ON DIPOLAR QUANTUM GASES IN THE UNSTABLE REGIME

JACOPO BELLAZZINI AND LOUIS JEANJEAN

ABSTRACT. We study the nonlinear Schrödinger equation arising in dipolar Bose-Einstein condensate in the unstable regime. Two cases are studied: the first when the system is free, the second when gradually a trapping potential is added. In both cases we first focus on the existence and stability/ instability properties of standing waves. Our approach leads to the search of critical points of a constrained functional which is unbounded from below on the constraint. In the free case, by showing that the constrained functional has a so-called *mountain pass geometry*, we prove the existence of standing states with least energy, the ground states, and show that any ground state is orbitally unstable. Moreover, when the system is free, we show that small data in the energy space scatter in all regimes, stable and unstable. In the second case, if the trapping potential is small, we prove that two different kind of standing waves appears: one corresponds to a *topological local minimizer* of the constrained energy functional and it consists in ground states, the other is again of *mountain pass type* but now corresponds to excited states. We also prove that any ground state is a *topological local minimizer*. Despite the problem is mass supercritical and the functional unbounded from below, the standing waves associated to the set of ground states turn to be orbitally stable. Actually, from the physical point of view, the introduction of the trapping potential stabilizes the system initially unstable. Related to this we observe that it also creates a gap in the ground state energy level of the system. In addition when the trapping potential is active the presence of standing waves with arbitrary small norm does not permit small data scattering. Eventually some asymptotic results are also given.

1. INTRODUCTION

In the recent years the so-called dipolar Bose-Einstein condensate, i.e a condensate made out of particles possessing a permanent electric or magnetic dipole moment, have attracted much attention, see e.g. [4, 5, 17, 21, 25, 27, 28]. At temperature much smaller than the critical temperature it is well described by the wave function $\psi(x, t)$ whose evolution is governed by the three-dimensional (3D) Gross-Pitaevskii equation (GPE), see e.g. [4, 5, 28, 31, 32],

$$(1.1) \quad ih\frac{\partial \psi(x, t)}{\partial t} = -\frac{h^2}{2m}\nabla^2\psi + W(x)\psi + U_0|\psi|^2\psi + (V_{dip} \star |\psi|^2)\psi, \quad x \in \mathbb{R}^3, \quad t > 0$$

where t is time, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is the Cartesian coordinates, $\star$ denotes the convolution, h is the Planck constant, m is the mass of a dipolar particle and $W(x)$ is an external trapping potential. In this paper we shall consider a harmonic potential

$$W(x) = \frac{m}{2}a^2 |x|^2$$

where a is the trapping frequency. $U_0 = 4\pi h^2 a_s/m$ describes the local interaction between dipoles in the condensate with a_s the $s-$wave scattering length (positive for repulsive interaction and negative for attractive interaction).

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The long-range dipolar interaction potential between two dipoles is given by

$$(1.2) \qquad V_{dip}(x) = \frac{\mu_0 \mu_{dip}^2}{4\pi} \frac{1 - 3cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3$$

where μ_0 is the vacuum magnetic permeability, μ_{dip} is the permanent magnetic dipole moment and θ is the angle between the dipole axis and the vector x. For simplicity we fix the dipole axis as the vector $(0, 0, 1)$. The wave function is normalized according to

$$(1.3) \qquad \int_{\mathbb{R}^3} |\psi(x, t)|^2 dx = N$$

where N is the total number of dipolar particles in the dipolar BEC.

This aim of this paper is twofold: first to study the existence of stationary states for (1.1) satisfying (1.3) and their stability properties, second to understand how the presence of the external trapping potential influences the dynamics of the system. In order to simplify the mathematical analysis we rescale (1.1) into the following dimensionless GPE,

$$(1.4) \qquad i\frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2}\nabla^2 \psi + \frac{a^2}{2}|x|^2 \psi + \lambda_1 |\psi|^2 \psi + \lambda_2 (K \star |\psi|^2)\psi, \quad x \in \mathbb{R}^3, \quad t > 0.$$

The dimensionless long-range dipolar interaction potential $K(x)$ is given by

$$K(x) = \frac{1 - 3cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3.$$

The corresponding normalization is now

$$(1.5) \qquad N(\psi(\cdot, t)) := ||\psi(\cdot, t)||_2^2 = \int_{\mathbb{R}^3} |\psi(x, t)|^2 dx = \int_{\mathbb{R}^3} |\psi(x, 0)|^2 dx = 1$$

and the physical parameters (λ_1, λ_2), which describes the strengh of the two nonlinearities, are given in (2.2). Note that despite the kernel K is highly singular it defines a smooth operator. More precisely the operator $u \to K \star u$ can be extended as a continuous operator on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$, see [11, Lemma 2.1]. Actually the local existence and uniqueness of solutions to (1.4) has been proved in [11].

From now on we deal with (1.4) under the condition (1.5) and we focus on the case when λ_1 and λ_2 fulfills the following conditions

$$(1.6) \qquad \begin{cases} \lambda_1 - \frac{4}{3}\pi\lambda_2 < 0, & \text{if} \quad \lambda_2 > 0; \\ \lambda_1 + \frac{8}{3}\pi\lambda_2 < 0, & \text{if} \quad \lambda_2 < 0. \end{cases}$$

These conditions which, following the terminology introduced in [11], define the *unstable regime* corresponds to the Figure 1.

To find stationary states we make the ansatz

$$(1.7) \qquad \psi(x, t) = e^{-i\mu t}u(x), \quad x \in \mathbb{R}^3$$

where $\mu \in \mathbb{R}$ is the chemical potential and $u(x)$ is a time-independent function. Plugging (1.7) into (1.4) we obtain the stationary equation

$$(1.8) \qquad -\frac{1}{2}\Delta u + \frac{a^2}{2}|x|^2 u + \lambda_1 |u|^2 u + \lambda_2 (K \star |u|^2)u + \mu u = 0$$
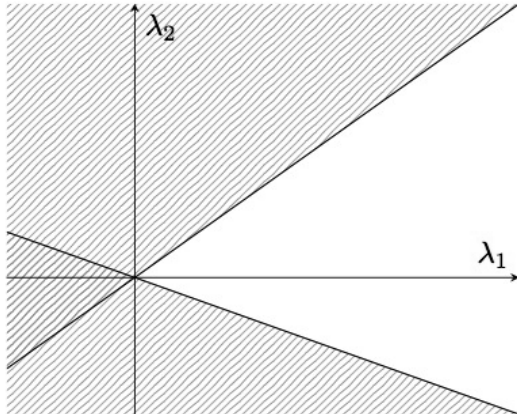
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FIGURE 1. The unstable regime given by (1.6) is the dark region ouside the cone.

and the corresponding constraint $u \in S(1)$ where

$$(1.9) \qquad S(1) = \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \ s.t. \ ||u||_2^2 = 1\}.$$

In the first part of the paper we consider the situation where the trapping potential is not active, namely when $a = 0$. The corresponding stationary equation is then just

$$(1.10) \qquad -\frac{1}{2}\Delta u + \lambda_1 |u|^2 u + \lambda_2 (K \star u^2) u + \mu u = 0, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}).$$

We recall, see [2], that the energy functional associated with (1.10) is given by

$$(1.11) \qquad E(u) := \frac{1}{2}||\nabla u||_2^2 + \frac{\lambda_1}{2}||u||_4^4 + \frac{\lambda_2}{2}\int_{\mathbb{R}^3}(K \star |u|^2)|u|^2 dx.$$

Any critical point of $E(u)$ constrained to $S(1)$ corresponds to a solution of (1.10) satisfying (1.9). The parameter $\mu \in \mathbb{R}$ being then found as the Lagrange multiplier.

As we shall prove, under assumption (1.6), the functional $E(u)$ is unbounded from below on $S(1)$. Actually when (1.6) is not satisfied, one speaks of the *stable regime*, the functional $E(u)$ is bounded from below on $S(1)$ and coercive, see [5, 10]. In that case one can prove that no standing waves exists, see Remark 4.1.

The problem of finding solutions to (1.10) was first considered in [2]. In [2, Theorem 1.1], assuming (1.6), Antonelli and Sparber obtain, for any $\mu > 0$, the existence of a real positive solution of (1.10), along with some symmetry, regularity and decay properties. To overcome the fact that $E(u)$ is unbounded from below on $S(1)$ they developped an approach in the spirit of Weinstein [30]. Namely their solutions are obtained as minimizers of the following scaling invariant functional

$$(1.12) \qquad J(v) := \frac{||\nabla v||_2^3 ||v||_2}{-\lambda_1 ||v||_4^4 - \lambda_2 \int_{\mathbb{R}^3}(K \star |v|^2)|v|^2}.$$

In [2] it is also shown that (1.6) are necessary and sufficient conditions to obtain a solution of (1.10).

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In this paper we propose an alternative approach. We directly work with $E(u)$ restricted to $S(1)$. We obtain our solution as a *mountain pass* critical point, Despite the fact the energy is unbounded from below on $S(1)$, if we restrict to states satisfying (1.9) that are stationary for the evolution equation (1.4), then the energy is bounded from below by a positive constant. We then show that this constant, corresponding to the mountain pass level, is reached and this will prove the existence of least energy states, also called ground states. As a direct consequence of this variational characterization and using a virial approach we manage to show that the associated standing waves are orbitally unstable.

Denoting the Fourier transform of u by $\mathcal{F}(u) := \int_{\mathbb{R}^3} u(x)e^{-ix\cdot\xi}dx$, the Fourier transform of K is given by

$$\hat{K}(\xi) = \frac{4}{3}\pi\left(\frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2}\right) \in [-\frac{4}{3}\pi, \frac{8}{3}\pi],$$

see [11, Lemma 2.3]. Then, thanks to the Plancherel identity, see for example [3, Theorem 1.25], one gets

$$(1.13) \qquad \lambda_1||v||_4^4 + \lambda_2 \int_{\mathbb{R}^3} (K \star |v|^2)|v|^2 = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2\hat{K}(\xi))|\hat{v^2}|^2 d\xi.$$

Thus we can rewrite $E(u)$ as

$$(1.14) \qquad E(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2}\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2\hat{K}(\xi))|\hat{u^2}|^2 d\xi.$$

In order to simplify the notation we define

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2\hat{K}(\xi))|\hat{u^2}|^2 d\xi.$$

$$Q(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2\hat{K}(\xi))|\hat{u^2}|^2 d\xi.$$

We also set $H := H^1(\mathbb{R}^N, \mathbb{C})$ and denotes by $||\cdot||$ the corresponding usual norm.

Despite the fact that we are primarily interested in solutions satisfying (1.9), for the mathematical treatment of the problem it is convenient to consider $E(u)$ on the set of constraints

$$S(c) = \left\{u \in H \ s.t. \ ||u||_2^2 = c\right\}.$$

Here $c > 0$ and the case $c = 1$ corresponds to the normalization (1.9). Given $c > 0$ we shall prove that $E(u)$ has a mountain pass geometry on $S(c)$, see [15] for a definition. More precisely we prove that there exists a $k > 0$ such that

$$(1.15) \qquad \gamma(c) := \inf_{g\in\Gamma(c)} \max_{t\in[0,1]} E(g(t)) > \max\{\max_{g\in\Gamma(c)} E(g(0)), \max_{g\in\Gamma(c)} E(g(1))\}$$

holds in the set

$$(1.16) \qquad \Gamma(c) = \{g \in C([0,1], S(c)) \ s.t. \ g(0) \in A_k, E(g(1)) < 0\},$$

where

$$A_k = \{u \in S(c) \ s.t. \ ||\nabla u||_2^2 \le k\}.$$

It it standard, see for example [15, Theorem 3.2], that the mountain pass geometry induces the existence of a Palais-Smale sequence at the level $\gamma(c)$. Namely a sequence $(u_n) \subset S(c)$ such that

$$E(u_n) = \gamma(c) + o(1), \quad ||E'|_{S(c)}(u_n)||_{H^{-1}} = o(1).$$

If one can show in addition the compactness of (u_n), namely that up to a subsequence, $u_n \to u$ in H, then a critical point is found at the level $\gamma(c)$. Actually under the assumptions of [2], in the unstable regime, we are able to prove the following

Theorem 1.1. *Let $c > 0$ and assume that (1.6) holds. Then $E(u)$ has a mountain pass geometry on $S(c)$ and there exists a couple $(u_c, \mu_c) \in H \times \mathbb{R}^+$ solution of (1.10) with $||u_c||_2^2 = c$ and $E(u_c) = \gamma(c)$. In addition $u_c \in S(c)$ is a ground state.*

Since our definition of ground states does not seem to be completely standard we now precise it.

Definition 1.1. *Let $c > 0$ be arbitrary, we say that $u_c \in S(c)$ is a ground state if*

$$E(u_c) = \inf\{E(u) \ s.t. \ u \in S(c), E'|_{S(c)}(u) = 0\}.$$

Namely a solution $u_c \in S(c)$ of (1.10) is a ground state if it minimize the energy functional $E(u)$ among all the solutions of (1.10) which belong to $S(c)$. We point out that with Definition 1.1 a ground state may exists even if $E(u)$ is unbounded from below on $S(c)$.

