stage of the proof, we have established the following key estimate: for all $\beta \in (0,1]$,

$$\nu_0 \int_{\|\xi\| \le 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi + \frac{\nu_0}{M_\beta} \int_{\|\xi\| > 1/\beta} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\ \le \frac{1+\alpha}{\alpha} C_0 C_1. \quad (15)$$

Since $\beta \in (0,1]$, the first term of the left hand side of the above inequality is bounded below as follows:

$$\begin{aligned} \nu_0 \int_{\|\xi\| \le 1/\beta} \|\xi\|^{2s} |\hat{f}_{\alpha,\beta}(\xi)|^2 \, \mathrm{d}\xi \ &\ge \ \nu_0 \int_{1 \le \|\xi\| \le 1/\beta} \|\xi\|^{2s} |\hat{f}_{\alpha,\beta}(\xi)|^2 \, \mathrm{d}\xi \\ &\ge \ \nu_0 \int_{1 \le \|\xi\| \le 1/\beta} |\hat{f}_{\alpha,\beta}(\xi)|^2 \, \mathrm{d}\xi. \end{aligned}$$

As for the second term, Lemma 10 (i) implies that

$$\frac{\nu_0}{M_\beta} \int_{\|\xi\| > 1/\beta} |\hat{f}_{\alpha,\beta}(\xi)|^2 \, \mathrm{d}\xi \ge \nu_0 \left(1 + \|\phi\|_{L^1(\mathbb{R}^d)}\right)^{-2} \int_{\|\xi\| > 1/\beta} |\hat{f}_{\alpha,\beta}(\xi)|^2 \, \mathrm{d}\xi.$$

We thus find that the left hand side of (15) is bounded below by

$$\nu_1 \int_{\|\xi\| \ge 1} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi, \quad \text{where} \quad \nu_1 := \nu_0 \left(1 + \|\phi\|_{L^1(\mathbb{R}^d)}\right)^{-2}.$$

Thus, for every $\beta \in (0,1]$,

$$\nu_1 \int_{\|\xi\| \ge 1} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \le \frac{1+\alpha}{\alpha} C_0 C_1. \quad (16)$$

Finally, notice that the truncated Fourier operator $T_{B_1^c} \colon L^2(V) \to \operatorname{ran} T_{B_1^c}$ is invertible with continuous inverse (see Proposition 4). Hence, writing $f_{\alpha,\beta}$ as $T_{B_1^c}^{-1} \mathbb{1}_{B_1^c} \hat{f}_{\alpha,\beta}$, we see that

$$\left\| f_{\alpha,\beta} \right\|^2 \le \left\| T_{B_1^c}^{-1} \right\|^2 \int_{\|\xi\| \ge 1} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi,$$

from which we deduce our L^2-estimate: for every $\beta \in (0,1]$,

$$\left\| f_{\alpha,\beta} \right\|_{L^2(V)}^2 \le \left\| T_{B_1^c}^{-1} \right\|^2 \frac{(1+\alpha) C_0 C_1}{\alpha \nu_1}. \quad (17)$$

Step 2: weak convergence. In order to prove that $f_{\alpha,\beta}$ converges weakly to $T_W^+ g$ as $\beta \downarrow 0$, it suffices to show that, for every positive sequence $(\beta_n)_{n\in\mathbb{N}^*}$ converging to zero, the sequence $(f_n)_{n\in\mathbb{N}^*}$ defined by

$$f_n := f_{\alpha,\beta_n}$$

has a subsequence which converges weakly to $T_W^+ g$.
Let $(\beta_n)_{n\in\mathbb{N}^*}$ and $(f_n)_{n\in\mathbb{N}^*}$ be as above. It results from Step 1 that (f_n) is bounded, and thus from the Weak Compactness Theorem that there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}^*}$ which converges weakly to some $f' \in L^2(V)$. We shall prove that $f' = T_W^+ g$. Recall that $T_W^+ g$ is the unique solution of

$$(\mathcal{P}_0) \quad \left| \begin{array}{rl} \text{minimize} & \dfrac{1}{2} \left\| g - T_W f \right\|^2_{L^2(W)} \\ \text{s.t.} & f \in L^2(V) \end{array} \right.$$

(see Remark 3). Since f_{n_k} is the solution of $(\mathcal{P}_{\alpha,\beta_{n_k}})$, for every $f \in L^2(V)$,

$$\begin{aligned} \frac{1}{2} \left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2_{L^2(W)} &+ \frac{\alpha}{2} \left\| (1-\hat{\phi}_{n_k}) \hat{f}_{n_k} \right\|^2_{L^2(\mathbb{R}^d)} \\ &\le \frac{1}{2} \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2_{L^2(W)} + \frac{\alpha}{2} \left\| (1-\hat{\phi}_{n_k}) \hat{f} \right\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where $\hat{\phi}_{n_k} := \hat{\phi}_{\beta_{n_k}}$. Consequently,

$$\begin{aligned} &\left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2_{L^2(W)} \\ &\qquad \le \left\| \hat{\phi}_{n_k} g - T_W f \right\|^2_{L^2(W)} + \alpha \left\| (1-\hat{\phi}_{n_k}) \hat{f} \right\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{18}$$

