# ON THE COMPOSITION OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. We prove that a Banach space $X$ has the Schur property if and only if every $X$-valued weakly differentiable function is Fréchet differentiable. We give a general result on the Fréchet differentiability of $f \circ T$, where $f$ is a Lipschitz function and $T$ is a compact linear operator. Finally we study, using in particular a smooth variational principle, the differentiability of the semi norm $\| \ \|_{lip}$ on various spaces of Lipschitz functions.

## 1. INTRODUCTION

If $X$ and $Y$ are two Banach spaces and $f$ is a map from an open subset $U$ of $Y$ into $X$. We shall say that $f$ is weakly differentiable at the point $y_0$ of $U$ if, for any $x^*$ in the dual space $X^*$, $x^* \circ f$ is differentiable at $y_0$. For a positive integer $k$, we will say as well that $f$ is weakly $C^k$, or weakly $k$ times continuously differentiable, if for any $x^*$ in $X^*$, $x^* \circ f$ is a $C^k$ function. It is stated in [3] and proved in [7] that a weakly $C^k$ function is always of class $C^{k-1}$. On the other hand, there are examples, as we will recall later, of nowhere differentiable weakly $C^1$ functions. In section 2, we will show that differentiability and weak differentiability are equivalent if and only if the space $X$ has the Schur property. We recall that a Banach space $X$ has the Schur property if any weakly convergent sequence in $X$ is norm convergent and that $\ell_1$ is the most classical example of an infinite dimensional Banach space with the Schur property.

We now define a few spaces of real valued functions that will be studied in sections 3 and 4.
We denote by $C_b(\mathbb{R})$ the Banach space of all bounded continuous real valued functions defined on $\mathbb{R}$ equipped with the supremum norm $\| \ \|_\infty$.
For $0 < \alpha \leq 1$, $Lip_b^\alpha(\mathbb{R})$ is the space of all bounded $\alpha$-Hölder functions equipped with the complete norm defined by

$$\|\phi\|_{\infty,\alpha} = \max(\|\phi\|_\infty, \|\phi\|_\alpha), \ \forall \phi \in Lip_b^\alpha(\mathbb{R})$$

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where,

$$\|\phi\|_\alpha = \sup_{x,y\in\mathbb{R},\ x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|^\alpha}.$$

When $\alpha = 1$, we simply denote this space by $Lip_b(\mathbb{R})$ and write $\|\phi\|_1 = \|\phi\|_{lip}$. If $s$ is a positive integer, $(C_b^s(\mathbb{R}), \|\ \|_{C^s})$ is the Banach space of all $C^s$ functions $\phi$ such that $\phi$, $\phi'$,..., $\phi^{(s)}$ are bounded on $\mathbb{R}$. The norm $\|\ \|_{C^s}$ is defined by

$$\|\phi\|_{C^s} = \max(\|\phi\|_\infty, \|\phi'\|_\infty, ..., \|\phi^{(s)}\|_\infty).$$

Let again $\alpha \in (0,1]$. We denote by $C_b^{1,\alpha}(\mathbb{R})$ the Banach space of all functions in $C_b^1(\mathbb{R})$ such that $\phi'$ is $\alpha$-Hölder, equipped with the norm

$$\|\phi\|_{1,\alpha} = \max(\|\phi\|_\infty, \|\phi'\|_\infty, \|\phi'\|_\alpha).$$

In section 3, we use the classical idea that compactness improves the Gâteaux differentiability of Lipschitz functions into Fréchet differentiability. We apply this idea to prove the generic Fréchet differentiability of $\|\ \|_\infty$ on the space of Lipschitz functions on $[0,1]$ and of $\|\ \|_{L^1}$ on the space of absolutely continuous functions on $[0,1]$. We also give a general result in Theorem 3.4.

Section 4 is essentially devoted to the study of the differentiability of the semi norm $R$ defined by $R(\phi) = \|\phi\|_{lip}$ on various Banach spaces of Lipschitz functions. First we give some negative results about its Fréchet differentiability on $C_b^1(\mathbb{R})$ and its Gâteaux differentiability on $Lip_b(\mathbb{R})$. Then, we apply the smooth variational principle of Deville, Godefroy and Zizler [5] and also results of Ghoussoub [6], to obtain positive statements about its generic Gâteaux differentiability on $C_b^1(\mathbb{R})$ and about its generic Fréchet differentiability on $C_b^k(\mathbb{R})$ for $k \geq 2$ or on $C_b^{1,\alpha}(\mathbb{R})$ for $0 < \alpha \leq 1$.

## 2. Weakly differentiable functions

Our first result can be seen as a converse of the "chain rule" on Banach spaces with the Schur property.

**Theorem 2.1.** *Let $X$ be a Banach space with the Schur property, $U$ be an open subset of a Banach space $Y$ and $y_0 \in U$. Suppose that $f : U \to X$ is weakly differentiable at $y_0$. Then $f$ is Fréchet differentiable at $y_0$.*

*Proof.* We may assume that $y_0 = 0$ and $f(0) = 0$. For $h \in Y$ and $x^* \in X^*$, we set

$$\Phi(h)(x^*) = \lim_{t\to 0} t^{-1}(x^* \circ f)(th) = D(x^* \circ f)(0)h. \tag{2.1}$$

Clearly, $\Phi(h)$ is linear on $X^*$. Let $(t_n)$ be a sequence of real numbers tending to 0. By assumption, we have that for any $x^*$ in $X^*$, the sequence