Remark 1.1. To prove that a Palais-Smale sequence converges a first step is to show that it is bounded and this is not given for free for $E(u)$. Note also that, due to the dipolar term, our functional is not invariant by rotations. This lack of symmetry also make delicate to prove the compactness of the sequences. To overcome both difficulties we shall prove the existence of one specific Palais-Smale sequence that fulfill $Q(u_n) = o(1)$. This localization property which follows the original ideas of [15], provides a direct proof of the H boundedness of the sequence and also, after some work, of its compactness.

Remark 1.2. Theorem 1.1 is in the spirit of some recent works [6, 20] in which constrained critical points are obtained for functionals unbounded from below on the constraint. We also refer to [26] for a closely related problem.

To prove Theorem 1.1 we establish that

$$\gamma(c) = \inf_{u \in V(c)} E(u)$$

where

$$V(c) = \{u \in S(c) \ s.t. \ Q(u) = 0\}.$$

As we shall see $V(c)$ contains all the critical points of $E(u)$ restricted to $S(c)$. Actually we also have

Lemma 1.1. *Let $c > 0$ be arbitrary, then $V(c)$ is a natural constraint, i.e each critical point of $E_{|V(c)}$ is a critical point of $E_{|S(c)}$.*

Let us denote the set of minimizers of $E(u)$ on $V(c)$ as

(1.17) $$\mathcal{M}_c := \{u_c \in V(c) \ s.t. \ E(u_c) = \inf_{u \in V(c)} E(u)\}.$$

Lemma 1.2. *Let $c > 0$ be arbitrary, then*

(i) *If $u_c \in \mathcal{M}_c$ then also $|u_c| \in \mathcal{M}_c$.*
(ii) *Any minimizer $u_c \in \mathcal{M}_c$ has the form $e^{i\theta}|u_c|$ for some $\theta \in \mathbb{S}^1$ and $|u_c(x)| > 0$ a.e. on $\mathbb{R}^3$.*

In view of Lemma 1.2 each element of $\mathcal{M}_c$ is a real positive function multiply by a constant complex factor.

Our next result connects the solutions found in [2] with the ones of Theorem 1.1.

Theorem 1.2. *Let $v \in H$ be, for some $\mu > 0$, the solution obtained in [2, Theorem 1.1]. Then setting $c = ||v||_2^2$ we have that $E(v) = \gamma(c)$.*

Remark 1.3. Since we do not know if nonnegative solutions of (1.10) are, up to translations, unique it is not possible to directly identify the solutions of [2] with the ones at the mountain pass level.

Concerning the dynamics, under (1.6) the global well posedness for (1.4) is not guaranteed in unstable regime. The problem is L^2 super-critical and energy estimates do not control the H norm. Conditions for blow-up has been discussed in [11]. However we are able to prove the following global existence result in an open nonempty set of H that contains not only small initial data.

Theorem 1.3. *Let $u_0 \in H$ be an initial condition associated to (1.4) with $c = ||u_0||_2^2$. If*

$$Q(u_0) > 0 \text{ and } E(u_0) < \gamma(c),$$

then the solution of (1.4) with $a = 0$ and initial condition u_0 exists globally in times.

For small data in the energy space we now show that scattering occurs independently of the values of λ_1 and λ_2. In particular it occurs in all regimes, stable and unstable.

Theorem 1.4. *Let $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$. There exists $\delta > 0$ such that if $||\psi_0|| < \delta$ then the solution $\psi(t)$ of (1.4) with $a = 0$ scatters in H. More precisely there exist $\psi_\pm \in H$ such that*

$$\lim_{t \to \pm\infty} ||\psi(t) - e^{it\frac{\Delta}{2}}\psi_\pm|| = 0.$$

Remark 1.4. In case of cubic NLS the classical strategy to show small data scattering in H is to prove that some $L_t^p W_x^{1,q}$ Strichartz admissible norm is uniformly bounded in time. In our case we follow the same strategy recalling that the additional nonlocal convolution term K describing the dipolar interaction is a continuous operator in L^p when $1 < p < \infty$. This permits to prove the boundedness of $L_{[0,\infty]}^{\frac{8}{3}}W_x^{1,4}$ and hence the scattering.

We now prove that the standing waves associated to elements in $\mathcal{M}_c$ are unstable in the following sense.

Definition 1.2. A standing wave $e^{i\omega t}v(x)$ is strongly unstable if for any $\varepsilon > 0$ there exists $u_0 \in H$ such that $||u_0 - v||_H < \varepsilon$ and the solution $u(t, \cdot)$ of the equation (1.4) with $u(0, \cdot) = u_0$ blows up in finite time.

Theorem 1.5. *For any $u \in \mathcal{M}_c$ the standing wave $e^{-i\mu_c t}u$ where $\mu_c > 0$ is the Lagrange multiplier, is strongly unstable.*

In a second part of the paper we analyse what happens to the system when one add, gradually, a confining potential. We are particularly interested in the existence of ground states and their stability but we shall also obtain the existence of excited states.

When $a > 0$ the functional associated to (1.8) becomes

(1.18) $$E_a(u) := \frac{1}{2}||\nabla u||_2^2 + \frac{a^2}{2}|||x|u||_2^2 + \frac{1}{2}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(\lambda_1 + \lambda_2\hat{K}(\xi))|\hat{u^2}|^2d\xi.$$

This functional being now defined on the space

(1.19) $$\Sigma = \left\{ u \in H \ s.t. \ \int |x|^2u^2dx < \infty \right\}.$$

The associated norm is

$$||u||_\Sigma^2 := ||u||_H^2 + |||x|u||_2^2.$$

It is standard, see [10, 11], that $E_a(u)$ is of class C^1 on Σ. Note that Σ has strong compactness properties which will be essential in our analysis. In particular the embedding $\Sigma \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $p \in [2,6)$, see for example [33, Lemma 3.1].

For simplicity we keep the notation $S(c)$ for the constraint which is now given by

$$S(c) = \left\{ u \in \Sigma \ s.t. \ ||u||_2^2 = c \right\}.$$

Definition 1.3. For $c > 0$ being given we say that $v \in S(c)$ is a topological local minimizer for $E_a(u)$ restricted to $S(c)$ if there exist an open subset $A \subset S(c)$ with $v \in A$, such that

(1.20) $$E_a(v) = \inf_{u \in A} E_a(u) \quad \text{and} \quad E_a(v) < \inf_{u \in \partial A} E_a(u).$$

Here the boundary is taken relatively to $S(c)$. If this occurs we say that v is a topological local minimizer for $E_a(u)$ on A.

Theorem 1.6. *Let $c > 0$ be given and assume that (λ_1, λ_2) satisfies (1.6). Then there exists a value $a_0 = a_0(\lambda_1, \lambda_2) > 0$ such that for any $a \in (0, a_0]$,*

(1) *$E_a(u)$ restricted to $S(c)$ admits a ground state u_a^1 and there exists a $k > 0$ such that u_a^1 is a topologial local minimizer for $E_a(u)$ on the set*

$$B_k = \{u \in S(c) \ s.t. \ ||\nabla u||_2^2 < k\}.$$

In addition any ground state for $E_a(u)$ restricted to $S(c)$ is a topological local minimizer for $E_a(u)$ on B_k.

(2) *$E_a(u)$ restricted to $S(c)$ admits a second critical point u_a^2 obtained at a mountain pass level and it corresponds to an excited state.*

(3) *The following properties hold*
   (a) *u_a^1 and u_a^2 are real, non negative.*
   (b) *For any $a \in (0, a_0]$, $0 < E_a(u_a^1) < E_a(u_a^2)$.*
   (c) *Any ground state $u_a \in S(c)$ for $E_a(u)$ on $S(c)$ satisfies $A(u_a) \to 0$ and $E_a(u_a) \to 0$ as $a \to 0$. Also $E_a(u_a^2) \to \gamma(c)$, where $\gamma(c) > 0$ is the least energy level of $E(u)$, the functional without the trapping potential.*
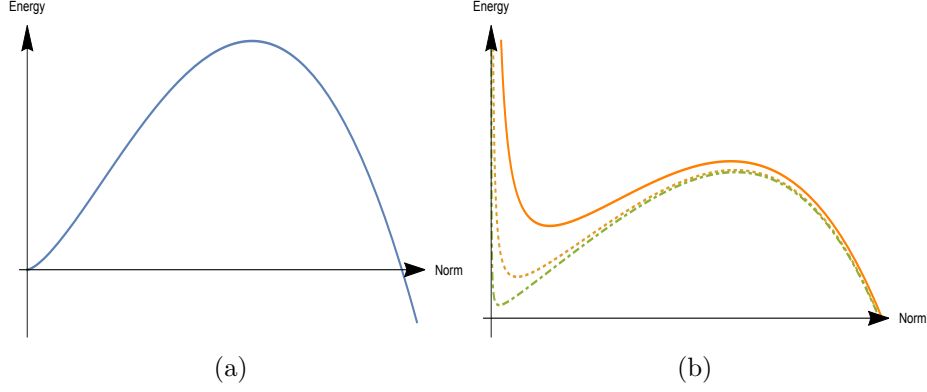
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FIGURE 2. Qualitative behavior of $E(u)$ (left) and $E_a(u)$ (right). In figure (b) the three curves mimic the behavior $E_a(u)$ for three different values of a.

Remark 1.5. The change of geometry of the constrained energy functional can be viewed as a consequence of the Heisenberg uncertainty principle, see e.g. [23],

$$(1.21) \qquad \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |x|^2 |u|^2 dx \right)^{\frac{1}{2}} \geq \frac{3}{2} \left(\int_{\mathbb{R}^3} |u|^2 dx \right).$$

Using (1.21) the energy functional $E_a(u)$, thanks to Gagliardo-Nirenberg inequality, fulfills

$$E_a(u) \geq \frac{1}{2}A(u) + \frac{9a^2c^2}{8A(u)} + \frac{1}{2}B(u) \geq \frac{1}{2}A(u) + \frac{9a^2c^2}{8A(u)} - CA(u)^{\frac{3}{2}}c^{\frac{1}{2}}$$

for some constant $C > 0$. The fact that $E_a(u)$ admits a topological local minimum is closely related to the previous inequality which implies in particular that

$$(1.22) \qquad \lim_{k \to 0} \inf_{u \in A_k} E_a(u) = +\infty.$$

A qualitative picture is given by Figure 2.

As a byproduct of Theorem 1.6 we are able to show that topological local minimizers, taking $a > 0$ fixed, fulfills $||u_a^1||_\Sigma \to 0$ when $c \to 0$. This fact implies

Corollary 1.1. *Under the assumption of Theorem 1.6 small data scattering cannot hold.*

Theorem 1.7. *Under the assumptions of Theorem 1.6 any ground state of $E_a(u)$ restricted to $S(c)$ is orbitally stable.*

The proof of Theorem 1.7 is simple. By Theorem 1.6 we know that any ground state is a topological local minimizer for $E_a(u)$ on B_k. By conservation of the energy and of the mass, for any initial data in B_k the trajectory remains in B_k (and in particular we have global existence). As a consequence of this it is possible to directly apply the classical arguments of Cazenave-Lions [13] which were developed to show the orbital stability of standing waves characterized as global minimizers. Note however that the energy $E_a(u)$ is unbounded from below on $S(c)$ for any $a \geq 0$.

Remark 1.6. From the physical point of view, Theorems 1.1, 1.6 and 1.7 show that the introduction of a small trapping potential leads to a stabilization of a system which was originally unstable. Up to our knowledge such physical phenomena had not been observed previously in laboratories or numerically. Note that such stabilizing effect is known to hold for lithium quantum gases (with a negative scattering lenght, attractive interactions), see [9]. We conjecture that as the trapping potential increases the system ceases however to be stable.

Remark 1.7. From Theorem 1.6 (1) we know that the ground state energy level corresponds to the one of the topological local minimizer u_a^1. Also from Theorem 1.6 (3) (c) we see that there is a discontinuity at $a = 0$ in the energy level of the ground state (which for $a = 0$ corresponds to $\gamma(c) > 0$). Thus the addition of a trapping potential, however small, create a *gap* in the ground state energy level of the system.

In contrast to the case $a = 0$ where the Lagrange parameter $\mu \in \mathbb{R}$ (namely the chemical potential) associated to any solution is strictly positive, see Lemma 3.1, we now have when $a > 0$,

Theorem 1.8. *Let $a \in (0, a_0]$ and u be a ground state for $E_a(u)$ restricted to $S(c)$. Then if $a > 0$ is sufficiently small $\mu \in \mathbb{R}$ as given in (1.8) satisfies $\mu < 0$.*

Finally we analyze what happen when (λ_1, λ_2) moves from *unstable region* towards the border of the *stable region*.

