A SHORT COURSE ON NON LINEAR GEOMETRY OF BANACH SPACES

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Abstract. In this course we show how some linear properties of Banach spaces, in particular properties related to the asymptotic structure of Banach spaces, are stable under coarse-Lipschitz embedddings or uniform homeomorphisms. We will focus on the recent use of some fundamental metric graphs or trees in the subject.

Foreword.

These notes are based on a series of five lectures given at the Winter school on "Functional and Harmonic Analysis" organized in Lens (France) in December 2010. I am glad to thank S. Grivaux and D. Li for the excellent organization of this school and for giving me the opportunity to present this series of lectures. I also would like to thank the participants and especially the students and young researchers for the very friendly and mathematically intense atmosphere they helped to create during this week.

A great part of this course will be based on Nigel Kalton's work. Unfortunately, Nigel Kalton passed away in August 2010. This is a terrible loss for his family, his friends and for Mathematics. One of the goals of this course is to give a flavour of some of the wonderful ideas he has left for the researchers in this field.

1. Introduction

The fundamental problem in non linear Banach space theory is to describe how the linear structure of a Banach space is (or is not) determined by its linear structure. In other words, we try to exhibit the linear properties of Banach spaces that are stable under some particular non linear maps. These non linear maps can be of very different nature: Lipschitz isomorphisms or embeddings, uniform homeomorphisms, uniform or coarse embeddings.

It is often said that the birth of the subject coincides with the famous theorem by Mazur and Ulam [34] in 1932 who showed that any onto isometry between two normed spaces is affine. Much later, another very important event for this area was the publication in 2000 of the authoritative book by Benyamini and Lindenstrauss [7]. Since then, there has been a lot of progress in various directions of this field.

In this series of lectures we will concentrate on uniform homeomorphisms and more generally on coarse-Lipschitz embeddings between Banach spaces (roughly speaking, a coarse-Lipschitz embedding is a map which is bi-Lipschitz for very large distances).

In section 2 we will very shortly visit the most important results on isometric embeddings. Section 3 is devoted to Lipschitz embeddings and isomorphisms. This subject is extremely vast and nicely exposed in the book by Benyamini and Lindenstrauss [7]. We will just recall the results that will be needed for the sequel such as the differentiability properties of Lipschitz maps and the applications to the Lipschitz classification of Banach spaces. Section 4 is the heart of this course. We will study uniform homeomorphisms and coarse-Lipschitz embeddings between Banach spaces. Our approach will be based on the use of particular graph metrics on the k-subsets of $\mathbb{N}$. These tools have been developed in the last few years by N.J. Kalton. We will see how to use them in order to obtain older results, such as the celebrated theorem by Johnson, Lindenstrauss and Schechtman [22] on the uniqueness of the uniform structure of ℓ_p for $1 < p < \infty$. We will also show how they yield new results on the stability of the asymptotic structure of Banach spaces. In section 5, we consider a few universality problems. We recall Aharoni's theorem on the universality of c_0 for separable metric spaces and Lipschitz embeddings. We address the question of its converse and a few variants of this problem. We study the Banach spaces that are universal for locally finite metric spaces and Lipschitz embeddings. We also focus on a theorem of Kalton, asserting that a Banach space universal for separable metric spaces and coarse embeddings cannot be reflexive. Finally, in section 6, we give a few examples of linear properties that can be characterized by purely metric conditions. We detail the proof of a recent characterization, due to Johnson and Schechtman [24], of super-reflexivity through the Lipschitz embeddability of diamond graphs.

2. Isometries

We begin this course with a quick review of the proof of Mazur-Ulam's theorem [34]. This will allow us to use for the first time the notion of metric midpoints whose variants will be very useful in the sequel.

Theorem 2.1. **(Mazur-Ulam 1932)** *Let X and Y be two real normed spaces and suppose that $f : X \to Y$ is a surjective isometry. Then f is affine.*

Proof. It is clearly enough to show that for any $x, y \in X$, $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$.
So let us fix $x, y \in X$ and define

$$K_0 = Mid(x,y) = \{u \in X,\ \|u-x\| = \|u-y\| = \frac{1}{2}\|x-y\|\},$$

the set of the so-called *metric midpoints* of x and y. Then we build by induction

$$K_{n+1} = \{u \in K_n,\ K_n \subset B(u, \frac{\text{diam }(K_n)}{2})\},$$

It is then easy to show by induction that $\frac{x+y}{2} \in K_n$ and that K_n is symmetric about $\frac{x+y}{2}$. Since diam (K_n) tends to 0, we obtain that $\cap_{n=0}^{\infty} K_n = \{\frac{x+y}{2}\}$.
On the other hand, starting, with $F_0 = Mid(f(x), f(y))$, we define similarly

$$F_{n+1} = \{v \in K_n,\ F_n \subset B(v, \frac{\text{diam }(F_n))}{2}\}.$$

Since the sets K_n and F_n are defined in purely metric terms and f is an isometry, we have that $f(K_n) = F_n$, for all $n \geq 0$. This yields our conclusion.