Let k tend to ∞ in this inequality. On the one hand, Lemma 11 implies that $\hat{\phi}_{n_k} g$ converges strongly to $\hat{\phi}(0) g = g$ in $L^2(W)$, as $k \to \infty$. Moreover, it is clear that $T_W f_{n_k}$ converges weakly to $T_W f'$. It follows that $\hat{\phi}_{n_k} g - T_W f_{n_k}$ converges weakly to $g - T_W f'$, so that

$$\left\| g - T_W f' \right\|^2_{L^2(W)} \le \liminf_{k\to\infty} \left\| \hat{\phi}_{n_k} g - T_W f_{n_k} \right\|^2_{L^2(W)}.$$

On the other hand, $\hat{\phi}_{n_k} g - T_W f$ converges strongly to $g - T_W f$ and $(1 - \hat{\phi}_{n_k}) \hat{f} = \hat{f} - \hat{\phi}_{n_k} \hat{f}$ converges strongly to $\hat{f} - \hat{f} = 0$ by Lemma 11 again. We deduce that the right hand side of (18) converges to $\| g - T_W f \|^2_{L^2(W)}$, so that

$$\left\| g - T_W f' \right\|^2_{L^2(W)} \le \left\| g - T_W f \right\|^2_{L^2(W)}.$$

Since $f \in L^2(V)$ is arbitrary, we see that f' must be the unique solution of $(\mathcal{P}_0)$.

Step 3: strong convergence. For every $h \in \mathbb{R}^d$ and every $f \in L^2(\mathbb{R}^d)$, let $\mathcal{T}_h f$ denote the translated function $x \mapsto f(x-h)$. Let $(\beta_n)_{n\in\mathbb{N}^*}$ and $(f_n)_{n\in\mathbb{N}^*}$ be as in Step 2. It results from Step 1 that $(f_n)_{n\in\mathbb{N}^*}$ is bounded. Moreover, since $(f_n)_{n\in\mathbb{N}^*} \subset L^2(V)$ and V is bounded, it is clear that

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}^*} \int_{\|x\|>R} \left| f_n(x) \right|^2 \,\mathrm{d}x = 0.$$

Now, if we can prove that

$$\sup_{n\in\mathbb{N}^*} \left\| \mathcal{T}_h f_n - f_n \right\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \|h\| \to 0, \tag{19}$$

a classical *relative compactness theorem* (see *e.g.* [4], Theorem 3.8 page 175) will then show that $(f_n)_{n\in\mathbb{N}^*}$ is precompact. Combined with the weak convergence of $(f_n)_{n\in\mathbb{N}^*}$ to $T_W^+ g$ obtained in Step 2, this will establish the claimed strong convergence. We thus proceed to prove (19).
We have $U\mathcal{T}_h f_{\alpha,\beta} = e^{-2i\pi\langle h,\cdot\rangle}\hat{f}_{\alpha,\beta}$ and Plancherel's Equality implies that

$$\begin{aligned}
\left\| \mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^d)} &= \left\| U\left(\mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta}\right) \right\|^2_{L^2(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} \left| e^{-2i\pi\langle h,\xi\rangle} - 1 \right|^2 \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\
&= I_1 + I_2,
\end{aligned} \tag{20}$$

where

$$I_1 := \int_{\|\xi\|\le 1/\beta} \left| e^{-2i\pi\langle h,\xi\rangle} - 1 \right|^2 \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi$$

and

$$I_2 := \int_{\|\xi\|> 1/\beta} \left| e^{-2i\pi\langle h,\xi\rangle} - 1 \right|^2 \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi.$$

Let us estimate the first integral:

$$\begin{aligned}
I_1 &= \int_{0<\|\xi\|\le 1/\beta} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \left\|\xi\right\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \\
&\le \sup_{\xi\neq 0} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} \int_{\|\xi\|\le 1/\beta} \left\|\xi\right\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi,
\end{aligned}$$

where $s' := \min\{1, s\}$. Using the change of variable $\xi' = \|h\|\xi$ (for $h \neq 0$), we have:

$$\sup_{\xi \neq 0} \frac{|e^{-2i\pi\langle h,\xi\rangle} - 1|^2}{\|\xi\|^{2s'}} = \|h\|^{2s'} \sup_{\xi' \neq 0} \frac{|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1|^2}{\|\xi'\|^{2s'}}.$$

Since $|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1| = \mathcal{O}(\|\xi'\|)$ near the origin, it is clear that there exists a positive constant C_2 such that, for every $h \in \mathbb{R}^d \setminus \{0\}$,

$$\sup_{\xi' \neq 0} \frac{|e^{-2i\pi\langle \|h\|^{-1}h,\xi'\rangle} - 1|^2}{\|\xi'\|^{2s'}} \leq C_2.$$

It follows that

$$I_1 \leq C_2 \|h\|^{2s'} \int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s'} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi.$$