Theorem 1.9. *Let $c > 0$ and assume that (1.6) holds. Calling $\lambda_1' = \lambda_1 - \frac{4}{3}\pi\lambda_2$ when $\lambda_2 > 0$ $(\tilde{\lambda}_1' = \lambda_1 + \frac{8}{3}\pi\lambda_2$ when $\lambda_2 < 0)$ we have when $\lambda_1' \to 0^-$ $(\tilde{\lambda}_1' \to 0^-$ respectively)*
   (1) *The H-norm of the mountain pass solution obtained in Theorem 1.1 goes to infinity.*
   (2) *We can allow any $a_0 > 0$ in Theorem 1.6.*

We have chosen not to consider in this paper the stability of the standing wave corresponding to u_a^2. We conjecture that it is strongly unstable. Note that, due to the fact that the geometry of $E_a(u)$ on $S(c)$ is more complex than the one of $E(u)$, in particular the analogue of Lemma 3.3 does not hold, the treatment of this question probably requires new ideas.

We end our paper by an Appendix in which we prove a technical result concerning the Palais-Smale sequences associated to $E_a(u)$.

In the sequel we mainly consider the first case of (1.6), namely $\lambda_2 > 0$, $\lambda_1 - \frac{4}{3}\pi\lambda_2 < 0$, the second case follows by a similar treatment.

2. DERIVATION OF OUR DIMENSIONLESS GPE

In order to obtain a dimensionless GPE from (1.1) we introduce the new variables

(2.1) $$\tilde{t} = t, \quad \tilde{x} = \gamma x \text{ where } \gamma = \sqrt{\frac{m}{h}}, \quad \tilde{\psi}(\tilde{x}, \tilde{t}) = \frac{1}{\sqrt{N}} \frac{1}{\gamma^{3/2}} \psi(x, t).$$

Plugging (2.1) into (1.1), dividing by $h\sqrt{N}\gamma^{3/2}$ and then removing all $\tilde{}$ we obtain the dimensionless GPE

$$i\frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2}\nabla^2 \psi + \frac{a^2}{2}|x|^2 \psi + \lambda_1 |\psi|^2 \psi + \lambda_2 (K \star |\psi|^2)\psi, \quad x \in \mathbb{R}^3, \quad t > 0,$$

under the normalization

$$N(\psi(\cdot, t)) := ||\psi(\cdot, t)||^2 = \int_{\mathbb{R}^3} |\psi(x, t)|^2 dx = \int_{\mathbb{R}^3} |\psi(x, 0)|^2 dx = 1.$$

Here

(2.2) $$\lambda_1 = 4\pi a_s N\gamma, \quad \lambda_2 = \frac{mN\mu_0\mu_{dip}^2}{4\pi h^2}\gamma$$

and the dimensionless long-range dipolar interaction potential $K(x)$ is given by

(2.3) $$K(x) = \frac{1 - 3cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3.$$

3. PROOF OF THEOREM 1.1

First we show that any constrained critical point belongs to $V(c)$ and that the associated Lagrange multiplier is strictly positive.

Lemma 3.1. *If v is a weak solution of*

$$-\frac{1}{2}\Delta u + \lambda_1 |u|^2 u + \lambda_2 (K \star u^2)u + \mu u = 0$$

then $Q(v) = 0$. If we assume $v \neq 0$ then $\mu > 0$.

Proof. The proof is essentially contained in [2]. It follows from Pohozaev identity that

$$\frac{1}{4}A(u) + \frac{3}{4}B(u) + \frac{3}{2}\mu||u||_2^2 = 0.$$

Moreover, multiplying the equation by u and integrating one obtains

$$\frac{1}{2}A(u) + B(u) + \mu||u||_2^2 = 0.$$

The two equalities imply that

$$Q(u) = A(u) + \frac{3}{2}B(u) = 0 \quad \text{and} \quad A(u) = 6\mu||u||_2^2.$$

$\square$

To understand the geometry of $E(u)$ on $S(c)$ we introduce the scaling

(3.1) $$u^t(x) = t^{\frac{3}{2}} u(tx), \quad t > 0.$$

Observing that $\mathcal{F}(u^t)^2(\xi) = \mathcal{F}u^2(\frac{\xi}{t})$ the energy rescales as

(3.2) $$t \to E(u^t) = \frac{t^2}{2} A(u) + \frac{t^3}{2} B(u).$$

Lemma 3.2. *Let $u \in S(c)$ be such that $\int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) |\hat{u^2}|^2 d\xi < 0$ then:*
(1) $A(u^t) \to \infty$ and $E(u^t) \to -\infty$, as $t \to \infty$.
(2) There exists $k_0 > 0$ such that $Q(u) > 0$ if $\|\nabla u\|_2 \leq k_0$.
(3) If $E(u) < 0$ then $Q(u) < 0$.

Proof. Using (3.2) and since it always holds that

(3.3) $$E(u) - \frac{1}{3} Q(u) = \frac{1}{6} A(u)$$

we get (1) and (3). Now thanks to Gagliardo-Nirenberg inequality and Plancherel identity there exists a constant $C > 0$ such that

$$Q(u) > A(u) + \frac{3}{2} \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} (\lambda_1 - \frac{4}{3}\pi\lambda_2) |\hat{u^2}|^2 d\xi = A(u) - C\|u\|_4^4 \geq A(u) - CA(u)^{\frac{3}{2}} \|u\|_2,$$

and this proves (2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our next lemma is inspired by [12, Lemma 8.2.5].

Lemma 3.3. *Let $u \in S(c)$ be such that $\int_{\mathbb{R}^3} (\lambda_1 + \lambda_2 \hat{K}(\xi)) |\hat{u^2}|^2 d\xi < 0$ then we have:*
(1) There exists a unique $t^\star(u) > 0$, such that $u^{t^\star} \in V(c)$;
(2) The mapping $t \longmapsto E(u^t)$ is concave on $[t^\star, \infty)$;
(3) $t^\star(u) < 1$ if and only if $Q(u) < 0$;
(4) $t^\star(u) = 1$ if and only if $Q(u) = 0$;
(5)

$$Q(u^t) \begin{cases} > 0, \ \forall \ t \ \in \ (0, t^*(u)); \\ < 0, \ \forall \ t \ \in \ (t^*(u), +\infty). \end{cases}$$

(6) $E(u^t) < E(u^{t^\star})$, for any $t > 0$ and $t \neq t^\star$;
(7) $\frac{\partial}{\partial t} E(u^t) = \frac{1}{t} Q(u^t), \ \forall t > 0$.

Proof. Since

$$E(u^t) = \frac{t^2}{2} A(u) + \frac{t^3}{2} B(u)$$

we have that

$$\frac{\partial}{\partial t} E(u^t) = tA(u) + \frac{3}{2} t^2 B(u) = \frac{1}{t} Q(u^t).$$

Now we denote

$$y(t) = tA(u) + \frac{3}{2} t^2 B(u),$$

and observe that $Q(u^t) = t \cdot y(t)$ which proves (7). After direct calculations, we see that:

$$\begin{aligned} y'(t) &= A(u) + 3tB(u); \\ y''(t) &= 3B(u). \end{aligned}$$

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From the expression of $y'(t)$ and the assumption $B(u) < 0$ we know that $y'(t)$ has a unique zero that we denote $t_0 > 0$ such that t_0 is the unique maximum point of $y(t)$. Thus in particular the function $y(t)$ satisfies:

(i) $y(t_0) = \max_{t>0} y(t)$;

(ii) $\lim_{t \to +\infty} y(t) = -\infty$;

(iii) $y(t)$ decreases strictly in $[t_0, +\infty)$ and increases strictly in $(0, t_0]$.

By the continuity of $y(t)$, we deduce that $y(t)$ has a unique zero $t^\star > 0$. Then $Q(u^{t^*}) = 0$ and point (1) follows. Points (2)-(4) and (5) are also easy consequences of (i)-(iii). Finally since $y(t) > 0$ on $(0, t^*(u))$ and $y(t) < 0$ on $(t^*(u), \infty)$ we get (6). $\qquad \square$

Proposition 3.1. *Let $(u_n) \subset S(c)$ be a bounded Palais-Smale sequence for $E(u)$ restricted to $S(c)$ such that $E(u_n) \to \gamma(c)$. Then there is a sequence $(\mu_n) \subset \mathbb{R}$, such that, up to a subsequence:*

(1) $u_n \rightharpoonup \bar{u}$ weakly in H;

(2) $\mu_n \to \mu$ in $\mathbb{R}$;

(3) $-\frac{1}{2}\Delta u_n + \lambda_1 |u_n|^2 u_n + \lambda_2 (K \star |u_n|^2)u + \mu u_n \to 0$ in H^{-1};

(4) $-\frac{1}{2}\Delta \bar{u} + \lambda_1 |\bar{u}|^2 \bar{u} + \lambda_2 (K \star |\bar{u}|^2)\bar{u} + \mu \bar{u} = 0$ in H^{-1}.

Proof. The proof of Proposition 3.1 is standard and we refer to [6, Proposition 4.1] for a proof in a similar context. $\qquad \square$

Proof of Theorem 1.1. The proof requires several steps.

Step 1: *$E(u)$ has a Mountain-Pass Geometry on $S(c)$.*

First let us show that letting

$$C_k = \{u \in S(c) \text{ s.t. } A(u) = k\}$$

it is possible to choose $0 < k < 2k_0$, where $k_0 > 0$ is given in Lemma 3.2, such that

$$(3.4) \qquad 0 \le \inf_{u \in A_k} E(u) \le \sup_{u \in A_k} E(u) < \inf_{u \in C_{2k}} E(u).$$

Indeed observe that by Gagliardo-Nirenberg inequality and Plancherel identity, for some positive constants $\tilde{C}_i$, $i = 1, \cdots, 4$,

$$(3.5) \qquad E(u) \le \frac{A(u)}{2} + \tilde{C}_1 \|u\|_4^4 \le \frac{A(u)}{2} + \tilde{C}_2 A(u)^{\frac{3}{2}} \|u\|_2.$$

$$E(u) \ge \frac{A(u)}{2} + \frac{1}{2}\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\lambda_1 - \frac{4}{3}\pi\lambda_2)|\hat{u}^2|^2 d\xi \ge \frac{A(u)}{2} - \tilde{C}_3 \|u\|_4^4 \ge \frac{A(u)}{2} - \tilde{C}_4 A(u)^{\frac{3}{2}} \|u\|_2.$$

The proof of (3.4) follows directly from these two estimates taking $k > 0$ small enough. Now for an arbitrary $v \in S(c)$ consider the scaling given by

$$(3.6) \qquad v^t(x) = t^{\frac{5}{4}} v(tx_1, tx_2, t^{\frac{1}{2}}x_3), \quad t > 0.$$

We have $v^t \in S(c)$ for all $t > 0$ and the energy rescales as

$$E(v^t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla_{x_1, x_2} v|^2 dx + \frac{t}{2}\int_{\mathbb{R}^3} |\nabla_{x_3} v|^2 dx + \frac{t^{\frac{5}{2}}}{2}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} (\lambda_1 + \frac{4}{3}\pi\lambda_2 \frac{2t^2\xi_3^2 - t^2\xi_1^2 - t^2\xi_2^2}{t^2\xi_1^2 + t^2\xi_2^2 + t\xi_3^2})|\hat{v}^2|^2 d\xi.$$

This expression of $E(v^t)$ follows observing that $\mathcal{F}(u^t)^2(\xi) = \mathcal{F}u^2(\frac{\xi_1}{t}, \frac{\xi_2}{t}, \frac{\xi_3}{\sqrt{t}})$ and by a change of variable. Now under (1.6) we have that

$$\lim_{t\to\infty} \lambda_1 + \frac{4}{3}\pi\lambda_2 \frac{2t\xi_3^2 - t^2\xi_1^2 - t^2\xi_2^2}{t^2\xi_1^2 + t^2\xi_2^2 + t\xi_3^2} = \lambda_1 - \frac{4}{3}\pi\lambda_2 < 0,$$

which implies that $\lim_{t\to\infty} E(v^t) = -\infty$ thanks to Lebesgue's theorem. Just note that in the second case in (1.6), $\lambda_2 < 0$, $\lambda_1 + \frac{8}{3}\pi\lambda_2 < 0$, the same conclusion follows choosing the scaling

(3.7) $$\tilde{v}^t(x) = \lambda^{\frac{5}{4}} v(t^{\frac{3}{4}}x_1, t^{\frac{3}{4}}x_2, tx_3), \quad t > 0.$$

Thus in both cases the class of paths $\Gamma(c)$ defined in (1.16) is non void. Now if $g \in \Gamma(c)$ there exists a $\bar{t} \in (0, 1)$ such that $g(\bar{t}) \in C_{2k}$. Thus

$$\max_{t\in[0,1]} E(g(t)) \geq E(g(\bar{t})) \geq \inf_{u\in C_{2k}} E(u) > \sup_{u\in A_k} E(u),$$

and this implies that $\gamma(c) > 0$ where $\gamma(c)$ is given in (1.15). Thus $E(u)$ admits on $S(c)$ a mountain pass geometry.

