THE COARSE GEOMETRY OF TSIRELSON'S SPACE AND APPLICATIONS

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ABSTRACT. The main result of this article is a rigidity result pertaining to the spreading model structure for Banach spaces coarsely embeddable into Tsirelson's original space T^*. Every Banach space that is coarsely embeddable into T^* must be reflexive and all its spreading models must be isomorphic to c_0. Several important consequences follow from our rigidity result. We obtain a coarse version of an influential theorem of Tsirelson: T^* does not coarsely contain c_0 nor ℓ_p for $p \in [1, \infty)$. We show that there is no infinite dimensional Banach space that coarsely embeds into every infinite dimensional Banach space. In particular, we disprove the conjecture that the separable infinite dimensional Hilbert space coarsely embeds into every infinite dimensional Banach space. The rigidity result follows from a new concentration inequality for Lipschitz maps on the infinite Hamming graphs and taking values in T^*, and from the embeddability of the infinite Hamming graphs into Banach spaces that admit spreading models not isomorphic to c_0. Also, a purely metric characterization of finite dimensionality is obtained.

CONTENTS

1. INTRODUCTION

1.1. **Motivation.** A natural and powerful approach to classify mathematical objects in a given category, is to discover properties that are invariant with respect to the isomorphisms of the category. The notion of topological dimension, which allows us to distinguish between $\mathbb{R}^2$ and $\mathbb{R}^3$ as topological spaces, is a prime example. These invariants can take various forms, and in metric geometry one usually seeks for properties that are stable under embeddings of a certain type, e.g. Lipschitz, uniform, or coarse. In this article we are concerned with a coarse invariant in the form of a concentration inequality for Lipschitz maps defined on infinite graphs with values into certain infinite dimensional Banach spaces.

Discovering useful and tractable new concentration inequalities is usually a challenge. In the Lipschitz category, several fundamental inequalities have been discovered, partially motivated by the investigation of the Ribe program and its applications in geometric group theory or theoretical computer science. We refer to the survey of A. Naor [25] for an enlightening account of this theory, and to the recent work of Naor and Schechtman ([28], [27] and [26]) where fundamental and powerful new inequalities were proved.

Strong interest in the coarse geometry of infinite dimensional Banach spaces arose in connection with the Novikov conjecture in Topology and a coarse version of the Baum-Connes conjecture in Noncommutative Geometry (see [30], [38], [13] and [36]). A very efficient and powerful technique for proving the coarse geometric Novikov conjecture for a bounded geometry metric space is to coarsely embed this space into a Banach space with some strong enough convexity and smoothness properties. Indeed, G. Yu [39] showed that a discrete metric space with bounded geometry that is coarsely embeddable into a Hilbert space satisfies the coarse geometric Novikov conjecture. This result was later generalized by Kasparov and Yu [23] who proved that the conclusion holds if the space is merely coarsely embeddable into a super-reflexive Banach space. It is worth mentioning that the coarse geometric Novikov conjecture implies Gromov's conjecture that a uniformly contractible Riemannian manifold with bounded geometry cannot have uniformly positive scalar curvature, and the zero-in-the spectrum conjecture stating that the Laplace operator acting on the space of all L^2-forms of a uniformly contractible Riemannian manifold has zero in its spectrum. Consequently, whether the separable Hilbert space is the Banach space into which it is the hardest to embed, became a very natural and intriguing question. More precisely, the following problem was raised (Problem 14 in [15], Problem 11.17 in [34]).

Main Problem. *Does ℓ_2 coarsely embed into every infinite dimensional Banach space?*

There is rather strong evidence that ℓ_2 is the space into which it is the most difficult to embed. It was shown in [33] that every locally finite metric subset of ℓ_2 admits a bi-Lipschitz embedding into every infinite dimensional Banach space, and in [6] that every proper subset of ℓ_2 (i.e. whose closed balls are compact) is almost Lipschitz embeddable into every infinite dimensional Banach space. Both proofs use Dvoretzky's theorem [11] and the

barycentric gluing technique introduced in [3]. Moreover, using Gaussian random variables one can embed ℓ_2 linearly isometrically into $L_p[0,1]$ for all $p \in [1,\infty)$. Nowak [29] proved that ℓ_2 coarsely embeds into ℓ_p for all $p \in [1,\infty)$ using Mazur maps. This technique was extended by Ostrovskii [33] to embed ℓ_2 coarsely into every Banach space with an unconditional basis and non-trivial cotype using the Mazur maps constructed by Odell and Schlumprecht in their solution to the distortion problem [31]. There are also Banach spaces with trivial cotype such as $(\sum_{n=1}^{\infty} \ell_\infty^n)_{\ell_p}$, or c_0 (which is a universal space for separable spaces and bi-Lipschitz embeddings [1]) that contain coarsely ℓ_2.

On the other hand, useful coarse invariants, such as asymptotic dimension or finite decomposition complexity, are not well suited to study the coarse geometry of infinite dimensional Banach spaces, and the number of properties at our disposal to prevent coarse embeddability between infinite dimensional Banach spaces is rather limited. There are essentially three obstructions. The first obstruction was discovered by Johnson and Randrianarivony [18] when they showed that ℓ_q does not coarsely embed into ℓ_2 when $q > 2$. Their approach relies heavily upon the fact that the target space is ℓ_2. The second obstruction is the metric cotype of Mendel and Naor [24]. For instance, the metric cotype argument can be used to rule out the coarse embeddability of ℓ_q into ℓ_p when $q > \max\{p,2\}$. The last one is Kalton's property $\mathcal{Q}$ [19] which serves as an obstruction to coarse embeddability into reflexive (or stable) Banach spaces[1]. Since ℓ_2 has the smallest possible cotype allowed for a Banach space, and is both reflexive and stable, none of the above obstructions can provide a negative answer to the main problem.

In this article the main problem is answered negatively, using the following approach. We show that there exists an infinite dimensional Banach space in which the sequence $(H_k^\omega)_{k \geq 1}$ of k-dimensional Hamming graphs (over a countable set) does not equi-coarsely embed. Since it is easy, and well known, that $(H_k^\omega)_{k \geq 1}$ equi-coarsely embeds into ℓ_2, the claim follows. Our work is inspired by a concentration inequality of Kalton and Randrianarivony [21]. They proved the following: Assume that f is a Lipschitz map from H_k^ω into a reflexive Banach space Y, for which there exists $p \in (1,\infty)$, such that for any y in Y and any weakly null sequence $(y_n)_{n=1}^\infty$ in Y we have

$$(1.1) \qquad \limsup_{n \to \infty} \|y + y_n\|_Y^p \leq \|y\|_Y^p + \limsup_{n \to \infty} \|y_n\|_Y^p.$$

Then for every two vertices $\bar{m}, \bar{n}$ in some subgraph isometric to H_k^ω, one has

$$(1.2) \qquad \|f(\bar{m}) - f(\bar{n})\|_Y \leq 3\mathrm{Lip}(f)k^{1/p}.$$

Since $\mathrm{diam}(H_k^\omega) = k$, inequality (1.2) says that a Lipschitz map on H_k^ω concentrates on a "full" subgraph. This concentration inequality was originally used by Kalton and Randrianarivony as an obstruction to coarse Lipschitz embeddability, but also provides information on compression exponents (cf. [4] and [9]). Reflexivity is crucial to obtain inequality (1.2). Indeed, since c_0 is Lipschitz universal, inequality (1.2) cannot hold for Lipschitz maps with

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[1]A variant of Kalton's property $\mathcal{Q}$ can be useful in the non-separable setting [20].

values in c_0, even though c_0 satisfies inequality (1.1) for every $p \in (1, \infty)$. In order to prevent coarse embeddability one would like to have for some $C \in (0, \infty)$, which does not depend on k, the stronger inequality

$$(1.3) \qquad \|f(\bar{m}) - f(\bar{n})\| \leq C\mathrm{Lip}(f).$$

If we could find a reflexive Banach space Y satisfying inequality (1.2) for $p = \infty$ that is,

$$(1.4) \qquad \limsup_{n\to\infty} \|y + y_n\| \leq \max\big\{\|y\|, \limsup_{n\to\infty} \|y_n\|\big\},$$

then the concentration inequality (1.3) would hold for Y. Unfortunately, if a Banach space Y satisfies inequality (1.4) then Y must contain an isomorphic copy of c_0 (see [22]). Thus it cannot be reflexive and moreover it contains a bi-Lipschitz copy of every separable metric space, which precludes any concentration inequality. As we shall see, this transition phase phenomenon is not necessarily an obstruction. There are actually Banach spaces that have a "c_0-like" asymptotic smoothness property which is strong enough to obtain the concentration inequality (1.3) but weak enough to coexist with reflexivity. A prime example of such space is Tsirelson's original space.

Tsirelson's original space T^* was the first example of a Banach space that does not contain any isomorphic copies of ℓ_p or c_0 [37]. The validity of the concentration inequality (1.3) for T^* is pivotal to prove an unexpected rigidity result which in turn implies that coarse and uniform versions of Tsirelson's theorem hold.

1.2. **Main results.** All unexplained notation can be found in Section 2. The main result of this article is the following rigidity result pertaining to the spreading model structure of Banach spaces coarsely embeddable into Tsirelson's original space T^*.

Theorem A. *Let X be an infinite dimensional Banach space. If X coarsely embeds into T^*, or if its unit ball B_X uniformly embeds into T^*, then X is reflexive and all its spreading models are isomorphic to c_0.*

Our rigidity result shows that Tsirelson's construction is actually extremely robust in the sense that nonlinear versions of Tsirelson's theorem hold. Indeed, for $p \in [1, \infty)$ all the spreading models of ℓ_p are linearly isometric to ℓ_p, and c_0 does not coarsely embed into a reflexive space (by a result of Kalton [19]), and thus Corollary B below is an immediate consequence of Theorem A.