$(t_n^{-1}(x^* \circ f)(t_n h))_{n\geq 0}$ is bounded in $\mathbb{R}$. Then it follows from the Banach-Steinhaus Theorem that the sequence $(t_n^{-1} f(t_n h))_{n\geq 0}$ is bounded in $X$. This implies that $\Phi(h)$ is continuous on $X^*$ and therefore is an element of $X^{**}$. Then, equation (2.1) means that $\Phi(h)$ is the weak$^*$ limit of $t^{-1}f(th)$ as $t$ tends to 0. Since $X$ has the Schur property, it is a weakly sequentially complete Banach space. So $\Phi(h) \in X$.
We now claim that $\Phi : Y \to X$ is linear. Indeed, it follows easily from the Hahn-Banach Theorem that otherwise there would exist $x^* \in X^*$ such that $D(x^* \circ f)(0)$ is not linear.
Let us now fix $x^*$ in $X^*$. We have that

$$\sup_{\|h\|\leq 1} |\Phi(h)(x^*)| = \sup_{\|h\|\leq 1} |D(x^* \circ f)(0)h| = \|D(x^* \circ f)(0)\|_{Y^*} < +\infty.$$

So we can use again the Banach-Steinhaus Theorem to get that $\Phi$ is a bounded linear operator from $Y$ into $X$.
Finally, we have by assumption that $\|h\|^{-1}(f(h) - \Phi(h))$ tends weakly to 0 as $\|h\|$ tends to 0. Using again that $X$ has the Schur property, we obtain that

$$\lim_{h\to 0} \|h\|^{-1}\|f(h) - \Phi(h)\| = 0.$$

Thus $\Phi$ is the Fréchet derivative of $f$ at 0.

□

As a corollary, we obtain the following:

**Corollary 2.2.** *Let $X$ be a Banach space with the Schur property, $U$ be an open subset of a Banach space $Y$ and $f$ be a function from $U$ into $X$. Then, the following assertions are equivalent:*
*(i) $f$ is a $C^1$ function.*
*(ii) $f$ is a weakly $C^1$ function.*

*Proof.* We only need to show that (ii) implies (i). So let us assume that (ii) holds. By Theorem 2.1, we already know that $f$ is Fréchet differentiable on $U$ and that for any $x^*$ in $X^*$, $D(x^* \circ f) = x^* \circ Df$ on $U$.
Assume now that $Df$ is not continuous at some point $y$ of $U$. Then there exist $\varepsilon > 0$, $(y_n)_{n\geq 0} \subset U$ so that $\lim_{n\to\infty} \|y_n - y\| = 0$ and $(h_n)_{n\geq 0} \subset Y$ with $\|h_n\| = 1$ satisfying

$$\forall n \geq 0 \quad \|(Df(y_n) - Df(y))h_n\|_X > \varepsilon. \tag{2.2}$$

But we have by assumption that for any $x^* \in X^*$

$$\lim_{n\to\infty} \|x^* \circ Df(y_n) - x^* \circ Df(y)\|_{Y^*} = 0.$$

This implies that $(Df(y_n) - Df(y))h_n$ tends weakly to 0. Since $X$ has the Schur property, this is in contradiction with (2.2).

□

The above corollary can be extended to the case of $C^k$ functions.

**Corollary 2.3.** *Let $X$ be a Banach space with the Schur property, $U$ be an open subset of a Banach space $Y$ and $f$ be a function from $U$ into $X$. Let $k$ be a positive integer and assume that $f$ is weakly $C^k$. Then $f$ is a $C^k$ function.*

*Proof.* The proof will be done by induction. So let us assume that the statement is true for $k \in \mathbb{N}$ and that $f$ is weakly $C^{k+1}$. Since the proof is very similar to the previous ones, we will only outline its main steps.
By [7] or by the induction hypothesis, we already know that $f$ is $C^k$. First we have to show that for any $y$ in $U$, $D^k f$ is Fréchet differentiable at $y$. As usual, we assume for convenience that $y = 0$ and $D^k f(0) = 0$.
For $h = (h_1, .., h_{k+1}) \in Y^{k+1}$ and $x^* \in X^*$, we denote

$$\Psi(h)(x^*) = \lim_{t\to 0} t^{-1}(x^* \circ D^k f)(th_1)(h_2, .., h_{k+1}) = D^{k+1}(x^* \circ f)(0)(h_1, .., h_{k+1}).$$

Following the argument in the proof of Theorem 2.1 we show that for any $h$ in $Y^{k+1}$, $\Psi(h) \in X$ and that $\Psi$ is $(k+1)$-linear and continuous from $Y^{k+1}$ into $X$. Then, using the Schur property we deduce that $\Psi = D^{k+1} f(0)$.
Finally, one can easily adapt the proof of Corollary 2.2 in order to show that $D^{k+1} f$ is continuous on $U$.

□

We will now show that the composition property described in Theorem 2.1 is a characterization of the Schur property. More precisely, we have:

**Proposition 2.4.** *Let $X$ be a Banach which does not have the Schur property. Then there is a Lipschitz function $f : [-1, 1] \to X$ which is not differentiable at $0$, but such that for any open neighborhood $U$ of $f(0)$ in $X$ and any function $g : U \to \mathbb{R}$ differentiable at $f(0)$, $g \circ f$ is differentiable at $0$.*

*Proof.* Since $X$ does not have the Schur property, there is a weakly null normalized sequence $(x_n)_{n\geq 0}$ in $X$. We define $f$ by $f(0) = 0$, $f(2^{-n}) = 2^{-n}x_n$, $f$ is affine on each interval $[2^{-(n+1)}, 2^{-n}]$ and $f$ is an even function.
Let $t = u2^{-n} + (1-u)2^{-(n+1)}$ and $t' = u'2^{-n} + (1-u')2^{-(n+1)}$, with $u$ and $u'$ in $[0, 1]$. Then

$$\|f(t) - f(t')\| = |u - u'| \, \|2^{-n}x_n - 2^{-(n+1)}x_{n+1}\| \leq 3\,|u - u'|\, 2^{-(n+1)} = 3\,|t - t'|. \tag{2.3}$$

So $f$ is 3-Lipschitz on each interval $[2^{-(n+1)}, 2^{-n}]$.
Furthermore, for $p < n$:

$$\|f(2^{-p}) - f(2^{-n})\| \leq 2^{-p} + 2^{-n} \leq 3\,|2^{-p} - 2^{-n}|. \tag{2.4}$$