Step 2: $\gamma(c) = \inf_{V(c)} E.$

Let $v \in V(c)$. Since $Q(v) = 0$ we get that $B(v) < 0$ and considering the scaling $v^t(x) = t^{\frac{3}{2}}v(tx)$, $t > 0$ we deduce from (3.2) that there exists a $t_1 << 1$ and a $t_2 >> 1$ such that $v^{t_1} \in A_k$ and $E(v^{t_2}) < 0$. Thus if we define

$$g(\lambda) = v^{(1-\lambda)t_1 + \lambda t_2} \quad \lambda \in [0, 1],$$

we obtain a path in $\Gamma(c)$. By the definition of $\gamma(c)$

$$\gamma(c) \leq \max_{\lambda\in[0,1]} E(g(\lambda)) = E(g(\frac{1-t_1}{t_2-t_1})) = E(v).$$

On the other hand any path $g(t)$ in $\Gamma(c)$ by continuity and Lemma 3.2 crosses $V(c)$. This shows that

$$\max_{t\in[0,1]} E(g(t)) \geq \inf_{u\in V(c)} E(u).$$

Step 3: *Existence of a bounded Palais-Smale sequence $(u_n) \subset S(c)$ at the level $\gamma(c)$.*

As stated in the Introduction the mountain pass geometry implies the existence of a sequence $(u_n) \subset S(c)$ such that

$$E(u_n) = \gamma(c) + o(1), \quad ||E'|_{S(c)}(u_n)||_{H^{-1}} = o(1).$$

By using an argument due to [15] we can strenghten this information and select a specific sequence localized around $V(c)$, namely such that $dist(u_n, V(c)) = o(1)$. To be more precise taking F as $V(c)$ in [15, Theorem 4.1] we obtain the existence of a sequence $(u_n) \subset S(c)$ such that

$$E(u_n) = \gamma(c) + o(1), \quad ||E'|_{S(c)}(u_n)||_{H^{-1}} = o(1), \quad dist(u_n, V(c)) = o(1).$$

The fact that taking $F = V(c)$ is possible follows from Steps 1 and 2.

Now, for any fixed $c > 0$, it follows directly from (3.3) that the set

$$L := \{u \in V(c), E(u) \leq \gamma(c) + 1\}$$

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is bounded. On the other hand $||dQ(\cdot)||_{H^{-1}}$ is bounded on any bounded set of H and thus in particular in a neighborhood of L. Now, for any $n \in \mathbb{N}$ and any $w \in V(c)$ we can write

$$Q(u_n) = Q(w) + dQ(au_n + (1-a)w)(u_n - w) = dQ(au_n + (1-a)w)(u_n - w)$$

where $a \in [0, 1]$. Thus choosing $(w_m) \subset V(c)$ such that

$$||u_n - w_m|| \to dist(u_n, V(c)) \text{ as } m \to \infty$$

we obtain, since $dist(u_n, V(c)) \to 0$, that $Q(u_n) = o(1)$. At this point, using again (3.3), we deduce that

$$E(u_n) = \frac{1}{6}||\nabla u_n||_2^2 + o(1)$$

which proves the boundedness of $(u_n) \subset S(c)$.

Step 4: *For all $c_1 \in (0, c)$ $\gamma(c_1) > \gamma(c)$.*

We use here the characterization

$$(3.8) \qquad \gamma(c) = \inf_{u \in S(c)} \max_{t>0} E(u^t).$$

To show (3.8) let us denote the right hand side by $\gamma_1(c)$. On one hand by Lemma 3.3 it is clear that for any $u \in S(c)$ such that $\max_{t>0} E(u^t) < \infty$ there exists a unique $t_0 > 0$ such that $u^{t_0} \in V(c)$ and $\max_{t>0} E(u^t) = E(u^{t_0})$. Now $E(u^{t_0}) \geq \gamma(c)$ by Step 2 and we thus get that $\gamma_1(c) \geq \gamma(c)$. On the other hand, for any $u \in V(c)$, $\max_{t>0} E(u^t) = E(u)$ and this readily implies that $\gamma_1(c) \leq \gamma(c)$.

Now, recording (3.2), we get after a simple calculation, that

$$(3.9) \qquad \max_{t>0} E(u^t) = \frac{2}{27} \frac{A(u)^3}{B(u)^2}.$$

Next take $u_1 \in S(c_1)$, such that

$$\max_{t>0} E(u_1^t) < \frac{c}{c_1}\gamma(c_1).$$

From the scaling $u_\theta(x) := \theta^{-\frac{1}{2}} u_1(\frac{x}{\theta})$ with $\theta > 0$, we have

$$||u_\theta||_2^2 = \theta^2 ||u_1||_2^2, \quad A(u_\theta) = A(u_1) \quad \text{and} \quad B(u_\theta) = \theta B(u_1).$$

Thus taking $\theta^2 = \dfrac{c}{c_1}$ we obtain that $u_\theta \in S(c)$ and it follows from (3.8) that

$$\gamma(c) \leq \max_{t>0} E(u_\theta^t) = \frac{2}{27} \frac{A(u_1)^3}{B(u_1)^2} \frac{c_1}{c} < \gamma(c_1).$$

Step 5: *Convergence of the Palais-Smale sequence $(u_n) \subset S(c)$.*

From Step 3 we know that there exists a bounded Palais-Smale sequence $(u_n) \subset S(c)$ such that $E(u_n) \to \gamma(c)$ and $Q(u_n) = o(1)$. Proposition 3.1 then implies that $u_n \rightharpoonup \bar{u}$ with $\bar{u}$ a solution of (1.10). Let us first show that we can assume $\bar{u} \neq 0$. Notice that

$$\int_{\mathbb{R}^3} (\lambda_1 - \frac{4}{3}\pi\lambda_2)|\hat{u}_n^2|^2 d\xi < \int_{\mathbb{R}^3} (\lambda_1 + \lambda_2\hat{K}(\xi))|\hat{u}_n^2|^2 d\xi = \frac{2}{3}Q(u_n) - \frac{2}{3}A(u_n) = o(1) - 4\gamma(c).$$

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This implies by Plancherel identity that $||u_n||_4 \geq C > 0$. At this point since $||u_n||_2^2 = c$, $||u_n||_6 \leq CA(u_n)^{\frac{1}{2}} < C$, the classical pqr-Lemma [14] implies that there exists a $\eta > 0$, such that

(3.10)
$$\inf_n |\{|u_n| > \eta\}| > 0.$$

Here $|\cdot|$ denote the Lebesgue measure. This fact, together with Lieb Translation lemma [22], assures the existence of a sequence $(x_n) \subset \mathbb{R}^3$ such that a subsequence of $u_n(\cdot + x_n)$ has a weak limit $\bar{u} \not\equiv 0$ in H. Now let us prove the strong convergence. Since $\bar{u}$ is non trivial and is a solution of (1.10) we can assume by Lemma 3.1 that $\bar{u} \in V(c_1)$ for some $0 < c_1 \leq c$. We recall that

(3.11) $\qquad A(u - \bar{u}) + A(\bar{u}) = A(u_n) + o(1), \quad B(u - \bar{u}) + B(\bar{u}) = B(u_n) + o(1).$

For a proof of the splitting property for $B(u)$ we refer to [2]. Since $E(u_n) \to \gamma(c)$ the splittings give

(3.12)
$$\frac{1}{2}A(u_n - \bar{u}) + \frac{1}{2}A(\bar{u}) + \frac{1}{2}B(u_n - \bar{u}) + \frac{1}{2}B(\bar{u}) = \gamma(c) + o(1)$$

and we also have

(3.13)
$$Q(u_n - \bar{u}) + Q(\bar{u}) = Q(u_n) + o(1).$$

Since $\bar{u} \in V(c_1)$ we have by Step 2 that $E(\bar{u}) \geq \gamma(c_1)$ and we deduce from (3.12) that

(3.14)
$$E(u_n - \bar{u}) + \gamma(c_1) \leq \gamma(c) + o(1).$$

At this point from (3.13), (3.14), Step 4 and using the fact that

$$\frac{1}{6}A(u_n - \bar{u}) = E(u_n - \bar{u}) - \frac{1}{3}Q(u_n - \bar{u})$$

we deduce that necessarily $c_1 = c$ and $A(u_n - \bar{u}) = o(1)$. This proves the strong convergence of $(u_n) \subset S(c)$ in H.

Step 5: *Conclusion*

Since $(u_n) \subset S(c)$ converges we deduce from Proposition 3.1 the existence of a couple $(u_c, \mu_c) \in H \times \mathbb{R}$ which satisfies (1.10) and such that $E(u_c) = \gamma(c)$. By Lemma 3.1 we see that $\mu_c > 0$. Still from Lemma 3.1 and using Step 2 we deduce that u_c is a ground state. $\square$

4. PROOFS OF LEMMAS 1.1 AND 1.2

In this section we show that $V(c)$ acts as a natural constraint and derive some properties of the set of ground states of $E(u)$ on $S(c)$.

Proof of Lemma 1.1. The fact that $V(c)$ is a C^1 manifold is standard by the implicit function theorem. Let u be a critical point of $E_{|V(c)}$, then there exist μ_1 and μ_2 such that

$$E'(u) - \mu_1 Q'(u) - 2\mu_2 u = 0.$$

We need to show that $\mu_1 = 0$. Notice that u fulfills the following equation

(4.1) $\qquad (1 - 2\mu_1)(-\Delta u) + 2(1 - 3\mu_1)\left(\lambda_1 |u|^2 u + \lambda_2 (K \star u^2)u\right) - 2\mu_2 u = 0.$

Multiplying (4.1) by u and integrating we get

(4.2) $\qquad (1 - 2\mu_1)A(u) + 2(1 - 3\mu_1)B(u) - 2\mu_2 ||u||_2^2 = 0.$

Also from Pohozaev identity

(4.3) $$\frac{1}{2}(1 - 2\mu_1)A(u) + \frac{3}{2}(1 - 3\mu_1)B(u) - 3\mu_2||u||_2^2 = 0.$$

Combining (4.2) and (4.3) we get

(4.4) $$(1 - 2\mu_1)A(u) + \frac{3}{2}(1 - 3\mu_1)B(u) = 0.$$

Now using the fact that $u \in V(c)$, i.e $A(u) + \frac{3}{2}B(u) = 0$, it follows from (4.4) that $\mu_1 A(u) = 0$. Thus necessarily $\mu_1 = 0$. $\qquad\square$

Proof of Lemma 1.2. Let $u_c \in H$ with $u_c \in V(c)$. Since $||\nabla|u_c|||_2 \leq ||\nabla u_c||_2$ we have that $E(|u_c|) \leq E(u_c)$ and $Q(|u_c|) \leq Q(u_c) = 0$. In addition, by Lemma 3.3, there exists $t_0 \in (0, 1]$ such that $Q(|u_c|^{t_0}) = 0$. Observe that, since $Q(u_c) = Q(|u_c|^{t_0}) = 0$, we have

$$E(|u_c|^{t_0}) = \frac{1}{6}A(|u_c|^{t_0}) = t_0^2 \cdot \frac{1}{6}A(|u_c|) = t_0^2 \cdot E(|u_c|) \leq t_0^2 E(u_c).$$

Thus if $u_c \in H$ is a minimizer of $E(u)$ on $V(c)$ we have

$$E(u_c) = \inf_{u \in V(c)} E(u) \leq E(|u_c|^{t_0}) \leq t_0^2 \cdot E(u_c),$$

which implies that $t_0 = 1$ since $t_0 \in (0, 1]$. Then $Q(|u_c|) = 0$ and we conclude that

(4.5) $$||\nabla|u_c|||_2 = ||\nabla u_c||_2 \quad \text{and} \quad E(|u_c|) = E(u_c).$$

Thus point (i) follows. Now since $|u_c|$ is a minimizer of $F(u)$ on $V(c)$ we know by Lemmas 1.1 and 3.1 that it satisfies (1.10) for some $\mu_c > 0$. By elliptic regularity theory and the maximum principle it follows that $|u_c| \in C^1(\mathbb{R}^3, \mathbb{R})$ and $|u_c| > 0$. At this point, using that $||\nabla|u_c|||_2 = ||\nabla u_c||_2$ the rest of the proof of point (ii) is exactly the same as in the proof of Theorem 4.1 of [18]. $\qquad\square$

Remark 4.1. Clearly in the stable regime, $B(u) \geq 0$, for any $u \in S(c)$. Then one always have that $Q(u) > 0$ on $S(c)$ and, in view of Lemma 3.1, $E(u)$ has no constrained critical points on $S(c)$. Thus no solution of (1.10) exists. In Step 1 of the Proof of Theorem 1.1 we show that (1.6) is a sufficient condition for the existence of a $u \in S(c)$, such that $E(u) < 0$ and thus $B(u) < 0$. Thus (1.6) is equivalent to the existence of at least one $u \in S(c)$ such that $B(u) < 0$.

5. Proof of Theorem 1.2

The aim of this section is to prove that the solutions obtained by [2] coincide with minimizers of $E(u)$ on $V(c)$. In [2, Theorem 1.1] the solutions of (1.10) are obtained as minimizer of the functional

$$J(v) := \frac{A(v)^{\frac{3}{2}}||v||_2}{-B(v)}.$$

Let us call u the minimizer of $J(v)$ that solves for $\mu > 0$

(5.1) $$-\frac{1}{2}\Delta u + \lambda_1|u|^2 u + \lambda_2(K \star u^2)u + \mu u = 0$$

and set $||u||_2^2 = c$. Our aim is to show that $E(u) = \gamma(c)$. Note that scaling properties of $J(u)$ allows to find a solution for any $\mu > 0$.