Since $\|\xi\|^{2s'} \leq 1 + \|\xi\|^{2s}$, we deduce that

$$\begin{aligned}
I_1 &\leq C_2 \|h\|^{2s'} \left(\int_{\|\xi\| \leq 1/\beta} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi + \int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \right) \\
&\leq C_2 \|h\|^{2s'} \left(\left\| \hat{f}_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^d)} + \int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \right) \\
&= C_2 \|h\|^{2s'} \left(\| f_{\alpha,\beta} \|^2_{L^2(V)} + \int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \right),
\end{aligned}$$

thanks to Plancherel's Equality. By (17), we deduce that, for $\beta \in (0, 1]$,

$$I_1 \leq C_2 \|h\|^{2s'} \left(\left\| T^{-1}_{B_1^c} \right\|^2 \frac{(1+\alpha)C_0C_1}{\alpha\nu_1} + \int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \right).$$

Since, by (15),

$$\int_{\|\xi\| \leq 1/\beta} \|\xi\|^{2s} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \mathrm{d}\xi \leq \frac{(1+\alpha)C_0C_1}{\alpha\nu_0} \leq \frac{(1+\alpha)C_0C_1}{\alpha\nu_1},$$

we finally get:

$$I_1 \leq C_2 \|h\|^{2s'} \frac{(1+\alpha)C_0C_1}{\alpha\nu_1} \left(1 + \left\| T^{-1}_{B_1^c} \right\|^2 \right).$$

It remains to estimate the second integral. By (15), we have:

$$I_2 \leq 4 \int_{\|\xi\|>1/\beta} \left| \hat{f}_{\alpha,\beta}(\xi) \right|^2 \, \mathrm{d}\xi \leq \frac{1+\alpha}{\alpha} \frac{C_0 C_1}{\nu_0} M_\beta \leq \frac{1+\alpha}{\alpha} \frac{C_0 C_1}{\nu_1} M_\beta.$$

Thus, from (20) and the above estimates for I_1 and I_2, we deduce that

$$\begin{aligned}
&\left\| \mathcal{T}_h f_{\alpha,\beta} - f_{\alpha,\beta} \right\|^2_{L^2(\mathbb{R}^d)} \\
&\leq \; C_2 \left\| h \right\|^{2s'} \frac{(1+\alpha) C_0 C_1}{\alpha \nu_1} \left(1 + \left\| T^{-1}_{B_1^c} \right\|^2 \right) + \frac{1+\alpha}{\alpha} \frac{C_0 C_1}{\nu_1} M_\beta \\
&\leq \; C_3 \left(\left\| h \right\|^{2s'} + M_\beta \right), \qquad (21)
\end{aligned}$$

where

$$C_3 := \max \left\{ \frac{(1+\alpha) C_0 C_1 C_2}{\alpha \nu_1} \left(1 + \left\| T^{-1}_{B_1^c} \right\|^2 \right), \frac{(1+\alpha) C_0 C_1}{\alpha \nu_1} \right\} > 0.$$

We are now ready to prove (19). By Lemma 10 (ii), for every $\varepsilon > 0$, there exists $n_0 \geq 1$ such that, for every $n \geq n_0$, $M_{\beta_n} \leq \varepsilon$. By (21), it follows that

$$\begin{aligned}
&\sup_{n \in \mathbb{N}^*} \left\| \mathcal{T}_h f_n - f_n \right\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad \leq \max \left\{ \max_{1 \leq n \leq n_0} \left\| \mathcal{T}_h f_n - f_n \right\|^2_{L^2(\mathbb{R}^d)}, C_3 \left(\left\| h \right\|^{2s'} + \varepsilon \right) \right\}. \qquad (22)
\end{aligned}$$

The L^2-continuity of translations implies that, for all $n \in \mathbb{N}^*$, $\|\mathcal{T}_h f_n - f_n\|^2_{L^2(\mathbb{R}^d)}$ tends to zero as $h \to 0$, so that

$$\max_{1 \leq n \leq n_0} \left\| \mathcal{T}_h f_n - f_n \right\|^2_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \quad h \to 0.$$

Consequently,

$$\limsup_{h \to 0} \sup_{n \in \mathbb{N}^*} \left\| \mathcal{T}_h f_n - f_n \right\|^2_{L^2(\mathbb{R}^d)} \leq C_3 \varepsilon.$$

Since ε is arbitrary, the proof of (19) is complete, and so is that of the theorem. ∎

We emphasize that Condition (6) restricts the validity of our convergence result to the image by T_W of the subspace $L^2(V) \cap H^s(\mathbb{R}^d)$. Since $L^2(V) \cap H^s(\mathbb{R}^d)$ is dense in $L^2(V)$, its image by T_W is dense in $L^2(W)$ (recall that $\operatorname{ran} T_W$ is dense in $L^2(W)$). However, a convergence result valid for all $g \in \mathcal{D}(T_W)$ is most certainly desirable. Such a result will be obtained in the next section. As we shall see, the price to be paid is, among other things, the reintroduction of α as a regularization parameter, along with β.