Using the triangular inequality, one can easily deduce from (2.3) and (2.4) that $f$ is 3-Lipschitz on $[-1,1]$.
Now, for $t \in [2^{-(n+1)}, 2^{-n}]$, there is $u \in [0,1]$ such that $t = u2^{-n} + (1-u)2^{-(n+1)} = (1+u)2^{-(n+1)}$. Then, for any $x^* \in X^*$:

$$t^{-1}(x^* \circ f)(t) = (1+u)^{-1}(2ux^*(x_n) + (1-u)x^*(x_{n+1})). \quad (2.5)$$

Since $(x_n)_{n\geq 0}$ is weakly null, it is clear from the above equation that for any $x^*$ in $X^*$, $x^* \circ f$ is differentiable at 0 and $(x^* \circ f)'(0) = 0$. Then it follows easily from the fact that $f$ is Lipschitz that for any real valued function $g$ which is differentiable at 0, $g \circ f$ is differentiable at 0 and $(g \circ f)'(0) = 0$.
Finally, if $f$ was differentiable at 0, we would have $f'(0) = 0$. But this is in contradiction with the fact that for any $n$, $\|2^n f(2^{-n})\| = 1$. □

We will finish this section with a related open question. Let us say that a Banach space $X$ has the near Radon-Nikodỳm property (in short near RNP) if every weakly $C^1$ function $f$ from $\mathbb{R}$ into $X$ has a point of differentiability. Our problem is to characterize the Banach spaces with the near RNP. Let us just give some elementary information concerning this question.

It follows from the Mean Value Theorem and the Banach Steinhaus Theorem, that a weakly $C^1$ function from $\mathbb{R}$ into $X$ is locally Lipschitz. So a Banach space with the Radon-Nikodỳm property has the near RNP. However, the converse is not true. Indeed there exists a subspace $S$ of $L^1$, constructed by J. Bourgain and H.P. Rosenthal in [4], without the Radon-Nikodỳm property but with the Schur property. Then, it follows from our Corollary 2.2 that $S$ has the near RNP.

On the other hand, as we already mentioned in the introduction, there are Banach spaces without the near RNP. The space $c_0$ is probably the simplest example . Indeed, let $f : \mathbb{R} \to c_0$ be given by $f(t) = (f_n(t))_{n\geq 1}$, where

$$\forall p \in \mathbb{N},\ f_{2p-1}(t) = \frac{\cos pt}{p} \text{ and } f_{2p}(t) = \frac{\sin pt}{p}.$$

Then $f$ is a well known example of a weakly $C^1$ but nowhere differentiable function (see [7] for instance).

Finally, we do not know if $L^1$ has the near RNP. We do not know either, any space not containing $c_0$ and without the near RNP.

## 3. Fréchet smoothness through compact operators

We start with the following elementary lemma:

**Lemma 3.1.** *Let $X$, $Y$ and $Z$ be three Banach spaces and let $T$ be a compact linear operator from $X$ into $Z$. Assume that a locally Lipschitz function $f$ from an open subset $U$ of $Z$ into $Y$ is Gâteaux differentiable at $Tx$ for some $x$ in $X$. Then the function $f \circ T$ is Fréchet differentiable at $x$.*

*Proof.* It is known and easy to see that a locally Lipschitz function is so called Hadamard differentiable at some point $x$ if and only if this function is Gâteaux differentiable at $x$. Therefore $f$ is Hadamard differentiable at $Tx$. In other words, for every compact Hausdorff subset $K$ of $Z$,

$$\lim_{t\to 0}\sup_{k\in K}\frac{\|f\big(Tx+tk\big)-f\big(Tx\big)-t\langle Df\big(Tx\big),k\rangle\|_Y}{|t|}=0$$

In particular, for $K=\overline{TB_X}$, we obtain,

$$\lim_{t\to 0}\sup_{h\in B_X}\frac{\|f\circ T(x+th)-f\circ T(x)-t\langle Df\big(Tx\big)\circ T,h\rangle\|_Y}{|t|}=0$$

This implies that $f\circ T$ is Fréchet differentiable at $x$. $\square$

We will now describe two natural applications of this lemma. Let us first denote by $\mu$ the Lebesgue measure on $[0,1]$, by $(L^1[0,1],\|\ \|_{L^1})$ the space of $\mu$-integrable functions with its natural norm and by $\mathcal{AC}[0,1]$ the space of all absolutely continuous functions on $[0,1]$. We equip this space with the complete norm $\|\ \|_{\mathcal{AC}}$ defined by $\|\phi\|_{\mathcal{AC}}=\|\phi\|_{L^1}+\|\phi'\|_{L^1}$. The space $(C[0,1],\|\ \|_\infty)$ is the space of all continuous functions on $[0,1]$ with the supremum norm and $Lip\,[0,1]$ is the space of all Lipschitz functions on $[0,1]$ equipped with the complete norm $\|\phi\|_{\infty,1}=\max(\|\phi\|_\infty,\|\phi\|_{lip})$.

First, we have

**Proposition 3.2.** *The function $\mathcal{N}: Lip\,[0,1]\to\mathbb{R}$ defined by $\mathcal{N}(g)=\|g\|_\infty$ is generically Fréchet differentiable in $(Lip[0,1],\|\ \|_{\infty,1})$*

*Proof.* It is well known that the supremum norm $\|\ \|_\infty$ is Gâteaux differentiable at $f\in C[0,1]$ if and only if $|f|$ attains its unique maximum at some point of $[0,1]$. On the other hand, by Ascoli-Arzela Theorem, the canonical embedding from $Lip\,[0,1]$ into $C[0,1]$ is a compact operator. Using Lemma 3.1, the norm $\|\ \|_\infty$ is Fréchet differentiable on $(Lip\,[0,1],\|\ \|_{\infty,1})$ at $f\in Lip\,[0,1]$ if the function $|f|$ attains its unique maximum at some point $x$ of $[0,1]$ (the converse is also true and can be easily checked). But the set of Lipschitz continuous functions $f$ on $[0,1]$ such that $|f|$ attains its unique maximum at some point, is dense in $(Lip\,[0,1],\|\ \|_{\infty,1})$. This implies that the norm $\|\ \|_\infty$ is Fréchet differentiable on a dense subset of $(Lip\,[0,1],\|\ \|_{\infty,1})$. Since $\|\ \|_\infty$ is a convex continuous function, we obtain that $\|\ \|_\infty$ is generically Fréchet differentiable on $(Lip\,[0,1],\|\ \|_{\infty,1})$. $\square$

**Remark:** This is a particular case of results in [1], where the approach is to use variational principles instead of compactness.