Corollary B. c_0 *and ℓ_p for $p \in [1, \infty)$ (resp. B_{c_0} and B_{ℓ_p}) do not coarsely embed (resp. uniformly embed) into T^*.*

We will say that a Banach space is *coarsely minimal* if it coarsely embeds into every infinite dimensional Banach space. The Main Problem asks whether ℓ_2 is coarsely minimal. In fact, Theorem A provides a much stronger negative solution to the Main Problem. Indeed, a coarsely minimal Banach space embeds into ℓ_2 and it must have non-trivial cotype using Mendel and Naor metric cotype notion [24], but by Theorem A it must also have trivial cotype, a contradiction.

Corollary C. *There is no coarsely minimal infinite dimensional Banach space.*

The proof of Theorem A relies essentially on two ingredients. The main ingredient, which is presented in Section 4, is a new concentration inequality for Lipschitz maps on the k-dimensional infinite Hamming graph and taking values into Tsirelson's original space T^*.

Theorem D. *Let $k \in \mathbb{N}$ and $f \colon ([\mathbb{N}]^k, d_H) \to T^*$ be a Lipschitz map. Then there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that for all $\bar{m}, \bar{n} \in [\mathbb{M}]^k$ one has*

$$(1.5) \qquad \|f(\bar{m}) - f(\bar{n})\| \leq 5\mathrm{Lip}(f).$$

The concentration inequality above is strong enough to rule out coarse and uniform embeddings, and can already be invoked to provide a negative solution to the Main Problem without the full strength of Theorem A.

The other ingredient, presented in Section 3, uses the structure of spreading models of a Banach space to prove positive embeddability results. Such results are obtained for three different metrics: the tree metric, the Hamming metric and the symmetric difference metric. Our embeddings are based on the existence of spreading models that are isomorphic, or not isomorphic, to c_0.

The proof of Theorem A, given in Section 4, is obtained by analyzing the tension between the concentration inequality for the Hamming graphs and the embeddability of the Hamming graphs into spaces that do not have spreading models isomorphic to c_0. For readers interested mostly in the geometry of Hilbert space we included in Remark 4.6 a short proof (which avoids the machinery of spreading models and only requires Theorem D) of the ℓ_2-case of Corollary B.

The analysis of the spreading model structure of infinite dimensional Banach spaces in Section 3, also lead us to a metric characterization of finite dimensionality in terms of equi-coarse embeddability of the sequence of countably branching trees of finite but arbitrarily large height, denoted by $(T_k^\omega)_{k \geq 1}$.

Theorem E. *For a Banach space Y the following assertions are equivalent:*
  (1) Y is finite-dimensional.
  (2) $(T_k^\omega)_{k \geq 1}$ does not equi-coarsely embed into Y.

We mention a few related open problems in the last section.

2. Preliminaries

2.1. **Nonlinear embeddings.** Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$. One defines

$$\rho_f(t) = \inf \big\{ d_Y(f(x), f(y)) : d_X(x, y) \geq t \big\},$$

and

$$\omega_f(t) = \sup\{ d_Y(f(x), f(y)) : d_X(x, y) \leq t \}.$$

Note that for every $x, y \in X$,

$$(2.1) \qquad \rho_f(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \omega_f(d_X(x, y)).$$

The moduli ρ_f and ω_f will be called the *compression modulus* and the *expansion modulus* of the embedding, respectively. For unbounded metric spaces, the map f is a *coarse embedding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$. When the metric spaces are not uniformly discrete a map $f \colon X \to Y$ is said to be a *uniform embedding* if $\lim_{t\to 0} \omega_f(t) = 0$ and $\rho_f(t) > 0$ for all $t > 0$, i.e. f is an injective uniformly continuous map whose inverse is uniformly continuous.

If one is given a family of metric spaces $(X_i)_{i\in I}$, one says that $(X_i)_{i\in I}$ *equi-coarsely (resp. equi-uniformly)* embeds into Y if there exist non-decreasing functions $\rho, \omega \colon [0,\infty) \to [0,\infty)$ and for all $i \in I$, maps $f_i \colon X_i \to Y$ such that $\rho \le \rho_{f_i} \le \omega_{f_i} \le \omega$, and $\lim_{t\to\infty} \rho(t) = \infty$ and $\omega(t) < \infty$ for all $t > 0$ (resp. $\lim_{t\to 0} \omega(t) = 0$ and $\rho(t) > 0$ for all $t > 0$).

2.2. Tree, symmetric difference and Hamming metrics.

In this section we define the metric spaces which will be needed and studied in the sequel. For any infinite subset $\mathbb{M}$ of $\mathbb{N}$, let $[\mathbb{M}]^\omega := \{A \subset \mathbb{M} : A \text{ infinite}\}$ and $[\mathbb{M}]^{<\omega} := \{A \subset \mathbb{M} : A \text{ finite}\}$. For $k \in \mathbb{N}$, we put $[\mathbb{M}]^{\le k} := \{A \subset \mathbb{M} : |A| \le k\}$ and $[\mathbb{M}]^k := \{A \subset \mathbb{M} : |A| = k\}$ where $|A|$ denotes the cardinality of the set A. Elements of $[\mathbb{M}]^\omega$ and $[\mathbb{M}]^{<\omega}$ will always be listed in an increasing order, *i.e.*, if we write $\bar{m} = \{m_1, m_2, \ldots\} \in [\mathbb{M}]^\omega$, or $\bar{m} = \{m_1, m_2, \ldots, m_l\} \in [\mathbb{M}]^{<\omega}$ we assume that $m_1 < m_2 < \ldots$ or $m_1 < m_2 < \ldots < m_l$, respectively. We define three metrics, d_T, $d_\triangle$ and d_H on $[\mathbb{N}]^{<\omega}$. The restrictions of these metrics to the sets $[\mathbb{M}]^{\le k}$, $[\mathbb{M}]^k$ for some $\mathbb{M} \in [\mathbb{N}]^\omega$ and some $k \in \mathbb{N}$, will still be denoted d_T, $d_\triangle$ and d_H.

The tree metric. For $\bar{m}, \bar{n} \in [\mathbb{N}]^{<\omega}$, $\bar{m} \ne \bar{n}$, let

$$(2.2) \qquad d_T(\bar{m}, \bar{n}) := |\bar{m}| + |\bar{n}| - 2|\bar{m} \wedge \bar{n}|,$$

where $\bar{m} \wedge \bar{n} := \{m_1, m_2, \ldots, m_l\}$ if $m_1 = n_1, m_2 = n_2, \ldots, m_l = n_l$ and $m_{l+1} \ne n_{l+1}$.

We denote by $\preceq_T$ the partial order of *extension* on $[\mathbb{N}]^{<\omega}$, *i.e.* $\bar{m} \preceq_T \bar{n}$ if $|\bar{m}| \le |\bar{n}|$ and for all $i \le |\bar{m}|$ one has $m_i = n_i$. One can then define a graph structure on $[\mathbb{N}]^{<\omega}$, which is our set of vertices, by declaring two vertices $\bar{m}, \bar{n}$ to be adjacent if $\bar{m}$ is the immediate predecessor of $\bar{n}$ or $\bar{n}$ is the immediate predecessor of $\bar{m}$. The graph obtained is the countably branching tree of infinite height, denoted $T_\infty^\omega(\mathbb{N})$ or simply T_∞^ω. The countably branching tree of height k, $[\mathbb{N}]^{\le k}$, will also be denoted T_k^ω. It is easy to see that the tree metric coincides with the graph metric.

The symmetric difference metric. For $\bar{m}, \bar{n} \in [\mathbb{N}]^{<\omega}$, $\bar{m} \ne \bar{n}$, let

$$(2.3) \qquad d_\triangle(\bar{m}, \bar{n}) := |\bar{m} \triangle \bar{n}|,$$

where $\bar{m} \triangle \bar{n} := (\bar{m} \setminus \bar{n}) \cup (\bar{n} \setminus \bar{m})$ is the symmetric difference. At some point, it will be useful for us to notice that one can define a graph structure on $[\mathbb{N}]^k$ by defining an adjacency relation as follows: two vertices $\bar{m}, \bar{n} \in [\mathbb{N}]^k$ are adjacent if and only if $|\bar{m} \setminus \bar{n}| = |\bar{n} \setminus \bar{m}| = 1$. This graph is the k-dimensional Johnson graph over a countable set, and will be denoted $J_k^\omega(\mathbb{N})$ or simply J_k^ω, and its graph metric d_J coincides with the metric $\frac{d_\triangle}{2}$.

The Hamming metric. For $\bar{m}, \bar{n} \in [\mathbb{N}]^{<\omega}$, let

$$(2.4) \qquad d_H(\bar{m}, \bar{n}) := \big|\{i \in \{1, 2, \ldots, \min(|\bar{m}|, |\bar{n}|)\} : m_i \neq n_i\}\big|$$
$$+ \max(|\bar{m}|, |\bar{n}|) - \min(|\bar{m}|, |\bar{n}|).$$

When restricted to the set $[\mathbb{N}]^k$, the metric d_H, can be seen as the graph metric on the Hamming graph over a countable alphabet, denoted $H_k^\omega(\mathbb{N})$ or simply H_k^ω, where two vertices are adjacent if they differ in exactly one coordinate.

Remark 2.1. Both graphs H_k^ω and J_k^ω are infinite versions of finite graphs that arose from the fundamental work from the mid-20th century in Coding Theory of R. W. Hamming and S. M. Johnson, respectively.