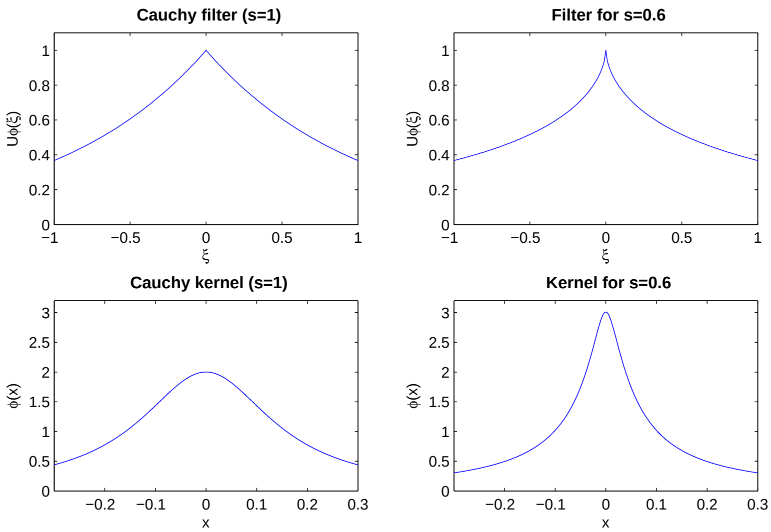

Figure 2: One-dimensional examples of filters of the form $\xi \mapsto \exp(-|\xi|^s)$, with associated point spread functions. In the case where $s = 1$, the point spread function is explicitly given by $\xi \mapsto 2/(1 + 4\pi\xi^2)$.

Remark 12. Among the filters $U\phi$ which satisfy Conditions (4) and (5), we find the functions $\xi \mapsto \exp(-\|\xi\|^s)$. For $s \in (0, 2]$, the corresponding Point Spread Functions $U^{-1}(\exp(-\|\xi\|^s))$ have nice morphological properties. In particular, they are positive (see [7]), isotropic, radially deacreasing (see [1]), and C^∞. For $s = 2$, ϕ is Gaussian, and for $s < 2$,

$$\phi(x) \sim_{\|x\| \to \infty} \|x\|^{-d-s}$$

up to a positive multiplicative constant (see [2]). Examples are given in Figure 2, in the cases $s = 1$ and $s = 0.6$. ∎

5 Hybrid approach

Theorem 13. *Let $\phi \in L^1(\mathbb{R}^d)$ be such that $\hat{\phi}(0) = \int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x \in (0, 1)$. Assume in addition that (5) holds true. Let $g \in \mathcal{D}(T_W^+)$. Let $(\alpha_n)_{n \in \mathbb{N}^*}$ and $(\beta_n)_{n \in \mathbb{N}^*}$ be sequences of positive reals converging to 0 which satisfy*

$$\sup_{\|\xi\| \leq \beta_n} \left| \hat{\phi}(\xi) - \hat{\phi}(0) \right|^2 = o(\alpha_n), \tag{23}$$

as $n \to \infty$. Let $f_n = f_{\alpha_n,\beta_n}$ be the unique solution of Problem $(\mathcal{P}_{\alpha_n,\beta_n})$. Then $\hat{\phi}(0)^{-1} f_n$ converges strongly to $T_W^+ g$, as $n \to \infty$.

Proof. As in the preceding proof, the first step consists in deriving an L^2-estimate for f_n. We use again the fact that f_n solves Problem $(\mathcal{P}_{\alpha_n,\beta_n})$ to write:

$$\begin{aligned}\frac{1}{2}\left\| \hat{\phi}_n g - T_W f_n \right\|^2_{L^2(W)} &+ \frac{\alpha_n}{2}\left\| (1-\hat{\phi}_n)\hat{f}_n \right\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \frac{1}{2}\left\| \hat{\phi}_n g - T_W T_W^+\big(\hat{\phi}(0) g\big) \right\|^2_{L^2(W)} + \frac{\alpha_n}{2}\left\| (1-\hat{\phi}_n) U T_W^+\big(\hat{\phi}(0) g\big) \right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Since $UT_W^+\big(\hat{\phi}(0)g\big) = \hat{\phi}(0) U T_W^+ g = \hat{\phi}(0)\tilde{g}$, the right hand side of the above inequality can be written as

$$\frac{1}{2}\left\| \big(\hat{\phi}_n - \hat{\phi}(0)\big) g \right\|^2_{L^2(W)} + \frac{\alpha_n \hat{\phi}(0)^2}{2}\left\| \big(1-\hat{\phi}_n\big)\tilde{g} \right\|^2_{L^2(\mathbb{R}^d)},$$

so that

$$\begin{aligned}\alpha_n &\left\| \big(1-\hat{\phi}_n\big)\hat{f}_n \right\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \left\| \big(\hat{\phi}_n - \hat{\phi}(0)\big) g \right\|^2_{L^2(W)} + \alpha_n \hat{\phi}(0)^2 \left\| \big(1-\hat{\phi}_n\big)\tilde{g} \right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Dividing the latter inequality by $\alpha_n \hat{\phi}(0)^2$ (which is clearly nonzero) yields