As another consequence of Lemma 3.1 we obtain

**Proposition 3.3.** *The function $\mathcal{M} : \mathcal{AC}[0,1] \to \mathbb{R}$ defined by $\mathcal{M}(g) = \|g\|_{L^1}$ for all $g \in \mathcal{AC}[0,1]$, is Fréchet differentiable at $f \in \mathcal{AC}[0,1]$ if and only if $\mu\big(\{x; f(x) = 0\}\big) = 0$. Moreover $\mathcal{M}$ is generically Fréchet differentiable on $\mathcal{AC}[0,1]$.*

*Proof.* The argument being similar, we just outline the main steps. The norm $\| \|_{L^1}$ is Gâteaux differentiable at $f \in L^1[0,1]$ if and only if $\mu(\{x, f(x) = 0\}) = 0$. It is also clear that if $f$ in $\mathcal{AC}[0,1]$ is such that $\mu(\{x, f(x) = 0\}) > 0$, then $\mathcal{M}$ is not Gâteaux differentiable at $f$. On the other hand, the canonical embedding from $\mathcal{AC}[0,1]$ into $L^1[0,1]$ is a compact operator. So the conclusion follows again from Lemma 3.1, the convexity of $\mathcal{M}$ and the density of the set $\{g \in \mathcal{AC}[0,1]; \mu(\{x, g(x) = 0\}) = 0\}$ in $\mathcal{AC}[0,1]$. □

We now turn to a general result:

**Theorem 3.4.** *Let $Y$ be a Banach space with the Radon-Nikodỳm Property and $Z$ be a separable Banach space. Let $X$ be a Banach space and suppose that there exists a compact operator $T$ from $X$ into $Z$ such that $TX$ is dense in $Z$. Then*
*(i) The set $TX$ is not Aronszajn null in $Z$.*
*(ii) For every locally Lipschitz function $f : Z \to Y$, there exists a subset $B$ of $Z$ which is Aronszajn null and such that $f \circ T$ is Fréchet-differentiable at every point of $T^{-1}(Z \setminus B)$.*

As an immediate corollary, we have

**Corollary 3.5.** *Let $(Z, \| \|_Z)$ be a separable Banach space and $(X, \| \|_X)$ be a Banach space such that $X$ is a dense subspace of $Z$. Suppose that the canonical embedding $i : X \to Z$ is compact. Then for every locally Lipschitz function $f : (Z, \| \|_Z) \to \mathbb{R}$, the restriction $f_{|X} : (X, \| \|_X) \to \mathbb{R}$ is Fréchet-differentiable at every point of a subset $A$ of $X$ which is not Aronszajn null in $Z$.*

The assertion (i) of Theorem 3.4 is a general and probably well known fact, that we state now and prove for the sake of completness.

**Proposition 3.6.** *Let $X$ be a Banach space, $Z$ be a separable Banach space and $T$ be a continuous linear operator from $X$ into $Z$. If $TX$ is dense in $Z$, then $TX$ is not Aronszajn null in $Z$.*

*Proof.* We will actually show that $TX$ is not cube null, which is an equivalent statement (see [2] for the definitions of these notions and the proof of their equivalence).
Let $(x_n)_{n\geq 1} \subset X$ such that $\sum \|x_n\|_X < \infty$, the vectors $Tx_n$ are linearly independant and their linear span is dense in $Z$. We now define $U$ from $Q = [0,1]^{\mathbb{N}}$ into $Z$ by

$$U(t) = \sum_{n\geq 1} t_n T x_n, \quad \text{for any } t = (t_n)_{n\geq 1} \in Q.$$

Since $X$ is complete and $T$ is continuous, we clearly have that $U(Q) \subset TX$. This shows that $TX$ is not cube null. $\square$

*Proof of Theorem 3.4.* Since $Z$ is a separable space, $Y$ is a space with the Radon-Nikodỳm Property and $f : Z \to Y$ is locally Lipschitz, it follows from Theorem 6.42. in [2] (different versions of this theorem were proved independantly by Aronszajn, Christensen and Mankiewicz) that the set $B$ on which $f$ is not Gâteaux differentiable is Aronszajn null. Then Lemma 3.1 implies that $f \circ T$ is Fréchet differentiable at every point of $\{x \in X : Tx \in Z \setminus B\}$. $\square$

Observe that the assumptions of Corollary 3.5 are satisfied for $X = \mathcal{AC}[0,1]$ and $Z = L^1[0,1]$ and for $X = Lip\,[0,1]$ and $Z = C[0,1]$. So in the above cases every Lipschitz function on $X$ which can be extended to a Lipschitz function on $Z$ has points of Fréchet differentiability. It is of particular interest when $X$ is not an Asplund space, which is the case in these situations.