2.3. **Spreading models.** For the convenience of the reader we briefly recall a few useful results from the theory of spreading models initiated by Brunel and Sucheston [10] in the 70's. We shall follow the exposition in [8]. An application of Ramsey's Theorem yields that every bounded sequence $(x_n)_{n=1}^\infty$ in a separable Banach space admits a subsequence $(y_n)_{n=1}^\infty$ such that for all $k \geq 1$, $(a_i)_{i=1}^k \subset \mathbb{R}$,

$$N(a_1, a_2, \ldots, a_k) := \lim_{n_1 < n_2 < \cdots < n_k \to \infty} \|a_1 y_{n_1} + a_2 y_{n_2} + \cdots + a_k y_{n_k}\|$$

exists, where the limit means that for all $\varepsilon > 0$, there is $r \in \mathbb{N}$ such that for all $r \leq n_1 < n_2 < \cdots < n_k$,

$$(2.5) \qquad \Big| \|a_1 y_{n_1} + a_2 y_{n_2} + \cdots + a_k y_{n_k}\| - N(a_1, a_2, \cdots, a_k) \Big| < \varepsilon.$$

It is easy to see that that if $(e_i)_{i=1}^\infty$ denotes the canonical basis of c_{00}, the vector space of sequences in $\mathbb{R}$ which eventually vanish, the formula $\|a_1 e_1 + a_2 e_2 + \cdots + a_k e_k\|_E := N(a_1, a_2, \cdots, a_k)$ defines a semi-norm, and that $(e_i)_{i=1}^\infty$ is a spreading sequence in the sense that for all $k \geq 1$, $(a_i)_{i=1}^k \subset \mathbb{R}$, and integers $n_1 < n_2 < \cdots < n_k$,

$$(2.6) \qquad \|a_1 e_{n_1} + a_2 e_{n_2} + \cdots + a_k e_{n_k}\|_E = \|a_1 e_1 + a_2 e_2 + \cdots + a_k e_k\|_E.$$

If $(x_n)_{n=1}^\infty$ does not have a converging subsequence it is simple to show that the semi-norm is actually a norm. The completion of c_{00} for the norm $\|\cdot\|_E$ is a Banach space E called a *spreading model of X generated by the sequence* $(x_n)_{n=1}^\infty$, and we will refer to the sequence $(e_i)_{i=1}^\infty$ as the *fundamental sequence* of the spreading model. The fundamental sequence $(e_i)_{i=1}^\infty$ is not necessarily a basis of E but if the generating sequence $(x_n)_{n=1}^\infty$ is normalized and weakly null then $(e_i)_{i=1}^\infty$ is a normalized 1-suppression unconditional basis ([8, Proposition 1, p. 24]). We call a basic sequence $(z_j)_{j=1}^\infty$ in a Banach space Z *c-suppression unconditional*, for some $c \geq 1$, if for all $(a_i)_{i=1}^\infty \in c_{00}$ and all $A \subset \mathbb{N}$

$$(2.7) \qquad \Big\| \sum_{i \in A} a_i z_i \Big\| \leq c \Big\| \sum_{i=1}^\infty a_i z_i \Big\|,$$

and in this case we denote by c_s the smallest number c satisfying (2.7). The basic sequence $(z_j)_{j=1}^\infty$ is called *c-unconditional* if for all $(a_i)_{i=1}^\infty \in c_{00}$ and

all $(\varepsilon_i)_{i=1}^\infty \subset \{-1,1\}$ we have

$$(2.8) \qquad \Big\| \sum_{i \in A} \varepsilon_i a_i z_i \Big\| \le c \Big\| \sum_{i=1}^\infty a_i z_i \Big\|,$$

and we denote in this case the smallest number c satisfying (2.8) by c_u. It is easy to see that basic sequences are suppression unconditional if and only if they are unconditional, and that in this case

$$(2.9) \qquad c_s \le c_u \le 2c_s.$$

The function $\varphi_E(k) := \|\sum_{i=1}^k e_i\|_E$ where $(e_i)_{i=1}^\infty$ is the fundamental sequence of a spreading model E of a Banach space is usually called the *fundamental function of E*. We deduce the following.

Proposition 2.2. *Let $(x_n)_{n=1}^\infty$ be a normalized weakly null sequence in a Banach space X. Then for every $\varepsilon > 0$ there is a normalized weakly null basic subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ with basis constant $(1+\varepsilon)$, generating a spreading model E with fundamental function φ_E, such that for all $k \ge 1$, for all $k \le n_1 < n_2 < \cdots < n_k$, and for all $(\varepsilon_i)_{i=1}^k \subset \{-1,1\}$ one has*

$$(2.10) \quad \frac{1}{2(1+\varepsilon)} \varphi_E(k) \le \frac{1}{1+\varepsilon} \Big\| \sum_{i=1}^k \varepsilon_i e_i \Big\|$$

$$\le \Big\| \sum_{i=1}^k \varepsilon_i y_{n_i} \Big\| \le (1+\varepsilon) \Big\| \sum_{i=1}^k \varepsilon_i e_i \Big\| \le 2(1+\varepsilon)\varphi_E(k).$$

We will be particularly interested in c_0-spreading models, where c_0 is the space of real valued sequences converging to 0 equipped with the sup-norm. The following proposition follows from [8, Lemma 1, p. 73] and [8, Lemma 4, p. 75].

Proposition 2.3. *X has a spreading model isomorphic to c_0 if and only if for all $\varepsilon > 0$, X has a spreading model E whose fundamental sequence $(e_i)_{i=1}^\infty$ is $(1+\varepsilon)$-equivalent to the canonical basis of c_0, i.e. that for all $k \ge 1$, $(a_i)_{i=1}^k \subset \mathbb{R}$,*

$$(2.11) \qquad \frac{1}{(1+\varepsilon)} \sup_{1 \le i \le k} |a_i| \le \Big\| \sum_{i=1}^k a_i e_i \Big\|_E \le (1+\varepsilon) \sup_{1 \le i \le k} |a_i|.$$

We will also need the following observation from [8].

Proposition 2.4. *[8, Proposition 3, p. 79] Let $(x_n)_{n=1}^\infty$ be a normalized weakly null sequence in a Banach space X that generates a spreading model E. Then E is not isomorphic to c_0 if and only if there exists a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that,*

$$(2.12) \qquad \lim_{k \to \infty} \inf_{n_1 < \cdots < n_k} \inf_{(\varepsilon_i)_{i=1}^k \in \{-1,1\}^k} \Big\| \sum_{i=1}^k \varepsilon_i y_{n_i} \Big\| = +\infty.$$

2.4. **Tsirelson spaces T and T^*.** We recall a construction of a space originally designed by Tsirelson [37], which is nowadays usually referred to as T^*, while T denotes its dual space and was described by Figiel and Johnson in [14]. Doing so, convenient notation and terminology that will be needed later on are introduced. Tsirelson's original construction of T^* has a geometric flavor while Figiel-Johnson construction of T is more analytic.

For $E, F \in [\mathbb{N}]^{<\omega}$, and $n \in \mathbb{N}$ we write $E < F$ if $\max(E) < \min(F)$, and $n \leq E$, or $n < E$, if $n \leq \min E$, or $n < \min E$, respectively. Here we set $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$. We call a sequence $(E_j)_{j=1}^n \subset [\mathbb{N}]^{<\omega}$ *admissible* if $n \leq E_1 < E_2 < \cdots < E_n$. For $x = \sum_{j=1}^\infty \xi_j e_j \in c_0$ we call $\mathrm{supp}(x) = \{j \in \mathbb{N} : \xi_j \neq 0\}$ the *support of x* and for $E \in [\mathbb{N}]^{<\omega}$ we write $E(x) = \sum_{j \in E} \xi_j e_j$. We call a finite sequence $(x_j)_{j=1}^n$ of elements in c_{00} a *block sequence* if $\mathrm{supp}(x_{j-1}) < \mathrm{supp}(x_j)$, for all $2 \leq j \leq n$. We define inductively for each $k \in \mathbb{N}_0$ a norm $\| \cdot \|_k$ on c_{00}. For $x = \sum_{j=1}^\infty \xi_j e_j \in c_{00}$ we put $\|x\|_0 = \max_{j \in \mathbb{N}} |\xi_j|$, and assuming that $\| \cdot \|_{k-1}$ has been defined for some $k \in \mathbb{N}$, we let for $x \in c_{00}$

$$\|x\|_k = \max\left(\|x\|_{k-1}, \frac{1}{2} \max\left\{\sum_{j=1}^n \|E_j(x)\|_{k-1} : (E_j)_{j=1}^n \text{ is admissible}\right\}\right).$$

Then we put for $x \in c_{00}$

$$\|x\| = \lim_{k \to \infty} \|x\|_k = \max_{k \in \mathbb{N}} \|x\|_k.$$

$\| \cdot \|$ is then a norm on c_{00} and T is defined to be the completion of c_{00} with respect to $\| \cdot \|$. As observed in [14], $\| \cdot \|$ satisfies the following implicit equation for all $x \in T$

$$\|x\| = \max\left(\|x\|_0, \frac{1}{2} \sup\left\{\sum_{j=1}^n \|E_j(x)\| : (E_j)_{j=1}^n \text{ is admissible}\right\}\right),$$

and the unit vector basis is a 1-unconditional basis of T. Then it was proven in [14] that T does not contain a subspace isomorphic to ℓ_1, which, together with the easy observation that T certainly does not contain a subspace isomorphic to c_0, yields by James' Theorem [16, Theorem 2] that T must be reflexive. Let us now denote the dual of T by T^* (the *original Tsirelson space*). We denote the unit basis of T^* by $(e_j^*)_{j=1}^\infty$ (which is a 1-unconditional basis of T^*) and its unit ball by B_{T^*}. Next to the property that T^* is reflexive, the following property of T^* will be essential for us:

$$(2.13) \quad \left\|\sum_{j=1}^n x_j^*\right\| \leq 2,$$

whenever $(x_j^*)_{j=1}^n \subset B_{T^*}$ is a block sequence, with $n \leq \mathrm{supp}(x_1^*)$.