$$\begin{aligned}&\left\| (1-\hat{\phi}_n) U\big(\hat{\phi}(0)^{-1} f_n\big) \right\|^2_{L^2(\mathbb{R}^d)} \\ &\qquad \leq \frac{\hat{\phi}(0)^{-2}}{\alpha_n}\left\| \big(\hat{\phi}_n - \hat{\phi}(0)\big) g \right\|^2_{L^2(W)} + \left\| \big(1-\hat{\phi}_n\big)\tilde{g} \right\|^2_{L^2(\mathbb{R}^d)} \\ &\quad \leq \hat{\phi}(0)^{-2} \sup_{\xi\in W} \frac{|\hat{\phi}(\beta_n\xi) - \hat{\phi}(0)|^2}{\alpha_n} \left\| g \right\|^2_{L^2(W)} + \left\| \big(1-\hat{\phi}_n\big)\tilde{g} \right\|^2_{L^2(\mathbb{R}^d)}. \qquad (24)\end{aligned}$$

Recall indeed that $\hat{\phi}_n(\xi) = \hat{\phi}(\beta_n \xi)$. Now, the function $\xi \mapsto |1-\hat{\phi}(\xi)|$ is continuous and tends to 1 at infinity (by the Riemann-Lebesgue lemma). Moreover Condition (5) and the assumption that $\hat{\phi}(0) \in (0,1)$ imply that it is positive on $\mathbb{R}^d$. It follows that there exists $\gamma > 0$ such that, for all $\xi \in \mathbb{R}^d$, $|1-\hat{\phi}(\xi)| \geq \gamma$. Hence, for all $\xi \in \mathbb{R}^d$, $|1-\hat{\phi}_n(\xi)| = |1-\hat{\phi}(\beta_n(\xi))| \geq \gamma$ and we deduce from (24) and Plancherel's Equality that

$$\left\| \big(1-\hat{\phi}_n\big) U\big(\hat{\phi}(0)^{-1} f_n\big) \right\|^2_{L^2(\mathbb{R}^d)} \geq \gamma^2 \left\| \hat{\phi}(0)^{-1} f_n \right\|^2_{L^2(V)}.$$

On the other hand, since W is bounded, Condition (23) implies that

$$\lim_{n\to\infty} \sup_{\xi\in W} \frac{|\hat{\phi}(\beta_n\xi) - \hat{\phi}(0)|^2}{\alpha_n} = 0, \tag{25}$$

so that

$$C := \sup_{n\in\mathbb{N}^*} \sup_{\xi\in W} \frac{|\hat{\phi}(\beta_n\xi) - \hat{\phi}(0)|^2}{\alpha_n} < \infty.$$

Consequently, for all $n \in \mathbb{N}^*$,

$$\begin{aligned}
\gamma^2 \left\| \hat{\phi}(0)^{-1} f_n \right\|^2_{L^2(V)} &\leq \hat{\phi}(0)^{-2} C \left\| g \right\|^2_{L^2(W)} + \left\| (1 - \hat{\phi}_n)\tilde{g} \right\|^2_{L^2(\mathbb{R}^d)} \\
&\leq \hat{\phi}(0)^{-2} C \left\| g \right\|^2_{L^2(W)} + \big(1 + \|\phi\|_{L^1(\mathbb{R}^d)}\big)^2 \left\| \tilde{g} \right\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which shows that the sequence $\big(\hat{\phi}(0)^{-1} f_n\big)_{n\in\mathbb{N}^*}$ is bounded in $L^2(V)$. Now, we leave it to the reader to check, with the same reasoning as in the proof of Theorem 9 (see Step 2), that $\hat{\phi}(0)^{-1} f_n$ converges weakly to $T_W^+ g$. It then remains to establish the strong convergence.

By Lemma 11, $(1 - \hat{\phi}_n)\tilde{g}$ converges strongly to $(1 - \hat{\phi}(0))\tilde{g}$, which implies that

$$\left\| (1 - \hat{\phi}_n)\tilde{g} \right\|_{L^2(\mathbb{R}^d)} \to \left\| (1 - \hat{\phi}(0))\tilde{g} \right\|_{L^2(\mathbb{R}^d)} \quad \text{as} \quad n \to \infty.$$

It then results from (24) and (25) that

$$\limsup_{n\to\infty} \left\| \big(1 - \hat{\phi}_n\big) U\big(\hat{\phi}(0)^{-1} f_n\big) \right\|^2_{L^2(\mathbb{R}^d)} \leq \left\| \big(1 - \hat{\phi}(0)\big)\tilde{g} \right\|^2_{L^2(\mathbb{R}^d)}. \tag{26}$$

We now prove that the function $\varphi_n := (1 - \hat{\phi}_n) U\big(\hat{\phi}(0)^{-1} f_n\big)$ converges weakly to $\big(1 - \hat{\phi}(0)\big)\tilde{g}$. For every $\psi \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
\left\langle \varphi_n , \psi \right\rangle_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \big(1 - \hat{\phi}_n(\xi)\big) U\big(\hat{\phi}(0)^{-1} f_n\big)(\xi) \overline{\psi(\xi)} \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} U\big(\hat{\phi}(0)^{-1} f_n\big)(\xi) \overline{\big(1 - \overline{U}\phi_n(\xi)\big)\psi(\xi)} \,\mathrm{d}\xi \\
&= \left\langle U\big(\hat{\phi}(0)^{-1} f_n\big) , \big(1 - \overline{U}\phi_n\big)\psi \right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{27}$$