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Since u satisfies (5.1) then, by Lemma 3.1, $Q(u) = 0$ and this implies that

$$E(u) = \frac{1}{6}A(u) \quad \text{and} \quad B(u) = -\frac{2}{3}A(u).$$

It then follows by a direct calculation that

(5.2)
$$J(u) = \frac{1}{4}6^{3/2}c^{1/2}E^{1/2}(u).$$

Now assume that u is not a minimizer of $E(u)$ on $V(c)$. Then denoting by $u_0 \in V(c)$ a minimizer of $E(u)$ on $V(c)$ (we know that it exists by Theorem 1.1) we have that $E(u_0) < E(u)$. Since $u_0 \in V(c)$ we also have that

$$A(u_0) = 6E(u_0) \quad \text{and} \quad B(u_0) = -4E(u_0).$$

Thus

(5.3)
$$J(u_0) = \frac{1}{4}6^{3/2}c^{1/2}E^{1/2}(u_0).$$

Comparing (5.2) and (5.3) we derive that $J(u_0) < J(u)$ which provides a contradiction with the fact that u minimizes $J(v)$.

6. Proof of Theorem 1.3

Let $u(x,t)$ be the solution of (1.4) with $u(x,0) = u_0$ and $T_{max} \in (0, \infty]$ its maximal time of existence. Then classically we have either

$$T_{max} = +\infty$$

or

(6.1)
$$T_{max} < +\infty \quad \text{and} \quad \lim_{t \to T_{max}} ||\nabla u(x,t)||_2^2 = \infty.$$

Since

$$E(u(x,t)) - \frac{1}{3}Q(u(x,t)) = \frac{1}{6}A(u(x,t))$$

and $E(u(x,t)) = E(u_0)$ for all $t < T_{max}$, if (6.1) happens then, we get

$$\lim_{t \to T_{max}} Q(u(x,t)) = -\infty.$$

Since $Q(u(x,0)) > 0$, by continuity it exists $t_0 \in (0, T_{max})$ such that $Q(u(x,t_0)) = 0$ with $E(u(x,t_0)) = E(u_0) < \gamma(c)$. This contradicts the definition $\gamma(c) = \inf_{u \in V(c)} E(u)$.

7. Proof of Theorem 1.4

We recall the Duhamel formula associated to the evolution equation (1.4) when $a = 0$

$$\psi(t) = U(t)\psi_0 - i\lambda_1 \int_0^t U(t-s)(|\psi|^2\psi)(s)ds - i\lambda_2 \int_0^t U(t-s)((K \star |\psi|^2)\psi)(s)ds$$

where

$$U(t) = e^{it\frac{\Delta}{2}}$$

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generates the time evolution of the linear Schrödinger equation. We also recall the Strichartz estimates in $\mathbb{R}^d$, $d \geq 3$

(7.1) $$||U(\cdot)\varphi||_{L^q_t L^r_x} \leq C||\varphi||_{L^2}$$

(7.2) $$||\int_0^t U(t-s)F(s)ds||_{L^q_t L^r_x} \leq C||F||_{L^{q'_1} L^{r'_1}}$$

where the pairs (q,r), (q_1, r_1) are admissible, i.e $2 \leq r \leq \frac{2d}{d-2}$ and $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$ (analogous for (q_1, r_1)). The local Cauchy theory for equation (1.4) is proved in [11].

Theorem 7.1 ([11]). *There exists $T > 0$ depending only on $||\psi_0||$ such that (1.4) with initial data ψ_0 has a unique solution $\psi \in X_T$, where*

$$X_T = \left\{ \psi \in C([0,T]; H^1(\mathbb{R}^3)); \quad \psi, \nabla\psi \in C([0,T]; L^2(\mathbb{R}^3)) \cap L^{\frac{8}{3}}([0,T]; L^4(\mathbb{R}^3)) \right\}.$$

For the proof of Theorem 1.4 we shall need the following

Proposition 7.1. *There exists $\delta > 0$ such that if $||\psi_0|| < \delta$ then the solution $\psi(t)$ of (1.4) is global and $\sup_t ||\psi|| < \infty$.*

Proof. The proof in the stable regime is a direct consequence of the energy conservation since $E(u)$ is then coercive [5, 10] and the global well-posedness holds for any initial data in H. Under conditions (1.6) there exists initial data that blows up in finite time and hence not all initial data have bounded kinetic energy for all times. We consider for simplicity the case $\lambda_2 > 0$, $\lambda_1 - \frac{4}{3}\pi\lambda_2 < 0$, the other case is identical. According to Theorem 1.3 we just need to prove that when $||\psi_0||$ is small one has $Q(\psi_0) > 0$ and $E(\psi_0) < \gamma(||\psi_0||_2^2)$. Observe that thanks to Plancherel identity we can write $B(u)$ as

$$B(u) = (\lambda_1 - \frac{4}{3}\pi\lambda_2)||u||_4^4 + \lambda_2 \int_{\mathbb{R}^3} (\tilde{K} \star |u|^2)|u|^2 dx$$

where the fourier transform of $\tilde{K}$ is $\hat{\tilde{K}} = 4\pi\frac{|\xi_3|^2}{|\xi|^2}$. Hence

$$Q(u) \geq A(u) + \frac{3}{2}(\lambda_1 - \frac{4}{3}\pi\lambda_2)||u||_4^4 \geq A(u)\left(1 + C(\lambda_1 - \frac{4}{3}\pi\lambda_2)A(u)^{\frac{1}{2}}||u||_2\right).$$

In particular $Q(u) > 0$ when $||u||$ is sufficiently small. Now consider the ground states energy $\gamma(c)$ and let $u_c \in S(c)$ be a groundstate. We have

$$0 = Q(u_c) \geq A(u_c)\left(1 + C(\lambda_1 - \frac{4}{3}\pi\lambda_2)A(u_c)^{\frac{1}{2}}||u_c||_2\right)$$

which implies that

$$\lim_{c \to 0} A(u_c) = +\infty.$$

Now since $E(u_c) = \frac{1}{6}A(u_c)$ we deduce that $\lim_{c \to 0} \gamma(c) = +\infty$. Thus when $||\psi_0||$ is small we certainly have $E(\psi_0) < \gamma(||\psi_0||_2^2)$ and the proof is completed. $\square$

Proof of Theorem 1.4. We follow the classical strategy to show that $||\psi||_{L^p_t W^{1,q}_x}$ is globally bounded in time: this implies scattering. The admissible pair that we use is $(p,q) = (\frac{8}{3}, 4)$

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with conjugated pair $(p', q') = (\frac{8}{5}, \frac{4}{3})$. By using the Duhamel formula we have

$$||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}} \leq ||U(t)\psi_0||_{L_t^{\frac{8}{3}} W_x^{1,4}} + \lambda_1 ||\int_0^t U(t-s)(|\psi|^2 \psi)(s)ds||_{L_t^{\frac{8}{3}} W_x^{1,4}} +$$

$$+ \lambda_2 ||\int_0^t U(t-s)((K \star |\psi|^2)\psi)(s)ds||_{L_t^{\frac{8}{3}} W_x^{1,4}}.$$

Using Strichartz estimates (7.1), (7.2) we get

$$||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}} \leq c||\psi_0||_{H^1} + c|||\psi|^2\psi||_{L_t^{\frac{8}{5}} W_x^{1,\frac{4}{3}}} +$$

$$+ c||(K \star |\psi|^2)\psi||_{L_t^{\frac{8}{5}} W_x^{1,\frac{4}{3}}}.$$

First we estimate the terms $|||\psi|^2\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}}$ and $||(K \star |\psi|^2)\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}}$. By Hölder inequality we have

$$|||\psi|^2\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq c||\psi||_{L_t^{\frac{8}{3}} L_x^4} |||\psi|^2||_{L_t^4 L_x^2} = c||\psi||_{L_t^{\frac{8}{3}} L_x^4} ||\psi||_{L_t^8 L_x^4}^2$$

and

$$||(K \star |\psi|^2)\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq c||\psi||_{L_t^{\frac{8}{3}} L_x^4} ||K \star |\psi|^2||_{L_t^4 L_x^2}^2 \leq c||\psi||_{L_t^{\frac{8}{3}} L_x^4} ||\psi||_{L_t^8 L_x^4}^2.$$

Notice that in the last step we used that $||K \star f||_p \leq c||f||_p$, namely the $L^p - L^p$ continuity of K established in [11, Lemma 2.1]. Now using Proposition 7.1 and Sobolev embedding

$$||\psi||_{L_t^8 L_x^4}^2 \leq ||\psi||_{L_t^{\frac{8}{3}} L_x^4}^{\frac{2}{3}} ||\psi||_{L_t^\infty L_x^4}^{\frac{4}{3}} \leq c||\psi||_{L_t^{\frac{8}{3}} L_x^4}^{\frac{2}{3}} ||\psi||_{L_t^\infty H_x^1}^{\frac{4}{3}} \leq c||\psi||_{L_t^{\frac{8}{3}} L_x^4}^{\frac{2}{3}}$$

and we obtain

$$||\psi|^2\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq c||\psi||_{L_t^{\frac{8}{3}} L_x^4}^{\frac{5}{3}}.$$

Now we estimate the terms $||\nabla(|\psi|^2\psi)||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}}$ and $||\nabla(K \star |\psi|^2)\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}}$. By Hölder inequality again and arguing as before we get

$$||\nabla(|\psi|^2\psi)||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq c||\nabla\psi||_{L_t^{\frac{8}{3}} L_x^4} |||\psi^2||_{L_t^4 L_x^2} = c||\nabla\psi||_{L_t^{\frac{8}{3}} L_x^4} ||\psi||_{L_t^8 L_x^4}^2 \leq c||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}}^{\frac{5}{3}}.$$

The term $||(K \star |\psi|^2)\nabla\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}}$ behaves identically

$$||(K \star |\psi|^2)\nabla\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq c||\nabla\psi||_{L_t^{\frac{8}{3}} L_x^4} ||K \star |\psi|^2||_{L_t^4 L_x^2} \leq c||\nabla\psi||_{L_t^{\frac{8}{3}} L_x^4} ||\psi||_{L_t^8 L_x^4}^2 \leq c||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}}^{\frac{5}{3}}.$$

The last term to compute is $||(K \star \nabla|\psi|^2)\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}}$. For this term we argue as before

$$||(K \star \nabla|\psi|^2)\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq c||\psi||_{L_t^8 L_x^4} ||K \star \nabla|\psi|^2||_{L_t^2 L_x^2} \leq c||\psi||_{L_t^8 L_x^4} ||(\nabla\psi)\psi||_{L_t^2 L_x^2}$$

and therefore by Hölder inequality

$$||(K \star \nabla|\psi|^2)\psi||_{L_t^{\frac{8}{5}} L_x^{\frac{4}{3}}} \leq ||\nabla\psi||_{L_t^{\frac{8}{3}} L_x^4} ||\psi||_{L_t^8 L_x^4}^2 \leq c||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}}^{\frac{5}{3}}.$$

Eventually we proved that

(7.3) $$||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}} \leq c||\psi_0||_{H^1} + c||\psi||_{L_t^{\frac{8}{3}} W_x^{1,4}}^{\frac{5}{3}}.$$

Now calling $||\psi||_{L_t^{\frac{8}{3}}W_x^{1,4}} = y$ and $||\psi_0|| = b$ and looking at the function $f(y) = y - b - y^{\frac{5}{3}}$ we notice that if b is sufficiently small then $\{y \ \ s.t. \ \ f(y) \leq 0\}$ has two connected components. This implies that choosing $||\psi_0||$ sufficiently small we obtain, for some $C > 0$,

$$(7.4) \qquad ||\psi||_{L_{[0,\infty]}^{\frac{8}{3}}W_x^{1,4}} \leq C.$$

Scattering follows now from classical arguments, see e.g [12]. To conclude it is indeed enough to show that $U(-t)\psi(t) \rightarrow \psi_+$ in $H^1(\mathbb{R}^3)$. Notice that for $0 < t < \tau$ and calling $g := \lambda_1|\psi|^2\psi + \lambda_2(K \star |\psi|^2)\psi$ and $v(t) := U(-t)\psi(t)$, by (7.4) one gets

$$||v(t) - v(\tau)||_{H^1} \leq c||g(\psi)||_{L_{[t,\tau]}^{\frac{8}{5}}W_x^{1,\frac{4}{3}}} \rightarrow_{t,\tau\rightarrow\infty} 0.$$

Therefore it exists ψ_+ such that $\lim_{t\rightarrow\infty}||v(t) - \psi_+|| = 0$ and hence

$$\lim_{t\rightarrow\infty}||\psi(t) - U(t)\psi_+|| = \lim_{t\rightarrow\infty}||U(-t)\psi(t) - \psi_+|| = 0.$$

$\square$

8. Proof of Theorem 1.5

The proof of Theorem 1.5 is standard and follows the original approach by Glassey [16] and Berestycki-Cazenave [8]. We recall the virial identity, see [11] and [24],

$$(8.1) \qquad \frac{d^2}{dt^2}||xv(t)||_2^2 = 2\int_{\mathbb{R}^3}|\nabla v|^2 dx + 3\int_{\mathbb{R}^3}\lambda_1|v|^4 + \lambda_2(K \star |v|^2)|v|^2 dx = 2Q(v)$$

and the fact that all real positive solutions of (1.10) belongs to Σ as given in (1.19). This follow from the decay estimates obtained in [2], see also [12].