Indeed, assume that $(x_j^*)_{j=1}^n$ is a block sequence in B_{T^*}, with $n \leq \mathrm{supp}(x_1^*)$, and let $x \in T$, $\|x\| = 1$, be such that $\left\|\sum_{j=1}^n x_j^*\right\| = \sum_{j=1}^n x_j^*(x)$. By the 1-unconditionality of $(e_j)_{j=1}^\infty$ in T we can assume that $\mathrm{supp}(x) \subset \bigcup_{j=1}^n E_j$,

where $E_j = \mathrm{supp}(x_j^*)$, for $j = 1, 2, \ldots, n$. Since $(E_j)_{j=1}^n$ is admissible we obtain from (2.4) that

$$\Big\| \sum_{j=1}^n x_j^* \Big\| = \sum_{j=1}^n x_j^*(x) = \sum_{j=1}^n x_j^*\big(E_j(x)\big) \leq \sum_{j=1}^n \big\| E_j(x) \big\| \leq 2\|x\| = 2,$$

which proves our claim (2.13).

Tsirelson space T^* is not the only reflexive space satisfying inequality (2.13). For every countable ordinal α, one can define T_α^* (see [32]), a higher order version of T^*, where the admissibility condition refers to Schreier families of order α. These Tsirelson-type spaces are all reflexive and satisfy inequality (2.13), and they are incomparable in the sense that T_α^* does not embed isomorphically into T_β^* whenever $\alpha \neq \beta$.

Remark 2.5. In this article, all the results that are valid for T^* also hold for every reflexive Banach space satisfying equation (2.13), so in particular for T_α^* for any countable ordinal α.

3. A METRIC CHARACTERIZATION OF FINITE DIMENSIONALITY

We start by proving some coarse embeddability results for the metric spaces introduced in Section 2.2. Here, we do not need that the metrics can be seen as graph metrics. In Lemma 3.1 below we show that if Y is an infinite dimensional Banach space admitting a spreading model E, generated by a weakly null sequence, then there always exists a 1-Lipschitz map into Y, defined on any of the spaces $([\mathbb{N}]^{<\omega}, d_T)$, $([\mathbb{N}]^{<\omega}, d_H)$, or $([\mathbb{N}]^{<\omega}, d_\triangle)$, whose compression modulus is essentially bounded from below by the fundamental function of the spreading model.

Lemma 3.1. *Let (X, d_X) be any of the following metric spaces: $([\mathbb{N}]^{<\omega}, d_\triangle)$, $([\mathbb{N}]^{<\omega}, d_T)$, or $([\mathbb{N}]^k, d_H)$, for $k \in \mathbb{N}$. Let Y be an infinite dimensional Banach space admitting a spreading model E, generated by a normalized weakly null sequence, then for every $\nu > 0$ there exists a map $f_\nu : (X, d_X) \to Y$ such that for all $x, y \in X$,*

$$(3.1) \qquad \frac{1}{8(1+\nu)} \varphi_E\left(d_X(x,y)\right) \leq \|f_\nu(x) - f_\nu(y)\|_Y \leq d_X(x,y).$$

Proof. Let $\varepsilon > 0$ such that $(1 + \varepsilon)^2 \leq (1 + \nu)$. By Proposition 2.2 there is a weakly null normalized basic sequence $(y_n)_{n=1}^\infty$ with basis constant not larger than $(1+\varepsilon)$, and thus the bimonotonicity constant is at most $2(1+\varepsilon)$, generating a spreading model E, such that for all $k \geq 1$, for all $k \leq n_1 < n_2 < \cdots < n_k$, and for all $(\varepsilon_i)_{i=1}^k \in \{-1, 1\}^k$ one has

$$(3.2) \qquad \Big\| \sum_{i=1}^k \varepsilon_i y_{n_i} \Big\| \geq \frac{1}{1+\varepsilon} \Big\| \sum_{i=1}^k \varepsilon_i e_i \Big\| \geq \frac{1}{2(1+\varepsilon)} \varphi_E(k).$$

We now consider the three different cases.

The symmetric difference metric case. Let $f : ([\mathbb{N}]^{<\omega}, d_\triangle) \to Y$ be defined by $f(\bar{n}) = \sum_{i=1}^k y_{n_i}$, for $\bar{n} = \{n_1, n_2, \ldots, n_k\} \in [\mathbb{N}]^{<\omega}$. The map f is

clearly 1-Lipschitz since after cancellations it follows for $\bar{m}, \bar{n} \in [\mathbb{N}]^{<\omega}$ that

$$\|f(\bar{m}) - f(\bar{n})\| = \Big\| \sum_{i=1}^{d_\triangle(\bar{m},\bar{n})} \varepsilon_i y_{q_i} \Big\|$$

for some $q_1 < q_2 < \cdots < q_{d_\triangle(\bar{m},\bar{n})}$ and $(\varepsilon_i)_{i=1}^{d_\triangle(\bar{m},\bar{n})} \in \{-1,1\}^{d_\triangle(\bar{m},\bar{n})}$. If $d = d_\triangle(\bar{m}, \bar{n}) = 2r$ is even, then $q_{r+1} \geq r$, and hence (3.2) and the assumption on the bimonotonicity constant of $(y_n)_{n=1}^\infty$ yield

$$\|f(\bar{m}) - f(\bar{n})\| \geq \frac{1}{2(1+\varepsilon)} \Big\| \sum_{i=r+1}^{d} \varepsilon_i y_{q_i} \Big\| \geq \frac{1}{2(1+\varepsilon)^2} \Big\| \sum_{i=r+1}^{d} \varepsilon_i e_i \Big\|$$

$$\geq \frac{1}{4(1+\varepsilon)^2} \varphi_E(r) \geq \frac{1}{8(1+\varepsilon)^2} \varphi_E(d).$$

If d is odd, the proof is similar, as will be seen while treating the next case.

The tree metric case. Let $\Phi \colon [\mathbb{N}]^{<\omega} \to \mathbb{N}$ be a bijection. Let $f \colon ([\mathbb{N}]^{<\omega}, d_T) \to Y$ be defined by $f(\bar{n}) = \sum_{\bar{u} \preceq \bar{n}} y_{\Phi(\bar{u})}$, for $\bar{n} \in [\mathbb{N}]^{<\omega}$. The map f is clearly 1-Lipschitz since after cancellations one has

$$\|f(\bar{m}) - f(\bar{n})\| = \Big\| \sum_{i=1}^{d_T(\bar{m},\bar{n})} \varepsilon_i y_{q_i} \Big\|,$$

for some $q_1 < q_2 < \cdots < q_{d_T(\bar{m},\bar{n})}$ and $(\varepsilon_i)_{i=1}^{d_T(\bar{m},\bar{n})} \in \{-1,1\}^{d_T(\bar{m},\bar{n})}$. Here we only detail the case when $d = d_T(\bar{m}, \bar{n})$ is odd, say $d = 2r - 1$, with $r \in \mathbb{N}$. Note that $q_r \geq r$. So, again (3.2) and the assumption on the bimonotonicity constant of $(y_n)_{n=1}^\infty$ imply that

$$\|f(\bar{m}) - f(\bar{n})\| \geq \frac{1}{2(1+\varepsilon)} \Big\| \sum_{i=r}^{d} \varepsilon_i y_{n_i} \Big\| \geq \frac{1}{2(1+\varepsilon)^2} \Big\| \sum_{i=r}^{d} \varepsilon_i e_i \Big\|$$

$$\geq \frac{1}{4(1+\varepsilon)^2} \varphi_E(r) \geq \frac{1}{8(1+\varepsilon)^2} \varphi_E(2r) \geq \frac{1}{8(1+\varepsilon)^2} \varphi_E(d).$$

The Hamming metric case. Let $\Phi \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection and let $f \colon ([\mathbb{N}]^k, d_H) \to Y$ be defined by $f(\bar{n}) = \frac{1}{2} \sum_{i=1}^{k} y_{\Phi(i,n_i)}$. The map f is clearly 1-Lipschitz since after cancellations one has

$$\|f(\bar{m}) - f(\bar{n})\| = \frac{1}{2} \Big\| \sum_{i=1}^{2d_H(\bar{m},\bar{n})} \varepsilon_i y_{q_i} \Big\|$$

for some $q_1 < q_2 < \cdots < q_{2d_H(\bar{m},\bar{n})}$ and $(\varepsilon_i)_{i=1}^{2d_H(\bar{m},\bar{n})} \in \{-1,1\}^{2d_H(\bar{m},\bar{n})}$. Necessarily $q_{d_H(\bar{m},\bar{n})} \geq d_H(\bar{m}, \bar{n})$, and hence, as in the other cases,

$$\|f(\bar{m}) - f(\bar{n})\| \geq \frac{1}{4(1+\varepsilon)} \Big\| \sum_{i=d_H(\bar{m},\bar{n})+1}^{2d_H(\bar{m},\bar{n})} \varepsilon_i y_{q_i} \Big\| \geq \frac{1}{8(1+\varepsilon)^2} \varphi_E\left(d_H(\bar{m}, \bar{n})\right).$$

$\square$

Remark 3.2. We restricted in Lemma 3.1 the Hamming metric d_H to $[\mathbb{N}]^k$, for $k \in \mathbb{N}$, because d_H is usually defined for strings of equal length. But it is not hard to show the existence of a map $f : [\mathbb{N}]^{<\omega} \to X$, satisfying condition (3.1), if we replace the value 8 by a larger number.