Since $\hat{\phi}(0)^{-1} f_n$ converges weakly to $T_W^+ g$, $U\big(\hat{\phi}(0)^{-1} f_n\big)$ converges weakly to $UT_W^+ g = \tilde{g}$, in $L^2(\mathbb{R}^d)$. Moreover, Lemma 11 implies that $\big(1 - \overline{U}\phi_n\big)\psi$ converges strongly to $\big(1 - \overline{U}\phi(0)\big)\psi$, in $L^2(\mathbb{R}^d)$. By the *weak-strong convergence theorem*, it follows from (27) that, for every $\psi \in L^2(\mathbb{R}^d)$,

$$\left\langle \varphi_n , \psi \right\rangle_{L^2(\mathbb{R}^d)} \to \left\langle \tilde{g} , \big(1 - \overline{U}\phi(0)\big)\psi \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle \big(1 - \hat{\phi}(0)\big)\tilde{g} , \psi \right\rangle_{L^2(\mathbb{R}^d)}$$

as $n \to \infty$. Consequently,

$$\left\| \big(1 - \hat{\phi}(0)\big)\tilde{g} \right\|_{L^2(\mathbb{R}^d)} \leq \liminf_{n\to\infty} \| \varphi_n \|_{L^2(\mathbb{R}^d)} . \tag{28}$$

From (26) and (28), we deduce that

$$\| \varphi_n \|_{L^2(\mathbb{R}^d)} \to \left\| \big(1 - \hat{\phi}(0)\big)\tilde{g} \right\|_{L^2(\mathbb{R}^d)} \quad \text{as} \quad n \to \infty,$$

and it follows that φ_n converges strongly to $\big(1 - \hat{\phi}(0)\big)\tilde{g}$ in $L^2(\mathbb{R}^d)$.
Finally, let us write $U(\hat{\phi}(0)^{-1} f_n) = (1 - \hat{\phi}_n)^{-1}\varphi_n$. Since, for every $n \in \mathbb{N}^*$, $|1 - \hat{\phi}_n|$ is bounded below by $\gamma > 0$, the function $(1 - \hat{\phi}_n)^{-1}$ converges locally uniformly on $\mathbb{R}^d$ to the constant $(1 - \hat{\phi}(0))^{-1}$ as $n \to \infty$, by being bounded by $1/\gamma$. This implies that $(1 - \hat{\phi}_n)^{-1}\varphi_n$ converges strongly to

$$\big(1 - \hat{\phi}_n\big)^{-1}\big(1 - \hat{\phi}(0)\big)\tilde{g} = \tilde{g}.$$

In other words, $U(\hat{\phi}(0)^{-1} f_n)$ converges strongly to $UT_W^+ g$. Since the Fourier operator is an isometry of L^2, this completes the proof of the strong convergence of $\hat{\phi}(0)^{-1} f_n$ to $T_W^+ g$. ▪

Figure 3 shows filters and their *complements*, in the cases where $\hat{\phi}(0) = 1$ and $\hat{\phi}(0) \in (0, 1)$. It should be noticed that, in our hybrid approach, the low frequency component of the object to be reconstructed (which correspond to the experimental domain W) undergoes a higher level of *tension* between the fit term and the regularization term. However, this extra tension can be made as small as desired by letting $\hat{\phi}(0)$ approach 1.

Appendix: Reminder on Tikhonov regularization

Throughout this appendix, F and G are infinite dimensional separable Hilbert spaces whose norms are both denoted by $\| \cdot \|$, and $T\colon F \to G$ is a compact, injective, linear operator. The adjoint (resp. generalized inverse) of T is denoted by $T^\star$ (resp. T^+). The closure of a set S is denoted by $\operatorname{cl} S$.

Theorem 14. (i) *There exists a real sequence $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ converging to zero and a Hilbert basis $\{f_k\}_{k\in\mathbb{N}^*}$ of F such that*