## 4. Spaces of differentiable functions on the real line

Let us recall that the supremum norm $\|\ \|_\infty$ is generically Fréchet differentiable on a large class of spaces of bounded continuous functions, like $(Lip_b(\mathbb{R}), \|\ \|_{\infty,1})$ or $(C^1_b(\mathbb{R}), \|\ \|_{C^1})$ (see [1]). As a consequence, we get for instance that the norm $\|\ \|_{C^1}$ is generically Fréchet differentiable on the open subset of $C^1_b(\mathbb{R})$:

$$\mathcal{O} = \{\phi \in C^1_b(\mathbb{R}) : \|\phi\|_\infty > \|\phi'\|_\infty\}.$$

In this section, we investigate the differentiability of the semi norm $R$ given by $R(\phi) = \|\phi\|_{lip}$ on various Banach spaces of Lipschitz functions defined on the real line. Notice that, by the Mean Value Theorem, the restriction of $R$ is defined on $C^1_b(\mathbb{R})$ by $R(\phi) = \|\phi'\|_\infty$.

We start with a negative result on the Fréchet differentiability of this function.

**Proposition 4.1.** *The function $R$ is nowhere Fréchet differentiable on $C^1_b(\mathbb{R})$.*

*Proof.* First we choose, as we may, an even function $b$ in $C^1_b(\mathbb{R})$ with support in $[-1,1]$, increasing on $[-1,0]$ such that $\|b\|_\infty = b(0) = 1$ and $1 \le \|b\|_{C^1} = \|b'\|_\infty \le 2$.
Consider now $\phi$ in $C^1_b(\mathbb{R})$. We will prove that $R$ is not Fréchet differentiable at $\phi$.
We pick $t$ in $(0,1)$ such that $tR(\phi) < \dfrac{1}{4}$ and $x_0$ in $\mathbb{R}$ satisfying

$$|\phi'(x_0)| \ge (1-t^2)R(\phi).$$

Assume for instance that $\phi'(x_0) = |\phi'(x_0)|$. Then there exists $\delta \in (0,1)$ so that

$$\forall x \in [x_0-\delta, x_0+\delta] \quad \phi'(x) \ge (1-2t^2)R(\phi).$$

Now, for $x$ in $\mathbb{R}$, we set $h(x) = t\delta\, b(\frac{x-x_0}{\delta})$. Easy computations show that $t \leq \|h\|_{C^1} = \|h'\|_\infty \leq 2t$. Notice that there is $y_0$ in $(x_0-\delta, x_0)$ and $z_0$ in $(x_0, x_0+\delta)$ such that $h'(y_0) = -h'(z_0) = \|h\|_{C^1}$. Then we have

$$\begin{aligned} R(\phi+h) + R(\phi-h) - 2R(\phi) &\geq (\phi'+h')(y_0) + (\phi'-h')(z_0) - 2R(\phi) \\ &\geq 2\|h\|_{C^1} - 4t^2R(\phi) \geq 2t(1-2tR(\phi)) \\ &\geq t \geq \frac{\|h\|_{C^1}}{2}. \end{aligned} \tag{4.6}$$

Since $R$ is convex, this concludes our proof. □

We now show that the situation is even worst on $Lip_b(\mathbb{R})$.

**Proposition 4.2.** *The function $R$ is nowhere Gâteaux differentiable on $Lip_b(\mathbb{R})$.*

*Proof.* Let $f \in Lip_b(\mathbb{R})$. We will prove that $R$ is not Gâteaux differentiable at $f$. We may assume that $R(f) = 1$. We need to introduce a few auxiliary functions. For $s < t \in \mathbb{R}$, we set

$$\Delta_{s,t}(f) = \frac{f(t)-f(s)}{t-s}.$$

For $\varepsilon > 0$ and $x \in \mathbb{R}$, we denote

$$\begin{aligned} D(x,\varepsilon) &= \{(s,t) \in (x-\varepsilon, x+\varepsilon) \times (x-\varepsilon, x+\varepsilon),\ s<t\}, \\ D^+(x,\varepsilon) &= \{(s,t) \in (x, x+\varepsilon) \times (x, x+\varepsilon),\ s<t\}, \\ D^-(x,\varepsilon) &= \{(s,t) \in (x-\varepsilon, x) \times (x-\varepsilon, x),\ s<t\}. \end{aligned}$$

We can now define:

$$u_f(x) = \lim_{\varepsilon\to 0} \sup_{(s,t)\in D(x,\varepsilon)} |\Delta_{s,t}(f)|,$$

$$v_f(x) = \lim_{\varepsilon\to 0} \sup_{(s,t)\in D(x,\varepsilon)} \Delta_{s,t}(f),$$

$$w_f(x) = \lim_{\varepsilon\to 0} \sup_{(s,t)\in D(x,\varepsilon)} -\Delta_{s,t}(f),$$

$$v_f^+(x) = \lim_{\varepsilon\to 0} \sup_{(s,t)\in D^+(x,\varepsilon)} \Delta_{s,t}(f),$$

$$w_f^+(x) = \lim_{\varepsilon\to 0} \sup_{(s,t)\in D^+(x,\varepsilon)} -\Delta_{s,t}(f).$$

We define similarly $v_f^-(x)$ and $w_f^-(x)$.
One can easily show that for any $x$ in $\mathbb{R}$,

$$u_f(x) = \max(v_f(x), w_f(x)),$$
$$v_f(x) = \max(v_f^+(x), v_f^-(x)), \text{ and } w_f(x) = \max(w_f^+(x), w_f^-(x)).$$

It is also clear that $u_f$ is bounded, upper semi continuous (u.s.c.) and that $\|u_f\|_\infty = R(f)$.