Lemma 3.1 yields a coarse embedding of $([\mathbb{N}]^{<\omega}, d_T)$, $([\mathbb{N}]^{<\omega}, d_\triangle)$, and an equi-coarse embedding of the sequence $([\mathbb{N}]^k, d_H)_{k \in \mathbb{N}}$ into every Banach space which admits at least one spreading model, that is not isomorphic to c_0 and is generated by a weakly null sequence. Recall that Proposition 2.4 in Section 2.3 insures that the associated fundamental function is unbounded. Then, the following proposition follows simply from Lemma 3.1.

Corollary 3.3. *The metric spaces $([\mathbb{N}]^{<\omega}, d_T)$, $([\mathbb{N}]^{<\omega}, d_\triangle)$, are all coarsely embeddable, and the sequence $([\mathbb{N}]^k, d_H)_{k \in \mathbb{N}}$ is equi-coarsely embeddable into a Banach space that has a spreading model E, generated by a normalized weakly null sequence, which is not isomorphic to c_0.*
More precisely, in all three cases, for any $\nu > 0$ there is a coarse embedding f_ν into Y such that f_ν is 1-Lipschitz and the modulus of compression of f_ν satisfies $\rho_{f_\nu}(t) \geq \frac{1}{8(1+\nu)} \varphi_E(t)$, for $t > 0$, where φ_E is the fundamental function of the spreading model E.

For the tree metric we can deduce the following embeddability result, even if all spreading models are equivalent to c_0.

Proposition 3.4. *If an infinite dimensional Banach space Y has a spreading model isomorphic to c_0 then $(T_k^\omega)_{k \geq 1}$ equi-bi-Lipschitzly embeds into Y. More precisely, for every $\nu > 0$ and every $k \in \mathbb{N}$ there exists a map $f_k : T_k^\omega \to Y$, such that for all $x, y \in T_k^\omega$,*

$$(3.3) \qquad \frac{1}{2(1+\nu)} d_T(x,y) \leq \|f_k(x) - f_k(y)\|_Y \leq d_T(x,y).$$

Proof. By Proposition 2.3 one may assume that there exists a normalized sequence $(y_n)_{n=1}^\infty$ such that for all $k \geq 1$, for all $k \leq n_1 < n_2 < \cdots < n_k$, and for all $(a_i)_{i=1}^k \subset \mathbb{R}$ one has

$$(3.4) \qquad \frac{1}{(1+\nu)} \sup_{1 \leq i \leq k} |a_i| \leq \Big\| \sum_{i=1}^k a_i y_{n_i} \Big\| \leq (1+\nu) \sup_{1 \leq i \leq k} |a_i|.$$

Let $\Phi_k : [\mathbb{N}]^{\leq k} \to \{2k, 2k+1, \dots\}$ be a bijection. For $k \in \mathbb{N}$ let $f_k : T_k^\omega \to Y$ be defined by $f_k(\bar{n}) = \sum_{\bar{u} \preceq \bar{n}} \sum_{\bar{s} \preceq \bar{u}} y_{\Phi(\bar{s})}$, for $\bar{n} \in [\mathbb{N}]^{\leq k}$. After cancellations one has

$$\|f_k(\bar{m}) - f_k(\bar{n})\| = \Big\| \sum_{i=1}^{d_T(\bar{m},\bar{n})} \alpha_i y_{q_i} \Big\|$$

for some $2k \leq q_1 < \cdots < q_{d_T(\bar{m},\bar{n})}$ and $(\alpha_i)_{i=1}^{d_T(\bar{m},\bar{n})} \subset \{1, \dots, d_T(\bar{m}, \bar{n})\}$ satisfying $\frac{1}{2} d_T(\bar{m}, \bar{n}) \leq \sup_{1 \leq i \leq d_T(\bar{m},\bar{n})} |\alpha_i| \leq d_T(\bar{m}, \bar{n})$. So

$$\|f_k(\bar{m}) - f_k(\bar{n})\| \geq \frac{1}{(1+\nu)} \sup_{1 \leq i \leq d_T(\bar{m},\bar{m})} |\alpha_i| \geq \frac{1}{2(1+\nu)} d_T(\bar{m}, \bar{n})$$

and

$$\|f_k(\bar{m}) - f_k(\bar{n})\| \leq (1+\nu) \sup_{1 \leq i \leq d_T(\bar{m}, \bar{n})} |\alpha_i| \leq (1+\nu)d_T(\bar{m}, \bar{n}),$$

which proves our claim.   □

Remark 3.5. If an infinite dimensional Banach space Y has a spreading model isomorphic to c_0 then, using a duality argument [8, Proposition 1, p. 80], the relationship between the ℓ_1^+-weakly null index and the Szlenk index [2, Theorem 4.2], and the embedding in [5, Theorem 2.6], one can show that T_∞^ω embeds bi-Lipschitzly into Y.

Moreover, since it is easy to see that every countable tree isometrically embeds into T_∞^ω we obtain that every countable tree coarsely embeds into any infinite dimensional Banach space.

In the following theorem, which includes Theorem E, a purely metric characterization of finite dimensionality in terms of graph preclusion in the coarse category is given.

Theorem 3.6. *Let Y be a Banach space. The following assertions are equivalent:*

    (1) Y is finite dimensional.
    (2) $(T_k^\omega)_{k \geq 1}$ does not equi-coarsely embed into Y.
    (3) T_∞^ω does not coarsely embed into Y.

Proof. We start with a self contained elementary proof of the main equivalence (1) $\iff$ (2), which avoids the use of Remark 3.5.
By compactness, a finite dimensional Banach space cannot contain an infinite sequence that is bounded and separated. On the other hand, if Y is an infinite dimensional Banach space, then by Rosenthal's ℓ_1-theorem, it either contains ℓ_1 isomorphically or it has a weakly Cauchy sequence which is not norm converging. Since ℓ_1 contains an isometric copy of T_∞^ω, we may assume that X contains a weakly Cauchy sequence $(x_n)_{n=1}^\infty$ which is not norm converging. After eventually passing to a subsequence of $(x_n)_{n=1}^\infty$, still denoted $(x_n)_{n=1}^\infty$, the sequence

$$(z_n)_{n=1}^\infty := \left(\frac{x_{2n-1} - x_{2n}}{\|x_{2n-1} - x_{2n}\|} \right)_{n=1}^\infty$$

is normalized and weakly null. Therefore, either $(z_n)_{n=1}^\infty$ has a subsequence that generates a spreading model isomorphic to c_0 and we apply Proposition 3.4 or $(z_n)_{n=1}^\infty$ has a subsequence that generates a spreading model not isomorphic to c_0 and we conclude with Corollary 3.3.
We now finish the circle of implications: (2) $\implies$ (3) is obvious and (3) $\implies$ (1) follows from Corollary 3.3 and Remark 3.5.   □

4. THE CONCENTRATION INEQUALITIES

All the results of this section will be valid for the spaces $[\mathbb{N}]^k$, $k \in \mathbb{N}$, equipped either with the Johnson metric $d_J = \frac{d_\triangle}{2}$ or the Hamming metric d_H. So throughout the section, $d_\bullet$ will denote either d_J or d_H. We start with a general structural result on Lipschitz maps from $([\mathbb{N}]^k, d_\bullet)$ into a reflexive space with a basis. For two vectors x and y in a Banach space

with a basis, we shall use freely the convenient notation $x \prec y$ to mean that $\mathrm{supp}(x) < \mathrm{supp}(y)$, and $k \preceq x$ when $k \leq \min(\mathrm{supp}(x))$, where the supports are with respect to the basis considered.

Proposition 4.1. *Let Y be a reflexive Banach space with a bimonotone basis $(e_i)_{i=1}^{\infty}$. Then, for all $k, r \in \mathbb{N}$, $\varepsilon > 0$, $\mathbb{M} \in [\mathbb{N}]^{\omega}$, and Lipschitz maps $f \colon ([\mathbb{M}]^k, d_{\bullet}) \to Y$ there exist $\mathbb{M}' \in [\mathbb{N}]^{\omega}$ and $y \in Y$ satisfying the following:*

For all $\bar{m} \in [\mathbb{M}']^k$ there exist $r \preceq y_{\bar{m}}^{(1)} \prec y_{\bar{m}}^{(2)} \prec \cdots \prec y_{\bar{m}}^{(k)}$, all with finite supports with respect to $(e_i)_{i=1}^{\infty}$, such that:

$$(4.1) \qquad \|y_{\bar{m}}^{(i)}\| \leq \mathrm{Lip}(f), \text{ for all } i \in \{1, 2, \ldots, k\},$$

and

$$(4.2) \qquad \|f(\bar{m}) - (y + y_{\bar{m}}^{(1)} + y_{\bar{m}}^{(2)} + \cdots + y_{\bar{m}}^{(k)})\| < \varepsilon.$$

Proof. We will show this result by induction on k. The proof is based on a standard gliding hump procedure and a few diagonal arguments that we have chosen to detail. For $n \in \mathbb{N}$, we shall denote P_n the basis projection on the linear span of $\{e_1, \ldots, e_n\}$.