$$\forall k \in \mathbb{N}^*, \qquad T^\star T f_k = \lambda_k f_k;$$

(ii) *$T^{-1}\colon \operatorname{ran} T \to F$ is unbounded, and so is T^+;*

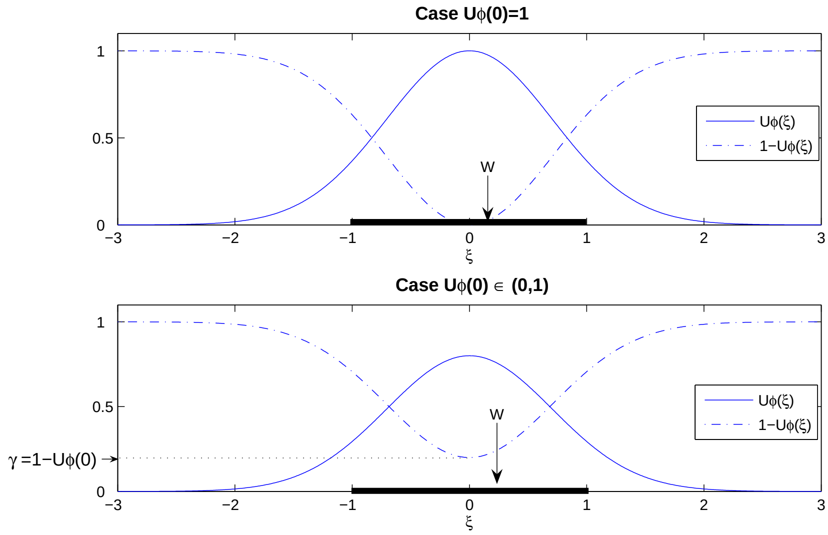

Figure 3: Gaussian filters and their *complements*, in the cases where $\hat{\phi}(0) = 1$ and $\hat{\phi}(0) \in (0, 1)$.

(iii) $\operatorname{ran} T$ *is not closed, so that* $\mathcal{D}(T^+) = \operatorname{ran} T + (\operatorname{ran} T)^\perp \subsetneq G$.

Proof. Clearly, $T^\star T$ is Hermitian, compact, positive and injective. Point (i) is then a particular instance of the spectral theorem for compact hermitian operators. Now for all $k \in \mathbb{N}^*$, $\|Tf_k\|^2 = \lambda_k$, so that

$$\inf_{\|f\|=1} \|Tf\| = 0.$$

This proves Point (ii). Finally, if $\operatorname{ran} T$ were closed, it would be a Banach space on its own, and the Open Mapping Theorem would imply continuity of T^{-1}. This would contradict Point (ii), thus Point (iii) is clear. ∎

The system

$$g_k := \frac{1}{\sqrt{\lambda_k}} T f_k, \qquad f_k = \frac{1}{\sqrt{\lambda_k}} T^\star g_k$$

is the so-called *Singular Value Decomposition* of T, and the numbers $\sqrt{\lambda_k}$ are the *Singular Values* of T.

Proposition 15. *The family* $\{g_k\}_{k \in \mathbb{N}^*}$ *is a Hilbert basis of* $\operatorname{cl} \operatorname{ran} T$.

PROOF. For all $k, l \in \mathbb{N}^*$,

$$\langle g_k, g_l\rangle = \frac{1}{\sqrt{\lambda_k \lambda_l}}\langle Tf_k, Tf_l\rangle = \frac{1}{\sqrt{\lambda_k \lambda_l}}\langle f_k, T^\star T f_l\rangle = \sqrt{\frac{\lambda_l}{\lambda_k}}\langle f_k, f_l\rangle = \delta_{kl},$$

where δ_{kl} is the Kronecker symbol. Thus $\{g_k\}_{k\in\mathbb{N}^*}$ is an orthonormal family in G. Let us show that $\operatorname{cl}\operatorname{vect}\{g_k\}_{k\in\mathbb{N}^*} = \operatorname{cl}\operatorname{ran} A$. Clearly, $\operatorname{vect}\{g_k\}_{k\in\mathbb{N}^*} \subset \operatorname{ran} T$, so that $\operatorname{cl}\operatorname{vect}\{g_k\}_{k\in\mathbb{N}^*} \subset \operatorname{cl}\operatorname{ran} T$. On the other hand, recall that $F = \{\sum_k \alpha_k f_k | \sum_k |\alpha_k|^2 < \infty\}$, so that

$$\operatorname{ran} A = \left\{ \sum_k \alpha_k \sqrt{\lambda_k} g_k \;\middle|\; \sum_k |\alpha_k|^2 < \infty \right\} = \left\{ \sum_k \beta_k g_k \;\middle|\; \sum_k \frac{|\beta_k|^2}{\lambda_k} < \infty \right\}.$$

Since $\sum_k |\beta_k|^2/\lambda_k < \infty \Longrightarrow \sum_k |\beta_k|^2 < \infty$, we see that

$$\operatorname{ran} A \subset \left\{ \sum_k \beta_k g_k \;\middle|\; \sum_k |\beta_k|^2 < \infty \right\} = \operatorname{cl}\operatorname{vect}(g_k),$$

and the result follows. ▪

Proposition 16. (i) *For all* $f \in F$, $Tf = \sum_{k\in\mathbb{N}^*} \lambda_k^{1/2}\langle f, f_k\rangle g_k$*;*

(ii) *for all* $g \in G$, $T^\star g = \sum_k \lambda_k^{1/2}\langle g, g_k\rangle f_k$*;*

(iii) *for all* $g \in \mathcal{D}(T^+)$, $T^+ g = \sum_k \lambda_k^{-1/2}\langle g, g_k\rangle f_k$*.*

PROOF. It is an easy exercise. ▪

Proposition 17. *Let* α *be a position number, and let* I *denote the identity of* F*. Then* $T^\star T + \alpha I$ *is bi-continuous and* $f_\alpha := (T^\star T + \alpha I)^{-1} T^\star g$ *is the unique minimizer of*

$$\mathcal{F}\colon f \mapsto \frac{1}{2}\|g - Tf\|^2 + \frac{\alpha}{2}\|f\|^2.$$