First we will assume that $u_f$ does not attain its supremum. Since $u_f$ is u.s.c., it implies that there exists a sequence $(x_n)$ in $\mathbb{R}$ such that $u_f(x_n)$ tends to $R(f) = 1$ and $|x_n|$ tends to $+\infty$. By using some symmetry arguments and extracting a subsequence, we may assume without loss of generality that

$$\forall n \in \mathbb{N},\ x_{n+1} > x_n + 2 \ \text{ and } u_f(x_n) = v_f^+(x_n).$$

Then, for each $n$ in $\mathbb{N}$, we can pick $s_n < t_n$ in $(x_n, x_n + 1)$ so that $\Delta_{s_n,t_n}(f)$ tends to 1. Once this is done, we choose $\varepsilon_n > 0$ such that

$$\forall n \in \mathbb{N},\ [x, x + \varepsilon_n] \cap [s_n, t_n] = \emptyset \ \text{ and } \sum_{n=1}^{\infty} 2\varepsilon_n \leq 1.$$

We now build a continuous, piecewise affine function $h$, with constant slope equals to 1 on each interval $(x_n - \varepsilon_n, x_n + \varepsilon_n)$, which is equal to 0 on $(-\infty, x_1 - \varepsilon_1]$ and constant on each $[x_n + \varepsilon_n, x_{n+1} - \varepsilon_{n+1}]$. Clearly, $h \in Lip_b(\mathbb{R})$ and $\|h\|_{\infty,1} = 1$. Moreover, for any $t \in (0, 1)$ and any $n$ in $\mathbb{N}$ we have

$$v_{f+th}^+(x_n) = v_f^+(x_n) + t \ \text{ and } \ \Delta_{s_n,t_n}(f - th) = \Delta_{s_n,t_n}(f).$$

So for any $t$ in $(0, 1)$, $R(f + th) = 1 + t$ and $R(f - th) \geq 1$. It follows that $R$ is not Gâteaux differentiable at $f$.

Secondly, we will assume that $u_f$ admits a maximum at some point $x_0$ in $\mathbb{R}$. For symmetry reasons, we may assume that $1 = R(f) = u_f(x_0) = v_f^+(x_0)$. Then, there is a strictly decreasing sequence $(s_n)_{n\geq 0}$ in $(x_0, x_0 + 1)$ which is tending to $x_0$ and such that $\Delta_{s_{2n+1},s_{2n}}(f)$ tends to $R(f) = 1$. We denote by $a_n$ the midpoint of $[s_{2n+1}, s_{2n}]$. Notice that $\Delta_{a_n,s_{2n}}(f)$ and $\Delta_{s_{2n+1},a_n}(f)$ also tend to 1.

We need to introduce a "hat" function $\phi$ defined by $\phi = 0$ on $(-\infty, -1]$, $\phi(x) = x + 1$ for $x$ in $[-1, 0]$ and $\phi$ is even. Then, for $\varepsilon > 0$ and $x \in \mathbb{R}$, we denote $\phi_\varepsilon(x) = \varepsilon\phi(x/\varepsilon)$. Notice that for any $\varepsilon \in (0, 1)$, $\|\phi_\varepsilon\|_{\infty,1} = R(\phi) = 1$. Now we define

$$h(x) = \sum_{n=0}^{\infty} \phi_{\frac{s_{2n}-s_{2n+1}}{2}}(x - a_n).$$

Again, $h \in Lip_b(\mathbb{R})$ and $\|h\|_{\infty,1} = 1$. Then, for any $t \in (0, 1)$ and any $n$ in $\mathbb{N}$:

$$\Delta_{s_{2n+1},a_n}(f + th) = \Delta_{s_{2n+1},a_n}(f) + t \ \text{ and } \ \Delta_{a_n,s_{2n}}(f - th) = \Delta_{a_n,s_{2n}}(f) + t.$$

It follows that for any $t \in (0, 1)$, $R(f + th) = R(f - th) = 1 + t$. Thus $R$ is not Gâteaux differentiable at $f$.

Up to a few symmetry arguments, we have considered all the possible situations and therefore shown that $R$ is nowhere Gâteaux differentiable on $Lip_b(\mathbb{R})$.

□

**Remark:** This proof, which of course works as well on $Lip[0,1]$, shows the importance in Corollary 3.5 of the assumption that the function on $X$ has a Lipschitz extension to $Z$.

So, our first motivation now will be to study the Gâteaux differentiability of $R$ on $C_b^1(\mathbb{R})$.

If $\mathcal{D}(\mathbb{R})$ is a Banach space of differentiable functions, then, for $x$ in $\mathbb{R}$, we denote by $\delta'_x$ the linear functional defined by $\delta'_x(\phi) = \phi'(x)$ for all $\phi \in \mathcal{D}(\mathbb{R})$. We also denote by $\varepsilon$ the signe function defined on $\mathbb{R} \setminus \{0\}$.

Our main result is

**Theorem 4.3.** *(a) The semi norm $R$ is generically Gâteaux differentiable on $(C_b^1(\mathbb{R}), \|.\|_{C^1})$. More precisely, the set of points of Gâteaux differentiability of $R$ is $G = \{\phi \in C_b^1(\mathbb{R})$ st $|\phi'|$ admits a strong maximum $\}$ which is a dense $G_\delta$ subset of $C_b^1(\mathbb{R})$. For $\phi \in G$, the derivative of the semi norm $R$ at $\phi$ is $\varepsilon(\phi'(x_0))\delta'_{x_0}$ where $x_0 \in \mathbb{R}$ is the point where the function $|\phi'|$ attains its strong maximum.*

*(b) The norm $\| \ \|_{C^1}$ of $C_b^1(\mathbb{R})$ is generically Gâteaux differentiable on $C_b^1(\mathbb{R})$.*

Thus, as the regularity of the function space gets better, so does the regularity of $R$. In fact, our methods will also yield some results on the Fréchet differentiability of $R$ on spaces of functions with higher order of smoothness.

**Theorem 4.4.** *Let $0 < \alpha \leq 1$ and $s \in \mathbb{N}$ such that $s \geq 2$. For $X = C_b^s(\mathbb{R})$ or $X = C_b^{1,\alpha}(\mathbb{R})$, $R$ is Fréchet differentiable on $\{\phi \in X \text{ st } |\phi'|$ admits a strong maximum $\}$ which is a dense $G_\delta$ subset of $X$ and the derivative of $R$ is given by the same formula.*

Before proceeding with the proof of the above theorems, we have to recall some useful background. The origin of the main tool of our proof goes back to a paper of Deville, Godefroy and Zizler [5] who proved a smooth variational principle based on the Baire category theorem. In [6], Ghoussoub, generalizes this result to an abstract class of function spaces. The Lemma 4.8 that we estabish below can be viewed as a new application of these works. The following definitions and Theorem 4.7 are taken from [6]. We also refer to the survey paper of Deville and Ghoussoub in [8].