For any $c > 0$, we define the set

$$\Theta = \left\{v \in H \setminus \{0\} \ s.t. \ E(v) < E(u_c), \ ||v||_2^2 = ||u_c||_2^2, \ Q(v) < 0\right\}.$$

The set Θ contains elements arbitrary close to u_c in H. Indeed, letting $v_0(x) = u_c^\lambda = \lambda^{\frac{3}{2}}u_c(\lambda x)$, with $\lambda < 1$, we see from Lemma 3.3 that $v_0 \in \Theta$ and that $v_0 \rightarrow u_c$ in H as $\lambda \rightarrow 1$. Let $v(t)$ be the maximal solution of (1.4) with initial datum $v(0) = v_0$ and $T \in (0,\infty]$ the maximal time of existence. Let us show that $v(t) \in \Theta$ for all $t \in [0,T)$. From the conservation laws

$$||v(t)||_2^2 = ||v_0||_2^2 = ||u_c||_2^2,$$

and

$$E(v(t)) = E(v_0) < E(u_c).$$

Thus it is enough to verify $Q(v(t)) < 0$. But $Q(v(t)) \neq 0$ for any $t \in (0,T)$. Otherwise, by the definition of $\gamma(c)$, we would get for a $t_0 \in (0,T)$ that $E(v(t_0)) \geq E(u_c)$ in contradiction with $E(v(t)) < E(u_c)$. Now by continuity of Q we get that $Q(v(t)) < 0$ and thus that $v(t) \in \Theta$ for all $t \in [0,T)$. Now we claim that there exists $\delta > 0$, such that

$$(8.2) \qquad Q(v(t)) \leq -\delta, \ \forall t \in [0,T).$$

Let $t \in [0, T)$ be arbitrary but fixed and set $v = v(t)$. Since $Q(v) < 0$ we know by Lemma 3.3 that $\lambda^\star(v) < 1$ and that $\lambda \longmapsto E(v^\lambda)$ is concave on $[\lambda^\star, 1)$. Hence

$$
\begin{aligned}
E(v^{\lambda^\star}) - E(v) &\leq (\lambda^\star - 1)\frac{\partial}{\partial \lambda}E(v^\lambda) \mid_{\lambda=1} \\
&= (\lambda^\star - 1)Q(v).
\end{aligned}
$$

Thus, since $Q(v(t)) < 0$, we have

$$
E(v) - E(v^{\lambda^\star}) \geq (1 - \lambda^\star)Q(v) \geq Q(v).
$$

It follows from $E(v) = E(v_0)$ and $v^{\lambda^\star} \in V(c)$ that

$$
Q(v) \leq E(v) - E(v^{\lambda^\star}) \leq E(v_0) - E(u_c)
$$

and this proves the claim. Now from the virial identity (8.1) we deduce that $v(t)$ must blow-up in finite time. Recording that v_0 has been taken arbitrarily close to u_c, this ends the proof of the theorem.

9. Proof of Theorem 1.6 and Corollary 1.1

This section is devoted to the proof of Theorem 1.6. Under the scaling given by (3.1) we have

$$(9.1) \quad t \to E_a(u^t) = \frac{t^2}{2}A(u) + \frac{a^2}{2t^2}D(u) + \frac{t^3}{2}B(u) \quad \text{where we have set} \quad D(u) = \int_{\mathbb{R}^3} |x|^2 u^2 dx.$$

Let us also define

$$
Q_a(u) := A(u) - a^2 D(u) + \frac{3}{2}B(u).
$$

The proof of Theorem 1.6 requires several steps.

Step 1: *There exist a $a_0 > 0$ such that, for any $a \in (0, a_0]$, $E_a(u)$ has a topological local minima/mountain-pass geometry on $S(c)$.*

From (3.4) we know that there exists a $k > 0$ such that

$$(9.2) \qquad\qquad 0 \leq \inf_{u \in A_k} E(u) \leq \sup_{u \in A_k} E(u) < \inf_{u \in C_{2k}} E(u).$$

Also by Theorem 1.1 there exists a $u_c \in V(c)$ such that $E(u_c) = \gamma(c)$. We consider the path

$$(9.3) \qquad\qquad t \to v^t(x) := t^{\frac{3}{2}} u_c(tx), \quad t > 0.$$

First we fix a $t_1 << 1$ such that $v^{t_1} \in A_k$. Then, taking $a > 0$ sufficiently small so that

$$
\frac{a^2}{2t_1^2}D(v^{t_1}) < \inf_{u \in C_{2k}} E(u) - \sup_{u \in A_k} E(u),
$$

we obtain that

$$(9.4) \qquad\qquad 0 < E_a(v^{t_1}) < \inf_{u \in C_{2k}} E(u) \leq \inf_{u \in C_{2k}} E_a(u).$$

Thus in view of (1.22) it is reasonable to search for a minima of $E_a(u)$ inside the set A_{2k}.

Now, since $D(v^t) \to 0$ as $t \to +\infty$, we still have that $E_a(v^t) \to -\infty$ as $t \to +\infty$. We fix a $t_2 >> 1$ such that $E_a(v^{t_2}) < 0$ and define

$$
\Gamma_a(c) = \{g \in C([0, 1], S(c)) \ s.t. \ g(0) = v^{t_1}, g(1) = v^{t_2}\}.
$$

Clearly $\Gamma_a(c) \neq \emptyset$ and from (9.4) it holds that

$$\gamma_a(c) := \inf_{g \in \Gamma_a(c)} \max_{t \in [0,1]} E_a(g(t)) > \max\{E_a(v^{t_1}), E_a(v^{t_2})\} > 0.$$

Namely $E_a(u)$ has a *mountain pass geometry* on $S(c)$.

Step 2: *Existence of a topologial local minimizer.*

Let us prove that there exists a $u_a^1 \in A_{2k}$ which satisfies

$$E_a(u_a^1) = \inf_{u \in A_{2k}} E_a(u) > 0.$$

Because of (9.4) necessarily $u_a^1 \notin C_{2k}$ and thus u_a^1 will be a topological local minimizer for $E_a(u)$ restricted to $S(c)$. Let $(u_n) \subset A_{2k}$ be an arbitrary minimizing sequence associated to

$$I_a(c) = \inf_{u \in A_{2k}} E_a(u).$$

This sequence, being in A_{2k}, is bounded and we can assume that it converges weakly to some u_a^1. To prove the strong convergence, we use the compactness of the embedding $\Sigma \hookrightarrow L^p(\mathbb{R}^3)$ for $p \in [2,6)$. This gives directly that $u_a^1 \in S(c)$. Also since, for some $C > 0$,

(9.5) $$|B(u_n - u_a^1)| \leq C\|u_n - u_a^1\|_4^4 = o(1)$$

we get that

$$E_a(u_a^1) \leq \liminf E_a(u_n) = I_a(c).$$

This implies that $E_a(u_a^1) = I_a(c)$ and $A(u_n - u_a^1) \to 0$. Thus $u_n \to u_a^1$ and u_a^1 is a minimizer of $I_a(c)$. Note that since $I_a(|u|) \leq I_a(u), \forall u \in S(c)$ we can assume without restriction that u_a^1 is real. More generally a description of the set of topological local minimizers as in Lemma 1.2 is available in a standard way, see for example [10].

Step 3: *Existence of a mountain-pass critical point.*

Let us suppose for a moment the existence of a *bounded* PS sequence such that $E_a(u_n) \to \gamma_a(c)$. The proof of such claim requires some work. We posponed it until the Appendix. The strong convergence then follows from the following equivalent of Proposition 3.1.

Proposition 9.1. *Let $(u_n) \subset S(c)$ be a bounded Palais-Smale in Σ for $E_a(u)$ restricted to $S(c)$ such that $E_a(u_n) \to \gamma_a(c)$. Then there is a sequence $(\mu_n) \subset \mathbb{R}$, such that, up to a subsequence:*
(1) $u_n \rightharpoonup u_a^2$ weakly in Σ;
(2) $\mu_n \to \mu$ in $\mathbb{R}$;
(3) $-\frac{1}{2}\Delta u_n + \frac{a^2}{2}|x|^2 u_n + \lambda_1 |u_n|^2 u_n + \lambda_2(K \star |u_n|^2)u_n + \mu u_n \to 0$ in Σ^{-1};
(4) $-\frac{1}{2}\Delta u_a^2 + \frac{a^2}{2}|x|^2 u_a^2 + \lambda_1 |u_a^2|^2 u + \lambda_2(K \star |u_a^2|^2)u + \mu u_a^2 = 0$ in Σ^{-1}.

Indeed, letting (3) and (4) act on u_n we get

$$\frac{1}{2}A(u_n) + \frac{a^2}{2}D(u_n) + B(u_n) + c\mu = o(1)$$

$$\frac{1}{2}A(u_a^2) + \frac{a^2}{2}D(u_a^2) + B(u_a^2) + c\mu = 0.$$

Thus by substraction and using the splittings (3.11) we get that

$$\frac{1}{2}A(u_n - u_a^1) + \frac{a^2}{2}D(u_n - u_a^2) + B(u_n - u_a^2) = o(1).$$

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Thus from (9.5) we deduce that $A(u_n - u_a^2) = o(1)$, $D(u_n - u_a^2) = o(1)$. Namely that $u_n \to u_a^2$.

Step 4: $E_a(u_a) \to 0$ and $E_a(u_a^2) \to \gamma(c)$ as $a \to 0$.

To prove that $E_a(u_a) \to 0$ as $a \to 0$ we just need to observe that $k > 0$ in (9.2) can be taken arbitrarily small and that from (3.5) we readily have that $\inf_{u \in C_{2k}} E(u) \to 0$ as $k \to 0$. Then the conclusion follows from (9.4) since $E_a(u_a) \leq E_a(v^{t_1})$.

To show that $E_a(u_a^2) \to \gamma(c)$ namely that $\gamma_a(c) \to \gamma(c)$ as $a \to 0$ it suffices to observe that, on one hand, since $\Gamma_a(c) \subset \Gamma(c)$ and $E_a(u) \geq E(u)$ for all $u \in \Sigma$, then $\gamma_a(c) \geq \gamma(c)$. On the other hand considering the path (9.3) we have, as $a \to 0$,

$$\gamma_a(c) \leq \sup_{t \in [t_1, +\infty]} E_a(v^t) \leq \sup_{t \in [t_1, +\infty]} E(v^t) + \frac{a^2}{2t_1^2} D(u_c) = \gamma(c) + \frac{a^2}{2t_1^2} D(u_c) \to \gamma(c).$$

Step 5: u_a^1 is a ground state. In addition any ground state $u_a \in S(c)$ for $E_a(u)$ on $S(c)$ is a topological local minimizer for $E_a(u)$ on A_{2k} and it satisfies $A(u_a) \to 0$ and $E_a(u_a) \to 0$ as $a \to 0$.

Notice that any constrained critical point v fulfills $Q_a(v) = 0$. From the definition of $E_a(u)$ and $Q_a(u)$ we get

$$E_a(v) - \frac{1}{3} Q_a(v) = \frac{1}{6} A(v) + \frac{5}{6} a^2 D(v)$$

which implies that

(9.6) $$E_a(v) \geq \frac{1}{6} A(v).$$

Now let u_a be a constrained critical point such that $E_a(u_a) \leq E_a(u_a^1)$. From Step 4 we know that $E_a(u_a^1) \to 0$ when $a \to 0$ and together with (9.6) this implies that $A(u_a) \to 0$. Notice that k does not depend on a and therefore $u_a \in A_{2k}$ when a is sufficiently small. By definition of u_a^1 we obtain the opposite inequality $E_a(u_a) \geq E_a(u_a^1)$.

Proof of Corollary 1.1. In order to show that small data scattering cannot hold under the assumption of Theorem 1.6 it is sufficient to prove, for $a > 0$ fixed, that our topological local minimizers u_a fulfill $\lim_{c \to 0} \|u_a\|_\Sigma = 0$. In turn it is sufficient to show, fixing an arbitrary $\delta > 0$, that for any sufficiently small $c > 0$, $u_a \in A_{2\delta}$ and $aD(u_a) \leq 2\delta$.