For $k = 1$ assume that we are given $r \in \mathbb{N}$, $\mathbb{M} \in [\mathbb{N}]^{\omega}$, a Lipschitz map $f \colon (\mathbb{M}, d_{\bullet}) \to Y$, and $\varepsilon > 0$. Since f is bounded and Y is reflexive, by weak compactness, there exists $y \in Y$ and $\mathbb{M}_0 \in [\mathbb{M}]^{\omega}$ such that $(f(m))_{m \in \mathbb{M}_0}$ converges weakly to y. From the weak lower semi-continuity of the norm of Y we deduce that $\|f(m) - y\| \leq \mathrm{Lip}(f)$ for all $m \in \mathbb{M}_0$. Since $(f(m) - y)_{m \in \mathbb{M}_0}$ is weakly null, one can find $m_1 \in \mathbb{M}_0$ such that

$$\|P_r(f(m) - y)\| < \frac{\varepsilon}{2}, \text{ for all } m \in \mathbb{M}_0, \text{ with } m \geq m_1.$$

Then for all for all $m \geq m_1$, $m \in \mathbb{M}_0$, there exists $s_m > r$ such that

$$\|(I - P_{s_m})(f(m) - y)\| < \frac{\varepsilon}{2}.$$

We now set $\mathbb{M}' := \{m \in \mathbb{M}_0 \colon m \geq m_1\}$ and $y_m^{(1)} := (P_{s_m} - P_r)(f(m) - y)$, for $m \in \mathbb{M}'$. For all $m \in \mathbb{M}'$, $r \preceq y_m^{(1)}$, $y_m^{(1)}$ has finite support and since the basis is bimonotone, $\|y_m^{(1)}\| \leq \mathrm{Lip}(f)$. Finally, we deduce that $\|f(m) - (y + y_m^{(1)})\| < \varepsilon$, from the triangle inequality.

Assume now that our statement is proved for $k \in \mathbb{N}$, and let $r \in \mathbb{N}$, a Lipschitz map $f \colon ([\mathbb{M}]^{k+1}, d_{\bullet}) \to Y$, and $\varepsilon > 0$ be given. Using that $f([\mathbb{M}]^{k+1})$ is a bounded subset of the reflexive space Y and a diagonal argument, we infer the existence of $\mathbb{M}_0 \in [\mathbb{M}]^{\omega}$ such that for all $\bar{m} \in [\mathbb{M}_0]^k$, $(f(\bar{m}, n))_{n \in \mathbb{M}_0}$ converges weakly to some $g(\bar{m}) \in Y$. Using again that the norm of Y is weakly lower semi-continuous, we have that $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$. This allows us to apply our induction hypothesis to the map $g \colon [\mathbb{M}_0]^k \to Y$ and to find $\mathbb{M}_1 \in [\mathbb{M}_0]^{\omega}$ and $y \in Y$, such that for all $\bar{m} \in [\mathbb{M}_1]^k$ there exist $r \preceq y_{\bar{m}}^{(1)} \prec y_{\bar{m}}^{(2)} \prec \cdots \prec y_{\bar{m}}^{(k)}$, all with finite supports, such that for all $i \in \{1, \ldots, k\}$ $\|y_{\bar{m}}^{(i)}\| \leq \mathrm{Lip}(f)$ and

$$(4.3) \qquad \|g(\bar{m}) - (y + y_{\bar{m}}^{(1)} + y_{\bar{m}}^{(2)} + \cdots + y_{\bar{m}}^{(k)})\| < \frac{\varepsilon}{3}.$$

We now fix $\bar{m} \in [\mathbb{M}_1]^k$. Note that the weak lower semi-continuity of the norm implies that for all $n \in \mathbb{M}_1$, $\|f(\bar{m}, n) - g(\bar{m})\| \leq \mathrm{Lip}(f)$. Denote

$r_{\bar{m}} = \max(\mathrm{supp}(y_{\bar{m}}^{(k)}))$. Since $(f(\bar{m},n) - g(\bar{m}))_{n \in \mathbb{M}_1}$ is weakly null, there exists $N_{\bar{m}} \in \mathbb{M}_1$ such that for all $n \in \mathbb{M}_1$, $n \geq N_{\bar{m}}$:

$$\left\| P_{r_{\bar{m}}}\big(f(\bar{m},n) - g(\bar{m})\big) \right\| < \frac{\varepsilon}{3}.$$

Then, for all $n \in \mathbb{M}_1$, $n \geq N_{\bar{m}}$, there exists $s_{\bar{m},n} > r_{\bar{m}}$ so that

$$\left\| (I - P_{s_{\bar{m},n}})\big(f(\bar{m},n) - g(\bar{m})\big) \right\| < \frac{\varepsilon}{3}.$$

We now set $y_{(\bar{m},n)}^{(k+1)} := (P_{s_{\bar{m},n}} - P_{r_{\bar{m}}})(f(\bar{m},n) - g(\bar{m}))$. We have that for all $n \in \mathbb{M}_1$, $n \geq N_{\bar{m}}$: $y_{(\bar{m},n)}^{(k+1)}$ has finite support, $y_{\bar{m}}^k \prec y_{(\bar{m},n)}^{(k+1)}$, $\|y_{(\bar{m},n)}^{(k+1)}\| \leq \mathrm{Lip}(f)$ and

$$\|f(\bar{m},n) - (y + y_{\bar{m}}^{(1)} + \cdots + y_{\bar{m}}^{(k)} + y_{(\bar{m},n)}^{(k+1)})\| < \varepsilon.$$

We conclude the proof with one last extraction argument.

If $\mathbb{M}_1 = \{m_1, m_2, \dots\}$, we define $\mathbb{M}' := \{m_1', m_2', \dots\}$ recursively as follows. For all $i \in \mathbb{N}$, $m_i' := m_{\Phi(i)}$ where $\Phi(1) = 1, \Phi(2) = 2, \dots, \Phi(k) = k$, and if $\Phi(i)$ has been chosen for $i \geq k$ then $\Phi(i+1)$ is picked such that $\Phi(i+1) > \Phi(i)$ and $m_{\Phi(i+1)} \geq N_{\bar{m}}$ for all $\bar{m}$ in the finite set $[\{m_1', m_2', \dots, m_i'\}]^k$. It is now easy to check that $\mathbb{M}'$, y, and for all $\bar{m} \in [\mathbb{M}']^k$ and $n \in \mathbb{M}'$, $n > \max(\bar{m})$, the vectors with finite support $y_{\bar{m} \cup \{n\}}^{(1)} := y_{\bar{m}}^{(1)}, \dots, y_{\bar{m} \cup \{n\}}^{(k)} := y_{\bar{m}}^{(k)}$, $y_{\bar{m} \cup \{n\}}^{(k+1)}$ satisfy the induction hypothesis for $k+1$.

$\square$

Remark 4.2. We have assumed that the basis of Y is bimonotone only for convenience. In the general case, one gets a similar result, with the only difference that $\|y_{\bar{m}}^{(i)}\| \leq C\mathrm{Lip}(f)$, where C is the bimonotonicity constant.

Remark 4.3. Proposition 4.1 is actually true if we only assume that Y is a Banach space with a boundedly complete Schauder finite dimensional decomposition. In that case, we just have to use weak*-compactness instead of weak-compactness.

Our concentration result for Lipschitz maps with values in T^* follows easily from Proposition 4.1. The following theorem is Theorem C when $d_\bullet = d_H$.

Theorem 4.4 (Concentration inequality). *Let $k \in \mathbb{N}$ and $f \colon ([\mathbb{N}]^k, d_\bullet) \to T^*$ be a Lipschitz map. Then there exists $\mathbb{M}' \in [\mathbb{N}]^\omega$ such that for all $\bar{m}, \bar{n} \in [\mathbb{M}']^k$ one has*

(4.4) $$\|f(\bar{m}) - f(\bar{n})\| \leq 5\mathrm{Lip}(f).$$

Proof. We assume, as we may, that $\mathrm{Lip}(f) > 0$. Then it follows from Proposition 4.1 with $Y = T^*$, $r = k$, $\mathbb{M} = \mathbb{N}$ and $\varepsilon = \frac{1}{2}\mathrm{Lip}(f)$ that there exists $\mathbb{M}' \in [\mathbb{N}]^\omega$, $y \in T^*$ such that for all $\bar{m} \in [\mathbb{M}']^k$ there exist $k \preceq y_{\bar{m}}^{(1)} \prec y_{\bar{m}}^{(2)} \prec \cdots \prec y_{\bar{m}}^{(k)}$, all with finite supports, satisfying (4.1) and (4.2). And hence, for all $\bar{m}, \bar{n} \in \mathbb{M}'$

$$\|f(\bar{m}) - f(\bar{n})\| \leq 2\varepsilon + \|y_{\bar{m}}^{(1)} + y_{\bar{m}}^{(2)} + \cdots + y_{\bar{m}}^{(k)}\| + \|y_{\bar{n}}^{(1)} + y_{\bar{n}}^{(2)} + \cdots + y_{\bar{n}}^{(k)}\|$$
$$\leq 5\mathrm{Lip}(f),$$

where for the last inequality we use property (2.13) of T^*.

$\square$

The concentration inequalities stated above immediately provide an obstruction to equi-coarse embeddability of the Johnson graphs $(J_k^\omega)_{k\geq 1}$ and the Hamming graphs $(H_k^\omega)_{k\geq 1}$. A rescaling argument can be used to provide an obstruction to equi-uniform embeddability of the rescaled metrics.

Corollary 4.5.