Let $(X,d)$ be a metric space and $(\mathcal{A}(X), \gamma)$ be a metric space of real valued functions defined on $X$. For any subset $F$ of $X$, we denote by $\mathcal{A}_F(X)$ the class of functions in $\mathcal{A}(X)$ that are bounded above on $F$. For $\phi \in \mathcal{A}_F(X)$, and $t > 0$, we define the *slice* $S(F, \phi, t)$ of $F$ by:

$$S(F, \phi, t) = \{x \in F; \phi(x) > \sup_F \phi - t\}.$$

**Definition 4.5.** The space $(X,d)$ is said to be *uniformly* $\mathcal{A}(X)$*-dentable* if for every non-empty set $F \subset X$, every $\phi \in \mathcal{A}_F(X)$, and every $\varepsilon > 0$ there exists $\psi \in \mathcal{A}_F(X)$ such that $\gamma(\phi,\psi) \leq \varepsilon$ and $t > 0$ such that $S(F,\psi,t)$ has diameter less than $\varepsilon$.

Then we associate to the metric space $(X,d)$, the space $\tilde{X} = X \times \mathbb{R}$ equipped with the pseudo-metric

$$\tilde{d}((x,\lambda)),(y,\mu)) = d(x,y).$$

We denote by $\tilde{\mathcal{A}}(\tilde{X})$ the class of all functions $\tilde{\phi}$ on $\tilde{X}$, with $\phi \in \mathcal{A}(X)$, where $\tilde{\phi}$ is defined by $\tilde{\phi}(x,\lambda) = \phi(x) - \lambda$. We equip $\tilde{\mathcal{A}}(\tilde{X})$ with the distance $\tilde{\gamma}(\tilde{\phi},\tilde{\psi}) = \gamma(\phi,\psi)$.

**Definition 4.6.** Let $(X,d)$ be a metric space. We say that the metric space $(\mathcal{A}(X),\gamma)$ is *admissible* if the three following conditions hold:
i) There exists $K > 0$ such that $\gamma(\phi,\psi) \geq K \sup\{|\phi(x) - \psi(x)|; x \in X\}$ for all $\phi$, $\psi \in \mathcal{A}(X)$.
ii)$(\mathcal{A}(X),\gamma)$ is a complete metric space.
iii) The product space $(\tilde{X},\tilde{d})$ is uniformly $\tilde{\mathcal{A}}(\tilde{X})$-dentable.

**Theorem 4.7. (Ghoussoub [6]):** *Let $(\mathcal{A}(X),\gamma)$ be an admissible class of functions on a complete metric space $(X,d)$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a bounded below, lower semicontinuous function with non empty domain. Then the set*

$$\{\phi \in \mathcal{A}(X) : f - \phi \text{ attains a strong minimum on } X\}$$

*is a dense $G_\delta$ subset of $\mathcal{A}(X)$.*

With these results in hand, we can now prove the following lemma:

**Lemma 4.8.** *Let $f : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be a bounded below lower semicontinuous function with non empty domain. Then for any positive integer $s$, the set*

$$\{\phi \in C_b^s(\mathbb{R}) : f - \phi' \text{ attains a strong minimum on } \mathbb{R}\}.$$

*is a dense $G_\delta$ subset of $C_b^s(\mathbb{R})$.*

*The same conclusion holds for the spaces $C_b^{1,\alpha}(\mathbb{R})$, with $\alpha \in (0,1]$.*

*Proof.* We only give the proof for the space $C_b^s(\mathbb{R})$. The other cases are similar. So let us consider the space $\mathcal{A}(\mathbb{R}) = \{\phi' : \phi \in C_b^s(\mathbb{R})\}$. We equip this space with the complete norm $\|\phi'\|_{\mathcal{A}(\mathbb{R})} = \|\phi\|_{C^s}$ for all $\phi \in C_b^s(\mathbb{R})$ such that $\phi(0) = 0$. By Theorem 4.7, it is enough to show that the space $\mathcal{A}(\mathbb{R})$ is admissible.

As we just mentioned, $\mathcal{A}(\mathbb{R})$ satisfies the condition ii) of Definition 4.6. It also clearly fulfils condition i). Thus it remains to show that the product space $\tilde{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$ is uniformly $\tilde{\mathcal{A}}(\mathbb{R} \times \mathbb{R})$-dentable.
So, let $\varepsilon \in (0,1)$ and let $\beta \in C_b^{s-1}(\mathbb{R})$ be a positive bump function on $\mathbb{R}$ such that supp($\beta$)$\subset [-\varepsilon/2,\varepsilon/2]$, $\|\beta\|_{C^{s-1}} = \varepsilon$ and $\beta(0) = \|\beta\|_\infty$. Now, let

$\mathcal{V} \in \tilde{\mathcal{A}}(\mathbb{R} \times \mathbb{R})$ be a function which is bounded above on a closed subset $F$ of $\mathbb{R} \times \mathbb{R}$. It follows from the definition of $\tilde{\mathcal{A}}(\mathbb{R} \times \mathbb{R})$, that there exists $\phi' \in \mathcal{A}(\mathbb{R})$ such that $\mathcal{V}(x,t) = \phi'(x) - t$ for all $(x,t)$ in $\mathbb{R} \times \mathbb{R}$. Let now $(x_0, t_0) \in F$ be such that

$$\phi'(x_0) - t_0 > \sup_F \mathcal{V} - \frac{\beta(0)}{2}.$$

Then we consider the functions $h$ and $\mathcal{W}$ defined by $h(x) = \int_0^x \beta(t - x_0)dt$ for all $x \in \mathbb{R}$ and $\mathcal{W}(x,t) = \phi'(x) + h'(x) - t$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.
Notice that $h'(x) = \beta(x - x_0)$. In particular $h'(x) = 0$ whenever $|x - x_0| \geq \varepsilon/2$ and $h'(x_0) = \beta(0)$. On the other hand,

$$\|\mathcal{W} - \mathcal{V}\|_{\tilde{\mathcal{A}}(\mathbb{R}\times\mathbb{R})} = \|h'\|_{\mathcal{A}(\mathbb{R})} = \|h\|_{C^s} = \varepsilon.$$