To establish this property we fix an arbitrary $u_0 \in S(1)$ with $u_0 \in A_\delta$ and consider again the mapping from $S(1)$ to $S(c^2)$ given by $u^c(x) = c^{-1/2} u(\frac{x}{c})$. After direct calculations we have that

$$\|u_0^c\|_2^2 = c^2, \ A(u_0^c) = A(u_0) = \delta, \ B(u_0^c) = c B(u_0) \ \text{and} \ D(u_0^c) = c^4 D(u_0).$$

This leads to

$$E_a(u_a^c) = \delta + a^2 c^4 D(u_0) + c B(u_0).$$

Thus on one hand, when $c > 0$ is sufficiently small, we have that

(9.7) $$E_a(u_a^c) < \frac{3\delta}{2}.$$

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On the other hand, we have that for $A(u) = 2\delta$ and $c > 0$ small enough

(9.8)
$$E_a(u) \geq E(u) \geq \frac{3A(u)}{2} = \frac{3\delta}{2}.$$

In view of (9.7) and (9.8) we deduce that $u_a \in A_{2\delta}$ for any $c > 0$ small enough. Clearly also (9.7) implies that $aD(u_0) \leq 2\delta$. $\qquad\square$

10. Proof of Theorem 1.7

In this section we prove Theorem 1.7 following the ideas of [13]. First of all we recall the definition of orbital stability. We define

$$S_a = \{e^{i\theta}u(x) \in S(c) \text{ s.t. } \theta \in [0, 2\pi), \|u\|_2^2 = c, \ E_a(u) = E_a(u_a^1), \ A(u) \leq 2k\}.$$

Notice that here we are considering only the set of topological local minimizers. We say that S_a is *orbitally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\psi_0 \in \Sigma$ with $\inf_{v \in S_a} \|v - \psi_0\|_\Sigma < \delta$ we have

$$\forall t > 0 \quad \inf_{v \in S_a} \|\psi(t, .) - v\|_\Sigma < \varepsilon$$

where $\psi(t, .)$ is the solution of (1.4) with initial datum ψ_0. In order to prove Theorem 1.7 we argue by contradiction, i.e we assume that there exists a $\varepsilon > 0$ a sequence of initial data $(\psi_{n,0}) \subset \Sigma$ and a sequence $(t_n) \subset \mathbb{R}$ such that the maximal solution ψ_n with $\psi_n(0, .) = \psi_{n,0}$ satisfies

$$\lim_{n \to +\infty} \inf_{v \in S_a} \|\psi_{n,0} - v\|_\Sigma = 0 \quad \text{and} \quad \inf_{v \in S_a} \|\psi_n(t_n, .) - v\|_\Sigma \geq \varepsilon.$$

Without restriction we can assume that $\psi_{n,0} \in S(c)$ such that $(\psi_{n,0})$ is a minimizing sequence for $E_a(u)$ inside A_{2k}. Also since $A(\psi_{n,0}) \leq 2k$ and

(10.1)
$$E_a(\psi_n(., t_n)) = E_a(\psi_{n,0}),$$

also $(\psi_n(., t_n))$ is a minimizing sequence for $E_a(u)$ inside A_{2k}. Indeed since

$$\inf_{u \in A_{2k}} E_a(u) < \inf_{u \in C_{2k}} E_a(u)$$

by continuity we have that $\psi_n(., t_n)$ lies inside A_{2k}. This proves in particular that ψ_n is global for $n \in \mathbb{N}$ large enough. Now since we have proved, in Step 2 of the proof of Theorem 1.6, that every minimizing sequence in A_{2k} has a subsequence converging in Σ to a topological local minimum on A_{2k} we reach a contradiction.

11. Proofs of Theorems 1.8 and 1.9

Proof of Theorem 1.8. Let $u \in S(c)$ be a topological local minimizer for $E_a(u)$ on A_{2k}. In particular it is a solution of

(11.1)
$$-\frac{1}{2}\Delta u + \frac{a^2}{2}|x|^2 u + \lambda_1 |u|^2 u + \lambda_2(K \star |u|^2)u + \mu u = 0.$$

Notice also that, as any critical point of $E_a(u)$ on $S(c)$, it satisfies since $Q_a(u) = 0$,

(11.2)
$$\mu\|u\|_2^2 = \frac{1}{6}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{5}{6}a^2 \int_{\mathbb{R}^3} |x|^2 |u|^2 dx.$$

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Finally observe that, thanks to Plancherel identity we can write $B(u)$ as

$$B(u) = (\lambda_1 - \frac{4}{3}\pi\lambda_2)||u||_4^4 + \lambda_2 \int_{\mathbb{R}^3} (\tilde{K} \star |u|^2)|u|^2 dx$$

where the fourier transform of $\tilde{K}$ is $\hat{\tilde{K}} = 4\pi \frac{|\xi_3|^2}{|\xi|^2}$. From now on we discuss separately the two cases $B(u) \geq 0$ and $B(u) < 0$.

Case $B(u) \geq 0$. Since $Q_a(u) = 0$ the fact that $B(u) \geq 0$ implies that $\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq a^2 \int_{\mathbb{R}^3} |x|^2|u|^2 dx$. Thus thanks to (11.2) we conclude that $\mu < 0$.

Case $B(u) < 0$. Any constrained critical point is a critical point of the free functional

$$J_a(u) := \frac{1}{2}E_a(u) + \frac{1}{2}\mu||u||_2^2.$$

Let us first compute $\langle J_a''(u)\varepsilon, \varepsilon \rangle$ where $\varepsilon \in H$ is real valued. It is easy to show that

$$\frac{1}{2}\langle J_a''(u)\varepsilon, \varepsilon \rangle = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\varepsilon|^2 dx + \frac{1}{4}a^2 \int_{\mathbb{R}^3} |x|^2|\varepsilon|^2 dx + \frac{3}{2}(\lambda_1 - \frac{4}{3}\pi\lambda_2) \int_{\mathbb{R}^3} |u|^2|\varepsilon|^2 dx +$$

(11.3)
$$+ \frac{\lambda_2}{2}\int_{\mathbb{R}^3} \left(\tilde{K} \star |u|^2 \right) |\varepsilon|^2 dx + \lambda_2 \int_{\mathbb{R}^3} \left(\tilde{K} \star |u\varepsilon| \right) |u\varepsilon| dx + \frac{1}{2}\mu||\varepsilon||_2^2.$$

Using the fact that u solves (11.1) we then get

$$\frac{1}{2}\langle J_a''(u)u, u \rangle = (\lambda_1 - \frac{4}{3}\pi\lambda_2) \int_{\mathbb{R}^3} |u|^4 dx + \lambda_2 \int_{\mathbb{R}^3} \left(\tilde{K} \star |u|^2 \right) |u|^2 dx = B(u).$$

Now we claim that

$$\langle E_a''(u)\varepsilon, \varepsilon \rangle \geq c \left(\int_{\mathbb{R}^3} |\nabla\varepsilon|^2 dx + a^2 \int_{\mathbb{R}^3} |x|^2\varepsilon^2 dx \right).$$

The claim clearly implies that $\mu < 0$. To prove the claim we shall use the fact, established in Step 5 of the proof of Theorem 1.6, that $\int_{\mathbb{R}^3} |\nabla u|^2 dx \to 0$ when $a \to 0$. For simplicity we consider only the case $\lambda_2 > 0$ and $\lambda_1 - \frac{4}{3}\pi\lambda_2 < 0$, the other one is identical. It suffices to look at the functional

$$\tilde{E}_a(u) := \frac{1}{2}||\nabla u||_2^2 + \frac{a^2}{2}|||x|u||_2^2 + \frac{1}{2}(\lambda_1 - \frac{4}{3}\pi\lambda_2) \int_{\mathbb{R}^3} |u|^4 dx.$$

Now, by Hölder and Sobolev inequalities we have

$$\langle \tilde{E}_a''(u)\varepsilon, \varepsilon \rangle \geq \int_{\mathbb{R}^3} |\nabla\varepsilon|^2 dx + a \int_{\mathbb{R}^3} |x|^2|\varepsilon|^2 dx + 6(\lambda_1 - \frac{4}{3}\pi\lambda_2)S^2(\int_{\mathbb{R}^3} |\nabla\varepsilon|^2 dx)||u||_3^2$$

(S is the Sobolev best constant $||\varepsilon||_6 \leq S||\nabla\varepsilon||_2$) and this implies that

$$\langle E_a''(u)\varepsilon, \varepsilon \rangle \geq \left(\int_{\mathbb{R}^3} |\nabla\varepsilon_1|^2 dx \right)(1 + 6(\lambda_1 - \frac{4}{3}\pi\lambda_2)S^2||u||_3^2) + a \int_{\mathbb{R}^3} |x|^2|\varepsilon_1|^2 dx$$

Thus if

(11.4)
$$||u||_3 \leq \frac{1}{S\sqrt{6(-\lambda_1 + \frac{4}{3}\pi\lambda_2)}}$$

we obtain that $\langle \tilde{E}_a''(u)\varepsilon, \varepsilon \rangle \geq 0$. But (11.4) happens when $a \to 0$ thanks to Gagliardo-Nirenberg inequality

$$||u||_3^2 \leq C||u||_2||\nabla u||_2$$

and the fact that $\int_{\mathbb{R}^3} |\nabla u|^2 dx \to 0$ when $a \to 0$. $\qquad\square$

Lemma 11.1 (Asymptotics for μ). *Let $a > 0$ be sufficiently small and u be a topological local minimizer for the constrained energy, then the following holds*

(1) $\lim_{a\to 0} \mu = 0$;

(2) $\mu < -\frac{3}{2}a$ *if* $B(u) \geq 0$;

(3) $\mu < -3a\sqrt{\frac{1}{4} + \frac{3}{2}S^2(\lambda_1 - \frac{4}{3}\pi\lambda_2)||u||_3^2}$ *if* $B(u) > 0$ *and* $\lambda_2 > 0$;

(4) $\mu < -3a\sqrt{\frac{1}{4} + \frac{3}{2}S^2(\lambda_1 + \frac{8}{3}\pi\lambda_2)||u||_3^2}$ *if* $B(u) > 0$ *and* $\lambda_2 < 0$.

Proof. The fact that $\lim_{a\to 0}\mu = 0$ follows easily from the relations

$$E_a(u) - \frac{1}{3}Q_a(u) = \frac{1}{6}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{5}{6}a^2\int_{\mathbb{R}^3}|x|^2|u|^2 dx$$

and

$$E_a(u) = \mu||u||_2^2 + \frac{5}{3}a^2\int_{\mathbb{R}^3}|x|^2|u|^2 dx.$$

The proof of the last three points follows from Heisenberg uncertainty principle written in the following form

(11.5) $$||\nabla u||_2^2 + \omega^2|||x|u||_2^2 - 3\omega||u||_2^2 \geq 0 \quad \forall u \in \Sigma, \omega > 0$$

Case $B(u) \geq 0$:
The fact that $\mu < -\frac{3}{2}a$ follows from (11.5) since u solves (11.1).
Case $B(u) < 0$ and $\lambda_2 > 0$:
Here we use (11.3). Using the fact that $\lambda_2 > 0$ we get

(11.6) $$0 > \frac{1}{2}\langle J_a''(u)u, u \rangle > \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{1}{4}a^2\int_{\mathbb{R}^3}|x|^2|u|^2 dx +$$
$$+ \frac{3}{2}(\lambda_1 - \frac{4}{3}\pi\lambda_2)\int_{\mathbb{R}^3}|u|^4 dx + \frac{1}{2}\mu||u||_2^2.$$

Now arguing as in the proof of Theorem 1.8 we obtain

$$0 > (\frac{1}{4} + \frac{3}{2}S^2(\lambda_1 - \frac{4}{3}\pi\lambda_2)||u||_3^2)\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{1}{4}a^2\int_{\mathbb{R}^3}|x|^2|u|^2 dx + \frac{1}{2}\mu||u||_2^2.$$

Calling $\beta = (\frac{1}{4} + \frac{3}{2}S^2(\lambda_1 - \frac{4}{3}\pi\lambda_2)||u||_3^2)$ we have from Heisenberg uncertainty principle (11.5)

$$\beta\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{1}{4\beta}a^2\int_{\mathbb{R}^3}|x|^2|u|^2 dx + \frac{1}{2\beta}\mu||u||_2^2\right) \geq \beta\left(\frac{3a}{2\sqrt{\beta}} + \frac{\mu}{2\beta}\right)||u||_2^2.$$

Remembering that $\beta > 0$ for $a > 0$ small we obtain the required estimate.
Case $B(u) < 0$ and $\lambda_2 < 0$:
This case is identical to the previous one just observing that we can write

$$B(u) = (\lambda_1 + \frac{8}{3}\pi\lambda_2)||u||_4^4 + \lambda_2\int_{\mathbb{R}^3}\left(K_1 \star |u|^2\right)|u|^2 dx$$

where $\hat{K_1} = -4\pi\frac{(\xi_1^2 + \xi_2^2)}{|\xi|^2}$. $\qquad\square$

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Proof of Theorem 1.9. We consider just the case $\lambda_2 > 0$. Since $\hat{K}(\xi) = \frac{4}{3}\pi\left(\frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2}\right) \in [-\frac{4}{3}\pi, \frac{8}{3}\pi]$ writing $\lambda_1 = \lambda_1' + \frac{4}{3}\pi\lambda_2$ we have, when $\lambda_1' \leq 0$,

$$(11.7) \qquad \lambda_1' \leq \lambda_1 + \lambda_2\hat{K}(\xi) \leq 4\pi\lambda_2.$$