(1) *The sequence of Hamming graphs $(H_k^\omega)_{k\geq 1}$ (resp. $([\mathbb{N}]^k, \frac{d_H}{k})_{k\geq 1}$) do not equi-coarsely (resp. equi-uniformly) embed into T^*.*

(2)*The sequence of Johnson graphs $(J_k^\omega)_{k\geq 1}$ (resp. $([\mathbb{N}]^k, \frac{d_J}{k})_{k\geq 1}$) do not equi-coarsely (resp. equi-uniformly) embed into T^*.*

Proof. We only treat the case of the Hamming graphs, the case of the Johnson graphs being similar. Assume by contradiction that $(H_k^\omega)_{k\geq 1}$ equi-coarsely embeds into T^*, *i.e.*, that there exist non-decreasing functions $\rho, \omega : [0,\infty) \to [0,\infty)$ and for each $k \in \mathbb{N}$ a function $f_k : ([\mathbb{N}]^k, d_H) \to T^*$, such that $\rho \leq \rho_{f_k} \leq \omega_{f_k} \leq \omega$, $\lim_{t\to\infty} \rho(t) = \infty$ and $\omega(t) < \infty$, for all $t > 0$. Since d_H is the graph distance on H_k^ω, f_k is actually $\omega(1)$-Lipschitz. By Theorem 4.4, for every $k \geq 1$, there exists $\mathbb{M}_k \in [\mathbb{N}]^\omega$ such that for all $\bar{m}, \bar{n} \in [\mathbb{M}_k]^k$ one has

$$(4.5) \qquad \|f_k(\bar{m}) - f_k(\bar{n})\| \leq 5\omega(1).$$

If $\bar{m}$ and $\bar{n}$ are chosen in $[\mathbb{M}_k]^k$ such that $d_H(\bar{m}, \bar{n}) = k$ then it implies that for every $k \geq 1$, $\rho(k) \leq 5\omega(1)$, which contradicts the properties of ω and ρ for k large enough.

Assume now by contradiction that $([\mathbb{N}]^k, \frac{d_H}{k})_{k\geq 1}$ equi-uniformly embeds into T^*, *i.e.*, there exist non-decreasing functions $\rho, \omega : [0,\infty) \to [0,\infty)$ and for each $k \in \mathbb{N}$ a function $f_k : ([\mathbb{N}]^k, \frac{d_H}{k}) \to T^*$, such that $\rho \leq \rho_{f_k} \leq \omega_{f_k} \leq \omega$, $\lim_{t\to 0} \omega(t) = 0$ and $\rho(t) > 0$, for all $t > 0$. Denote the identity map from $([\mathbb{N}]^k, d_H)$ to $([\mathbb{N}]^k, \frac{d_H}{k})$ by i_k. Then, $\text{Lip}(f_k \circ i_k) = \omega_{f_k \circ i_k}(1) \leq \omega(\frac{1}{k})$. By Theorem 4.4, for all $k \geq 1$, there exist $\mathbb{M}_k \in [\mathbb{N}]^\omega$ such that $\|f_k(\bar{m}) - f_k(\bar{n})\| \leq 5\omega(\frac{1}{k})$, for all $\bar{m}, \bar{n} \in [\mathbb{M}_k]^k$. Therefore, considering again $\bar{m}$ and $\bar{n}$ in $[\mathbb{M}_k]^k$ such that $d_H(\bar{m}, \bar{n}) = k$, we get that for all $k \geq 1$ one has $0 < \rho(1) \leq 5\omega(\frac{1}{k})$, which is a contradiction for large enough k. $\square$

Remark 4.6. As previously mentioned, Corollary B follows from the more involved rigidity phenomenon depicted in Theorem A. Nevertheless, for the reader only interested in the geometry of Hilbert space we include an abridged proof of the ℓ_2-case. The proof follows simply from Corollary 4.5 and, in the coarse setting, the fact that the Johnson graphs equi-coarsely embeds into ℓ_2. Indeed, for $k \in \mathbb{N}$ consider the map

$$f_k : ([\mathbb{N}]^k, d_J) \to \ell_2, \qquad \bar{n} \mapsto \sum_{i=1}^k e_{n_i},$$

where $(e_n)_{n=1}^\infty$ denotes the canonical basis of ℓ_2. Then

$$(4.6) \qquad \|f_k(\bar{m}) - f_k(\bar{n})\|_2 = \sqrt{2d_J(\bar{m}, \bar{n})}.$$

For our claim in the uniform category define for $k \in \mathbb{N}$

$$g_k \colon \left([\mathbb{N}]^k, \frac{d_J}{k} \right) \to B_{\ell_2}, \quad \bar{n} \mapsto \frac{1}{\sqrt{k}} \sum_{i=1}^{k} e_{n_i},$$

then

(4.7)
$$\|g_k(\bar{m}) - g_k(\bar{n})\|_2 = \sqrt{\frac{2 d_J(\bar{m}, \bar{n})}{k}},$$

and we again invoke Corollary 4.5 to conclude the proof of our claim.

Reflexivity is usually not preserved under nonlinear embeddings, as witnessed by Ribe's example in [35], where he showed that the two separable Banach spaces $(\sum_{n=1}^{\infty} \ell_{p_n})_{\ell_2}$ and $(\sum_{n=1}^{\infty} \ell_{p_n})_{\ell_2} \oplus \ell_1$, where $(p_n)_{n \geq 1}$ is a sequence in $(1, \infty)$ such that $\lim_{n \to \infty} p_n = 1$, are uniformly homeomorphic. Using the Kalton-Randrianarivony concentration inequality, it was shown in [5] that if X coarse Lipschitz embeds into a reflexive Banach space that is asymptotically uniformly smooth then X must be reflexive. Actually X must have the Banach-Saks property as recently observed in [9]. Another consequence of the concentration inequality gives that reflexivity is stable under coarse embeddability into Banach spaces possessing the "c_0-like" asymptotic smoothness property of Tsirelson space T^*. This trade-off, in order to preserve reflexivity, between faithfulness of the embedding and smoothness properties of the target space is a rather interesting phenomenon.

Proposition 4.7. *Let X be a Banach space. If X coarsely embeds, or B_X uniformly embeds, into T^* then X is reflexive.*

Proof. Assume that X is not reflexive. Then, by James's characterization of reflexive spaces [17] there exists a sequence $(x_n)_{n=1}^{\infty} \subset B_X$ such that for all $k \geq 1$ and $\bar{n} \in [\mathbb{N}]^{2k}$,

(4.8)
$$\left\| \sum_{i=1}^{k} x_{n_i} - \sum_{i=k+1}^{2k} x_{n_i} \right\| \geq \frac{k}{2}.$$

For every $k \in \mathbb{N}$, the map $\varphi_k \colon H_k^{\omega}(\mathbb{N}) \to X$ defined as $\varphi_k(\bar{n}) = \sum_{i=1}^{k} x_{n_i}$, for $\bar{n} \in [\mathbb{N}]^k$, is clearly 2-Lipschitz. Assume by contradiction that there exists a coarse embedding $f \colon X \to T^*$. It follows again from the properties of graph distances that $f \circ \varphi_k$ is $\omega_{f \circ \varphi_k}(1)$-Lipschitz. Since $\omega_{f \circ \varphi_k}(1) \leq \omega_f(2)$, it follows from Theorem 4.4 that for all $k \geq 1$, there exists $\mathbb{M}_k \in [\mathbb{N}]^{\omega}$, such that $\|f \circ \varphi_k(\bar{m}) - f \circ \varphi_k(\bar{n})\| \leq 5\omega_f(2)$, for all $\bar{m}, \bar{n} \in [\mathbb{M}_k]^k$. In particular for all $k \geq 1$, if $\bar{m}, \bar{n} \in [\mathbb{M}_k]^k$ are such that $m_1 < m_2 < \cdots < m_k < n_1 < n_2 < \cdots < n_k$, we obtain by (4.8) that $\rho_f(\frac{k}{2}) \leq 5\omega_f(2)$, which is impossible for large enough k.

For the proof in the uniform setting, we use the map $\varphi_k \colon H_k^{\omega} \to B_X$ defined as $\varphi_k(\bar{n}) = \frac{1}{k} \sum_{i=1}^{k} x_{n_i}$, which is clearly $\frac{2}{k}$-Lipschitz. Assume by contradiction that there exists a uniform embedding $f \colon B_X \to T^*$. Then, $\mathrm{Lip}(f \circ \varphi_k) \leq \omega_f(\frac{2}{k})$. By Theorem 4.4, for all $k \geq 1$, there exist $\mathbb{M}_k \in [\mathbb{N}]^{\omega}$ such that $\|f \circ \varphi_k(\bar{m}) - f \circ \varphi_k(\bar{n})\| \leq 5\omega_f(\frac{2}{k})$, for all $\bar{m}, \bar{n} \in [\mathbb{M}_k]^k$. In particular for all $k \geq 1$, again if $\bar{m}, \bar{n} \in [\mathbb{M}_k]^k$ are such that $m_1 < m_2 <$

$\cdots < m_k < n_1 < n_2 < \cdots < n_k$, we get by (4.8) that $0 < \rho_f(\frac{1}{2}) \le 5\omega_f(\frac{2}{k})$, which is again impossible if k is sufficiently large. $\qquad\square$

Remark 4.8. The conclusion of Proposition 4.7 can be slightly strengthened as already observed in [9]. Indeed, using [7, Proposition 2, p. 273] one can show that X has the Banach-Saks property.

Recall that Tsirelson's space T^* has the following important properties:
- T^* is reflexive.
- T^* does not contain isomorphic copies of ℓ_p, for any $p \in [1, \infty)$ nor of c_0.
- All the spreading models of T^* are isomorphic to c_0.

Theorem A and its Corollary 4.9 below show that Tsirelson's construction is surprisingly rigid.