We can now consider the following non empty slice of $F$:

$$S = \{(x,t) \in F;\ \mathcal{W}(x,t) > \sup_F \mathcal{W} - \frac{\beta(0)}{2}\}.$$

If $(x,s) \in F$ and $|x - x_0| \geq \varepsilon/2$, then $h'(x) = 0$ and $(x,s)$ cannot belong to $S$. It follows that the diameter of $S$ for $\tilde{d}$ is at most $\varepsilon$. This proves the uniform dentability of $\tilde{\mathcal{A}}(\mathbb{R} \times \mathbb{R})$.

□

We now turn to the proofs of the main results of this section.

*Proof of Theorem 4.3:* We start with the assertion $(a)$. Let $G = \{\phi \in C_b^1(\mathbb{R})$ st $|\phi'|$ admits a strong maximum $\}$. It is standard to show that $G$ is a $G_\delta$ subset of $C_b^1(\mathbb{R})$. One can for instance prove, following [5], that

$$G = \bigcap_{n=1}^{\infty} \{\phi \in C_b^1(\mathbb{R}),\ \exists x \in \mathbb{R} :\ |\phi'(x)| > \sup_{|y-x|>1/n} |\phi'(y)|\}.$$

Let now $G' = \{\phi \in C_b^1(\mathbb{R}),\ \phi'$ has a strong maximum$\}$ and $G" = \{\phi \in C_b^1(\mathbb{R}),$ $-\phi'$ has a strong maximum$\}$. It follows from Lemma 4.8 that $G'$ and $G"$ are dense $G_\delta$ subsets of $C_b^1(\mathbb{R})$. Consider now $O' = \{\phi \in C_b^1(\mathbb{R}),\ \sup_{\mathbb{R}} \phi' > \sup_{\mathbb{R}} -\phi'\}$ and $O" = \{\phi \in C_b^1(\mathbb{R}),\ \sup_{\mathbb{R}} \phi' < \sup_{\mathbb{R}} -\phi'\}$. It is clear that $O'$ and $O"$ are open in $C_b^1(\mathbb{R})$, that $O' \cup O"$ is dense in $C_b^1(\mathbb{R})$ and that $G = (G' \cap O') \cup (G" \cap O")$. Therefore $G$ is also dense in $C_b^1(\mathbb{R})$.
In order to prove that $G$ coincides with the set of the points of Gâteaux differentiability of $R$ one only has to imitate the proof of the fact that the set of points of Gâteaux differentiability of $\|\ \|_\infty$ on $C_b(\mathbb{R})$ is $\{\phi \in C_b(\mathbb{R})$st $|\phi|$ admits a strong maximum $\}$. Of course, the formula for the derivative is part of the argument.

We will now prove the assertion $(b)$. We know from [1] that the supremum norm: $\phi \mapsto \|\phi\|_\infty$ is Fréchet differentiable on a dense $G_\delta$ subset $G_1$ of $(C_b^1(\mathbb{R}), \|\ \|_{C^1})$ and we proved above that the semi norm $\phi \mapsto \|\phi'\|_\infty$ is Gâteaux differentiable on a dense $G_\delta$ subset $G_2$ of $(C_b^1(\mathbb{R}), \| \ \|_{C^1})$. On the other hand, it is easy to see that the closed subset $F = \{\phi \in C_b^1(\mathbb{R}) : \|\phi\|_\infty = \|\phi'\|_\infty\}$ has an empty interior. It implies that the norm $\| \ \|_{C^1} = \max(\|\phi\|_\infty, \|\phi'\|_\infty)$ is Gâteaux differentiable at every point of $(G_1 \cap G_2)\backslash F$, which is a dense $G_\delta$ subset of $(C_b^1(\mathbb{R}), \| \ \|_{C^1})$. □

*Proof of Theorem 4.4:* The argument is very similar to the previous one. Let us just indicate that the improvement on the smoothness of $R$ relies essentially on the generic Fréchet differentiability of $\| \ \|_\infty$ on $C_b^{s-1}(\mathbb{R})$ or $Lip_b^\alpha(\mathbb{R})$ (see [1]). □

## References

[1] M. Bachir, A non convex analogue to Fenchel duality, *J. of Functional Analysis*, **181** (2001), 300-312.

[2] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Vol 1, *American Mathematical Society, Colloquium Publications*, **48**.

[3] N. Bourbaki, Eléments de Mathématique, Fasc. XXXIII, Variétés différentielles et analytiques, Fascicule de résultats, *Actualités Sci. Indus.*, no **1333**, Hermann, Paris (1971).

[4] J. Bourgain and H.P. Rosenthal, Martingales valued in certain subspaces of $L^1$, *Israel J. of Math.*, **37** (1980), 54-75.

[5] R. Deville, G. Godefroy and V. Zizler, A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimentions, *J. of Functional Analysis*, **111** (1993), 197-212.

[6] N. Ghoussoub, Duality and Perturbation Methods in Critical Point Theory., *Cambridge Tracts in Mathematics*, Vol. **64**, Cambridge University Press.

[7] J.M. Gutiérrez, J.G. Llavona, Composition operators between algebras of differentiable functions, *Trans. Amer. Math. Soc.*, **338** (1993), 769-782.

[8] Handbook of the Geometry of Banach spaces, Vol. 1, Johnson and Lindenstrauss edts, North Holland (2001).

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