Thus

$$
\begin{aligned}
Q(u) &\geq A(u) + \frac{3}{2}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(\lambda_1 - \frac{4}{3}\pi\lambda_2)|\hat{u}^2|^2 d\xi \geq A(u) + C\lambda_1'||u||_4^4 \\
&\geq A(u) + \lambda_1' C A(u)^{3/2}||u||_2.
\end{aligned}
$$

In particular, for any $k > 0$, taking $\lambda_1' < 0$ sufficiently close to 0 it follows that $Q(u) > 0$ on A_k. Recording that $Q(u) = 0$ for any critical point this proves Point (1). To prove Point (2) first observe that,

$$(11.8) \qquad E(u) \leq \frac{A(u)}{2} + 4\pi\lambda_2||u||^4 \leq \frac{A(u)}{2} + \lambda_2 C A(u)^{3/2}||u||_2$$

and thus for any $k > 0$,

$$(11.9) \qquad \sup_{u \in A_k} E(u) \quad \text{does not depend on } \lambda_1'.$$

Also from (11.7) we have that

$$(11.10) \qquad E(u) \geq \frac{A(u)}{2} + \frac{\lambda_1'}{2}||u||^4 \geq \frac{A(u)}{2} + \lambda_1' C A(u)^{3/2}||u||_2$$

and then a direct calculation shows that

$$(11.11) \qquad \sup_{k>0}\inf_{u \in C_k} E(u) \to +\infty$$

as $\lambda_1' \to 0^-$. Now fix a $v \in S(c)$ with $A(v) = 1$. We have

$$E_a(v) \leq \sup_{u \in A_1} E(u) + \frac{a^2}{2}D(v).$$

From (11.9) and (11.11) we deduce that, for any $a > 0$,

$$E_a(v) < \sup_{k>0}\inf_{u \in C_k} E(u)$$

if $|\lambda_1'|$ is sufficiently small. Arguing as in Step 1 of the Proof of Theorem 1.6 this proves Point (2). $\qquad\square$

12. Appendix

In Step 3 of the proof of Theorem 1.6 we have assumed that $E_a(u)$ constrained to $S(c)$ possesses a bounded Palais-Smale sequence at the level $\gamma_a(c)$. We now prove that it is indeed the case and we also derive additional properties of this sequence.

Lemma 12.1. *For any fixed $c > 0$ and any $a \in (0, a_0]$ there exists a sequence $(u_n) \subset S(c)$ and a sequence $(v_n) \subset \Sigma$ of real, non negative functions such that*

(12.1)
$$\begin{cases} E_a(u_n) \to \gamma_c(c) > 0, \\ \|E'_a|_{S(c)}(u_n)\|_{\Sigma^{-1}} \to 0, \\ Q_a(u_n) \to 0, \\ \|u_n - v_n\|_\Sigma \to 0, \end{cases}$$

as $n \to \infty$. Here Σ^{-1} denotes the dual space of Σ.

From the definition of $E_a(u)$ and $Q_a(u)$ we get that

$$E_a(u) - \frac{1}{3}Q_a(u) = \frac{1}{6}A(u) + \frac{5}{6}a^2 D(u)$$

and thus we immediately deduce that the Palais-Smale sequence given by Lemma 12.1 is bounded. Note also that since $\|u_n - v_n\|_\Sigma \to 0$, if we manage to show that $u_n \to u$ strongly in Σ then the limit $u \in S(c)$ will be a real, non negative, function.

Roughly speaking what Lemma 12.1 says is that it is possible to incorporate into the variational problem the information that any critical point must satisfy the constraint $Q_a(u) = 0$. For previous works in that direction we refer to [19, 20]. Clearly it is possible to prove Step 3 of Theorem 1.1 by this approach, but here we have choosen to use the approach of [15] for its simplicity. The proof of the lemma is inspired from [20, Lemma 3.5]. Before proving Lemma 12.1 we need to introduce some notations and to prove some preliminary results. For any fixed $\mu > 0$, we introduce the auxiliary functional

$$\widetilde{E}_a : S(c) \times \mathbb{R} \to \mathbb{R}, \qquad (u, s) \mapsto E_a(H(u, s)),$$

where $H(u, s)(x) := e^{\frac{N}{2}s} u(e^s x)$, and the set of paths

$$\widetilde{\Gamma}_a(c) := \left\{ \widetilde{g} \in C([0, 1], S(c) \times \mathbb{R}) : \ \widetilde{g}(0) = (v^{t_1}, 0), \ \widetilde{g}(1) = (v^{t_2}, 0) \right\},$$

where $v^{t_1}, v^{t_2} \in S(c)$ are defined in the proof of Theorem 1.6 (remember that they are real non negative). Observe that setting

$$\widetilde{\gamma}_a(c) := \inf_{\widetilde{g} \in \widetilde{\Gamma}_a(c)} \max_{t \in [0,1]} \widetilde{E}_a(\widetilde{g}(t)),$$

we have that

(12.2)
$$\widetilde{\gamma}_a(c) = \gamma_a(c).$$

Indeed, by the definitions of $\widetilde{\gamma}_a(c)$ and $\gamma_a(c)$, (12.2) follows immediately from the observation that the maps

$$\varphi : \Gamma_a(c) \longrightarrow \widetilde{\Gamma}_a(c), \ g \longmapsto \varphi(g) := (g, 0),$$

and

$$\psi : \widetilde{\Gamma}_a(c) \longrightarrow \Gamma_a(c), \ \widetilde{g} \longmapsto \psi(\widetilde{g}) := H \circ \widetilde{g},$$

satisfy

$$\widetilde{E}_a(\varphi(g)) = E_a(g) \ \text{ and } \ E_a(\psi(\widetilde{g})) = \widetilde{E}_a(\widetilde{g}).$$

In the proof of Lemma 12.1, the lemma below which has been established by the Ekeland variational principle in [19, Lemma 2.3] is used. Hereinafter we denote by X the set $\Sigma \times \mathbb{R}$ equipped with the norm $\| \cdot \|_X^2 = \| \cdot \|_\Sigma^2 + | \cdot |_\mathbb{R}^2$ and denote by X^{-1} its dual space.

Lemma 12.2. *Let $\varepsilon > 0$. Suppose that $\widetilde{g}_0 \in \widetilde{\Gamma}_a(c)$ satisfies*

$$\max_{t\in[0,1]} \widetilde{E}_a(\widetilde{g}_0(t)) \le \widetilde{\gamma}_a(c) + \varepsilon.$$

Then there exists a pair of $(u_0, s_0) \in S(c) \times \mathbb{R}$ such that:

(1) $\widetilde{E}_a(u_0, s_0) \in [\widetilde{\gamma}_a(c) - \varepsilon, \widetilde{\gamma}_a(c) + \varepsilon]$;

(2) $\min_{t\in[0,1]} \|(u_0, s_0) - \widetilde{g}_0(t)\|_X \le \sqrt{\varepsilon}$;

(3) $\|\widetilde{E}'_a|_{S(c)\times\mathbb{R}}(u_0, s_0)\|_{X^{-1}} \le 2\sqrt{\varepsilon}$, *i.e.*

$$|\langle \widetilde{E}'_a(u_0, s_0), z\rangle_{X^{-1}\times X}| \le 2\sqrt{\varepsilon}\,\|z\|_X,$$

holds for all $z \in \widetilde{T}_{(u_0,s_0)} := \{(z_1, z_2) \in X, \langle u_0, z_1\rangle_2 = 0\}$.

Proof of Lemma 12.1. For each $n \in \mathbb{N}$, by the definition of $\gamma_a(c)$, there exists a $g_n \in \Gamma_a(c)$ such that

$$\max_{t\in[0,1]} E_a(g_n(t)) \le \gamma_a(c) + \frac{1}{n}.$$

Observe that $|g_n| \in \Gamma_a(c)$ and because $E_a(|u|) \le E_a(u)$ for all $\in \Sigma$ we have

$$\max_{t\in[0,1]} E_a(|g_n(t)|) \le \max_{t\in[0,1]} E_a(g_n(t)).$$

Since $\widetilde{\gamma}_a(c) = \gamma_a(c)$, then for each $n \in \mathbb{N}$, $\widetilde{g}_n := (|g_n|, 0) \in \widetilde{\Gamma}_a(c)$ satisfies

$$\max_{t\in[0,1]} \widetilde{E}_a(\widetilde{g}_n(t)) \le \widetilde{\gamma}_a(c) + \frac{1}{n}.$$

Thus applying Lemma 12.2, we obtain a sequence $\{(w_n, s_n)\} \subset S(c) \times \mathbb{R}$ such that:

(i) $\widetilde{E}_a(w_n, s_n) \in [\gamma_a(c) - \frac{1}{n}, \gamma_a(c) + \frac{1}{n}]$;

(ii) $\min_{t\in[0,1]} \|(w_n, s_n) - (|g_n(t)|, 0)\|_X \le \frac{1}{\sqrt{n}}$;

(iii) $\|\widetilde{E}'_a|_{S_{(c)}\times\mathbb{R}}(w_n, s_n)\|_{X^{-1}} \le \frac{2}{\sqrt{n}}$.

For each $n \in \mathbb{N}$, let $t_n \in [0,1]$ be such that the minimum in (ii) is reached. We claim that setting $u_n := H(w_n, s_n)$ and $v_n := |g_n(t_n)|$ the corresponding sequences satisfy (12.1). Indeed, first, from (i) we have that $E_a(u_n) \to \gamma_a(c)$, since $E_a(u_n) = E_a(H(w_n, s_n)) = \widetilde{E}_a(w_n, s_n)$. Secondly, by simple calculations, we have that

$$Q_a(u_n) = \langle \widetilde{E}'_a(w_n, s_n), (0, 1)\rangle_{X^{-1}\times X},$$

and $(0, 1) \in \widetilde{T}_{(w_n, s_n)}$. Thus (iii) yields that $Q_a(u_n) \to 0$. To verify that $\|E'_a|_{S(c)}(u_n)\|_{\Sigma^{-1}} \to 0$, it suffices to prove, for $n \in \mathbb{N}$ sufficiently large, that

$$(12.3) \qquad |\langle E'_a(u_n), \phi\rangle_{\Sigma^{-1}\times\Sigma}| \le \frac{4}{\sqrt{n}}\,\|\phi\|_\Sigma, \quad \text{for all } \phi \in T_{u_n},$$

where $T_{u_n} := \{\phi \in \Sigma, \langle u_n, \phi\rangle_2 = 0\}$. To this end, we note that, for each $\phi \in T_{u_n}$, setting $\widetilde{\phi} = H(\phi, -s_n)$, one has by direct calculations that

$$\langle E'_a(u_n), \phi\rangle_{\Sigma^*\times\Sigma} = \langle \widetilde{E}'_a(w_n, s_n), (\widetilde{\phi}, 0)\rangle_{X^{-1}\times X}.$$

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If $(\widetilde{\phi}, 0) \in \widetilde{T}_{(w_n, s_n)}$ and $\|(\widetilde{\phi}, 0)\|_X^2 \leq 4\|\phi\|_\Sigma^2$ for $n \in \mathbb{N}$ sufficiently large, then from (iii) we conclude that (12.3) holds. To check this claim one may observe that $(\widetilde{\phi}, 0) \in \widetilde{T}_{(w_n, s_n)} \Leftrightarrow \phi \in T_{u_n}$, and that from (ii) we have

$$(12.4) \qquad |s_n| = |s_n - 0| \leq \min_{t \in [0,1]} \|(w_n, s_n) - (|g_n(t)|, 0)\|_X \leq \frac{1}{\sqrt{n}},$$

by which we deduce that

$$\begin{aligned}
\|(\widetilde{\phi}, 0)\|_X^2 &= \|\widetilde{\phi}\|_\Sigma^2 = \||\phi\|_2^2 + e^{-2s_n}\|\nabla\phi\|_2^2 + e^{2s_n}\||x|\phi\|_2^2 \\
&\leq 2 \|\phi\|_\Sigma^2,
\end{aligned}$$

holds for $n \in \mathbb{N}$ large enough. Thus (12.3) has been proved. Finally, since $\|(w_n, s_n) - (v_n, 0)\|_X \to 0$ we have in particular that $\|w_n - v_n\|_\Sigma \to 0$. Thus from (12.4) and since

$$\|u_n - v_n\|_\Sigma = \|H(w_n, s_n) - v_n\|_\Sigma \leq \|H(w_n, s_n) - w_n\|_\Sigma + \|w_n - v_n\|_\Sigma,$$

we conclude that $\|u_n - v_n\|_\Sigma \to 0$ as $n \to \infty$. At this point, the proof of the lemma is complete. $\qquad\square$

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JACOPO BELLAZZINI
UNIVERSITÀ DI SASSARI
VIA PIANDANNA 4, 07100 SASSARI, ITALY
E-mail address: jbellazzini@uniss.it

LOUIS JEANJEAN
LABORATOIRE DE MATHÉMATIQUES (UMR 6623)
UNIVERSITÉ DE FRANCHE-COMTÉ
16, ROUTE DE GRAY 25030 BESANÇON CEDEX, FRANCE
E-mail address: louis.jeanjean@univ-fcomte.fr