Moreover, f_α *depends continuously on* $g \in G$*.*

PROOF. By Theorem 14(i), $T^\star T + \alpha I$ is diagonalizable, with eigenvalues $\lambda_1 + \alpha \geq \lambda_2 + \alpha \geq \ldots > \alpha$. It is then clear that $T^\star T + \alpha I$ is bi-continuous. Now, let $f \in F$ and let $h := f - f_\alpha$. For all $t \in \mathbb{R}$, let

$$\Phi(t) := \mathcal{F}(f_\alpha + th).$$

Clearly, $\Phi(t)$ is of the form $at^2 + bt + c$, with a, b, c real and a nonnegative. Thus Φ is convex. Moreover,

$$\Phi'(0) = \mathrm{Re}\left(\langle T^\star T f_\alpha - T^\star g + \alpha f_\alpha, h\rangle\right) = 0.$$

Thus, Φ reaches its minimum at 0. In particular, $\Phi(0) \leq \Phi(1)$, that is to say, $\mathcal{F}(f_\alpha) \leq \mathcal{F}(f)$. Since this is true for every $f \in F$, we deduce that f_α minimizes $\mathcal{F}$. The uniqueness of the minimizer is a consequence of the strict convexity of $\mathcal{F}$. As for the continuous dependence, it results immediately from the continuity of the operator $(T^\star T + \alpha I)^{-1}$. ∎

Theorem 18. *Let f_α be defined as above. Then f_α converges strongly to $f^+ := T^+ g$ as $\alpha \downarrow 0$.*

Proof. Let us first evaluate the difference $f^+ - f_\alpha$. We have:

$$\begin{aligned}
(T^\star T + \alpha I)(f^+ - f_\alpha) &= T^\star T f^+ + \alpha f^+ - (T^\star T + \alpha I)(T^\star T + \alpha I)^{-1} T^\star g \\
&= T^\star g + \alpha f^+ - T^\star g \\
&= \alpha f^+,
\end{aligned}$$

whence the equality $f^+ - f_\alpha = \alpha(T^\star T + \alpha I)^{-1} f^+$. Next, we express $(T^\star T + \alpha I)^{-1}$ in terms of the orthogonal projections P_k onto the one-dimensional subspaces $\mathrm{vect}\, f_k$, where $\{f_k\}_{k\in\mathbb{N}^*}$ is the Hilbert basis exhibited in Theorem 14. Since

$$\forall k \in \mathbb{N}^*,\ T^\star T f_k = \lambda_k f_k \quad \text{and} \quad \forall f \in F,\ f = \sum_{k\in\mathbb{N}^*} \langle f, f_k\rangle f_k,$$

one has, for all $f \in F$, $T^\star T f = \sum_{k\in\mathbb{N}^*} \lambda_k \langle f, f_k\rangle f_k = \sum_{k\in\mathbb{N}^*} \lambda_k P_k f$. Therefore,

$$(T^\star T + \alpha I) f = \sum_{k\in\mathbb{N}^*} (\lambda_k + \alpha) P_k f.$$

Now

$$\begin{aligned}
\sum_{j\in\mathbb{N}^*} \frac{1}{\lambda_j + \alpha} P_j (T^\star T + \alpha I) f &= \sum_{j\in\mathbb{N}^*} \frac{1}{\lambda_j + \alpha} P_j \left(\sum_{k\in\mathbb{N}^*} (\lambda_k + \alpha) P_k f \right) \\
&= \sum_{j\in\mathbb{N}^*} \frac{1}{\lambda_j + \alpha} (\lambda_j + \alpha) P_j f, \\
&= f,
\end{aligned}$$

where the convergence of the series in the left hand side is justified by the computation. Consequently,

$$(T^\star T + \alpha I)^{-1} = \sum_{k\in\mathbb{N}^*} \frac{1}{\lambda_k + \alpha} P_k.$$

Finally, let us prove that $f_\alpha \to f^+$ as $\alpha \to 0$. One has

$$\begin{aligned} \left\| f^+ - f_\alpha \right\|^2 &= \left\| \alpha (T^\star T + \alpha I)^{-1} f^+ \right\|^2 \\ &= \left\| \sum_{k\in\mathbb{N}^*} \frac{\alpha}{\lambda_k + \alpha} P_k f^+ \right\|^2 \\ &= \sum_{k\in\mathbb{N}^*} \left(\frac{\alpha}{\lambda_k + \alpha} \right)^2 |\langle f^+, f_k \rangle|^2. \end{aligned}$$

Since the series is convergent, and since, for all $k \in \mathbb{N}^*$,

$$\left(\frac{\alpha}{\lambda_k + \alpha} \right)^2 \left| \langle f^+, f_k \rangle \right|^2 \to 0 \quad \text{as} \quad \alpha \to 0,$$

a dominated convergence argument shows that $f_\alpha \to f^+$ as $\alpha \to 0$. ∎

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