Proof of Theorem A. Assume first that $g : X \to T^*$ is a coarse embedding. By Corollary 4.5 the sequence $(H_k^\omega)_{k \ge 1}$ does not equi-coarsely embed into X. Since Proposition 4.7 insures that X is reflexive, it follows from Rosenthal's ℓ_1 theorem that every spreading model of X can be generated by a weakly null sequence. Finally it follows from Corollary 3.3 that every spreading model of X is isomorphic to c_0.

Assume now that $g : B_X \to T^*$ is a uniform embedding. Assume also by contradiction that X admits a spreading model which is not isomorphic to c_0. By Proposition 4.7, X is reflexive. Thus, it follows from Rosenthal's ℓ_1 theorem and Proposition 2.4 that there exists a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X such that $\lim_{k \to \infty} \psi(k) = \infty$, where

$$(4.9) \qquad \psi(k) = \inf_{n_1 < n_2 < \cdots < n_k} \inf_{(\varepsilon_i)_{i=1}^k \in \{-1,1\}} \Big\| \sum_{i=1}^k \varepsilon_i x_{n_i} \Big\|.$$

Define now $\varphi_k \colon ([\mathbb{N}]^k, \frac{d_J}{k}) \to T^*$ by

$$(4.10) \qquad \varphi_k(\bar{n}) = g\Big(\frac{1}{\psi(2k)} \sum_{i=1}^k x_{n_i} \Big), \quad \text{for } \bar{n} \in [\mathbb{N}]^k.$$

Note that $\omega_{\varphi_k}(\frac{1}{k}) \le \omega_g(\frac{2}{\psi(2k)})$. For every $k \in \mathbb{N}$ there exists $\mathbb{M}_k \in [\mathbb{N}]^\omega$ by Theorem 4.4, such that

$$(4.11) \qquad \|\varphi_k(\bar{m}) - \varphi_k(\bar{n})\| \le 5\omega_g\Big(\frac{2}{\psi(k)} \Big), \quad \text{whenever } \bar{m}, \bar{n} \in [\mathbb{M}_k]^k.$$

But, it follows from the definition of ψ that $\|\varphi_k(\bar{m}) - \varphi_k(\bar{n})\| \ge \rho_g(1)$, whenever $d_J(\bar{m}, \bar{n}) = k$. So for all $k \in \mathbb{N}$, we have $5\omega_g(\frac{2}{\psi(k)}) \ge \rho_g(1) > 0$. This contradicts the fact that $\lim_{k \to \infty} \psi(k) = \infty$ and $\lim_{t \to 0} \omega_g(t) = 0$. $\quad\square$

As an immediate corollary of Theorem A, we have the following non linear versions of Tsirelson's theorem.

Corollary 4.9.

(1) Neither ℓ_p for $p \in [1, \infty)$ nor c_0 coarsely embeds into T^*.
(2) Neither B_{ℓ_p} for $p \in [1, \infty)$ nor B_{c_0} uniformly embeds into T^*.

5. Final comments and open problems

Since T^* clearly has trivial cotype, it follows from Mendel-Naor metric cotype obstruction that T^* is not coarsely embeddable into any Banach space that coarsely embeds into a Banach space with non-trivial type. Therefore, T^* and ℓ_p, $p \in [1, \infty)$, are coarsely incomparable in the sense that T^* is not coarsely embeddable into ℓ_p and ℓ_p is not coarsely embeddable into T^*. To the best of our knowledge this provides the first pairs of coarsely incomparable Banach spaces.

Corollary 5.1. *Assume that X is a Banach space such that*
  (1) $(H_k^\omega)_{k \geq 1}$ (or $(J_k^\omega)_{k \geq 1}$) equi-coarsely embeds into X,
  (2) X coarsely embeds into a Banach space with non-trivial type.
Then, X and T^* are coarsely incomparable.

Let us conclude with a few open questions.

Problem 5.1. Does ℓ_2 coarsely embed into every super-reflexive Banach space?

In view of Ostrovskii's result [33], which we referenced in the introduction, a counter-example to Problem 5.1 would have to be a Banach space that is super-reflexive not containing any unconditional basic sequence. It seems that the only known such space is Ferenczi's space [12].

Problem 5.2. Does ℓ_2 coarsely embed into any Banach space which has a spreading model that is not equivalent to c_0?

References

[1] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of c_0^+*, Israel J. Math. **19** (1974), 284–291.

[2] D. Alspach, R. Judd, and E. Odell, *The Szlenk index and local l_1-indices*, Positivity **9** (2005), no. 1, 1–44.

[3] F. Baudier, *Metrical characterization of super-reflexivity and linear type of Banach spaces*, Arch. Math. **89** (2007), 419–429.

[4] _____, *Quantitative nonlinear embeddings into Lebesgue sequence spaces*, J. Topol. Anal. **8** (2016), no. 1, 117-150.

[5] F. Baudier, N. J. Kalton, and G. Lancien, *A new metric invariant for Banach spaces*, Studia Math. **199** (2010), 73-94.

[6] F. Baudier and G. Lancien, *Tight embeddability of proper and stable metric spaces*, Anal. Geom. Metr. Spaces **3** (2015), 140–156.

[7] B. Beauzamy, *Banach-Saks properties and spreading models*, Math. Scand. **44** (1979), 357–384.

[8] B. Beauzamy and J.-T. Lapresté, *Modèles étalés des espaces de Banach*, Travaux en Cours. [Works in Progress], Hermann, Paris, 1984.

[9] B. M. Braga, *Asymptotic structure and coarse Lipschitz geometry of Banach spaces*, Studia Math. **237** (2017), no. 1, 71–97.

[10] A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory **7** (1974), 294–299.

[11] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), 1961, pp. 123–160.

[12] V. Ferenczi, *A uniformly convex hereditarily indecomposable Banach space*, Israel J. Math. **102** (1997), 199–225.

[13] S. C. Ferry, A. Ranicki, and J. Rosenberg, *A history and survey of the Novikov conjecture*, Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 1995, pp. 7–66.

[14] T. Figiel and W. B. Johnson, *A uniformly convex Banach space which contains no l_p*, Compositio Math. **29** (1974), 179–190.

[15] G. Godefroy, G. Lancien, and V. Zizler, *The non-linear geometry of Banach spaces after Nigel Kalton*, Rocky Mountain J. Math. **44** (2014), no. 5, 1529–1583.

[16] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) **52** (1950), 518–527.

[17] _____, *Some self-dual properties of normed linear spaces*, Symposium on Infinite-Dimensional Topology (Louisiana State Univ., Baton Rouge, La., 1967), 1972, pp. 159–175. Ann. of Math. Studies, No. 69.

[18] W. B. Johnson and N. L. Randrianarivony, *l_p $(p > 2)$ does not coarsely embed into a Hilbert space*, Proc. Amer. Math. Soc. **134** (2006), 1045–1050.

[19] N. J. Kalton, *Coarse and uniform embeddings into reflexive spaces*, Quart. J. Math. (Oxford) **58** (2007), 393–414.

[20] _____, *Lipschitz and uniform embeddings into ℓ_∞*, Fund. Math. **212** (2011), no. 1, 53–69.

[21] N. J. Kalton and N. L. Randrianarivony, *The coarse Lipschitz structure of $\ell_p \oplus \ell_q$*, Math. Ann. **341** (2008), 223–237.

[22] N. J. Kalton and D. Werner, *Property (M), M-ideals, and almost isometric structure of Banach spaces*, J. Reine Angew. Math. **461** (1995), 137–178.

[23] G. Kasparov and G. Yu, *The coarse geometric Novikov conjecture and uniform convexity*, Adv. Math. **206** (2006), 1–56.

[24] M. Mendel and A. Naor, *Metric cotype*, Ann. of Math.(2) **168** (2008), 247–298.

[25] A. Naor, *An introduction to the Ribe program*, Jpn. J. Math. **7** (2012), no. 2, 167–233.

[26] _____, *Discrete Riesz transforms and sharp metric X_p inequalities*, Ann. of Math. (2) **184** (2016), no. 3, 991–1016.

[27] A. Naor and G. Schechtman, *Pythagorean powers of hypercubes*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 3, 1093–1116.

[28] _____, *Metric X_p inequalities*, Forum Math. Pi **4** (2016), e3, 81pp.

[29] Nowak P. W., *On coarse embeddability into l_p-spaces and a conjecture of Dranishnikov*, Fund. Math. **189** (2006), no. 2, 111–116.

[30] P. Nowak and G. Yu, *Large scale geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2012.

[31] E. Odell and T. Schlumprecht, *The distortion problem*, Acta Math. **173** (1994), 259–281.

[32] E. Odell, Th. Schlumprecht, and A. Zsák, *Banach spaces of bounded Szlenk index*, Studia Math. **183** (2007), no. 1, 63–97.

[33] M. I. Ostrovskii, *Coarse embeddability into Banach spaces*, Topology Proc. **33** (2009), 163–183.

[34] _____, *Metric embeddings*, De Gruyter Studies in Mathematics, vol. 49, De Gruyter, Berlin, 2013. Bilipschitz and coarse embeddings into Banach spaces.

[35] M. Ribe, *Existence of separable uniformly homeomorphic nonisomorphic Banach spaces*, Israel J. Math. **48** (1984), 139–147.

[36] J. Rosenberg, *Novikov's conjecture*, Springer, [Cham], 2016.

[37] B. S. Tsirel'son, *It is impossible to imbed l_p of c_0 into an arbitrary Banach space*, Funkcional. Anal. i Priložen. **8** (1974), 57–60 (Russian).

[38] A. Valette, *Introduction to the Baum-Connes conjecture*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.

[39] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), 201–240.

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