□

We finish this very short section by mentioning an important recent result by G. Godefroy and N.J. Kalton [15] on isometries.

Theorem 2.2. (Godefroy-Kalton 2003) *Let X and Y be separable Banach spaces and suppose that $f : X \to Y$ is an <u>into</u> isometry, then X is linearly isometric to a subspace of Y.*

Let us just give a few indications on the scheme of the proof. Clearly, we may assume that Y is equal to the closed linear span of $f(X)$. The first step is due to Figiel in [13] who proved that there exists a linear quotient map $Q : Y \to X$ such that $\|Q\| = 1$ and $Qf = I_X$. Then, 35 years later Godefroy and Kalton proved that if X is a separable Banach space and if $Q : Y \to X$ is a continuous linear quotient map with a Lipschitz lifting f (that is $Qf = I_X$), then Q admits a linear lifting L, with $\|L\| = Lip(f)$. The map L is the desired linear isometric embedding.

Remarks.
(a) Of course, the map f does not need to be linear. For instance $f(t) = (t, \sin t)$ is an isometric embedding from $\mathbb{R}$ into $\ell_\infty^2 = (\mathbb{R}^2, \| \, \|_\infty)$.
(b) The above result is false in the non separable case. It is also proved in [15] that ℓ_∞ is isometric to a subset of the free space $\mathcal{F}(\ell_\infty)$ but not isomorphic to any subspace of $\mathcal{F}(\ell_\infty)$.

3. Lipschitz embeddings and isomorphisms

We start with the following

Definition 3.1. Let (M, d) and (N, δ) be two metric spaces. The map $f : M \to N$ is a *Lipschitz embedding* if

$$\exists A, B > 0 \;\; \forall x, y \in M \;\;\; Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

We denote $M \overset{Lip}{\hookrightarrow} N$.
If $C \geq B/A$, we shall write $M \overset{C}{\hookrightarrow} N$.
If moreover f is onto we will say that it is a *Lipschitz isomorphism* and denote $M \overset{Lip}{\sim} N$.

The subject of this section is very vast. We will just summarize the classical results that we need for the sequel of this course. We refer the reader to the book [7] for a complete exposition.

3.1. **Differentiability results.** The first natural idea in order to get a linear map from a Lipschitz map is to use differentiation. It is indeed well known that a Lipschitz map between finite dimensional real normed spaces is almost everywhere differentiable. It is no longer true when the target space is infinite dimensional. This justifies the following definition.

Definition 3.2. A Banach space Y is said to have the Radon-Nikodỳm Property (RNP) if every Lipschitz map $f : \mathbb{R} \to Y$ is a.e. differentiable.

Let us just mention that separable dual spaces and reflexive spaces have RNP (this is due to Gelfand [14]). Notice also that the spaces c_0 and L^1 do not have (RNP). This last remak can easily be checked by considering the maps $f : \mathbb{R} \to c_0$ and $g : \mathbb{R} \to L^1$ defined by $f(t) = (e^{int}/n)_{n=1}^{\infty}$ and $g(t) = \mathbb{1}_{[0,t]}$.

The next obstacle is the lack of a Haar measure on infinite dimensional Banach spaces. This difficulty was overcome by various authors in the seventies, by defining a suitable notion of negligible set in this setting. Let us mention one of them, without further explanation. A Borel subset A of a separable Banach space X is said to be *Gauss null* if $\mu(A) = 0$, for every non degenerate Gaussian measure on X. We refer the reader to [7] (chapter 6) for all the details. Let us just point the two main features of this notion: it is stable under countable unions and X is not Gauss-null. We can now state the following fundamental differentiability result.

Theorem 3.3. (Aronszajn, Christensen, Mankiewicz 1970's) *Let X be a separable Banach space, Y a Banach space with the (RNP) and $f : X \to Y$ a Lipschitz map. Then there is Gauss-null subset A of X such that f is Gâteaux-differentiable at every point of $X \setminus A$.*

We can immediately deduce

Corollary 3.4. *Let X be a separable Banach space and Y a Banach space with the (RNP) such that $X \overset{Lip}{\hookrightarrow} Y$. Then X is linearly isomorphic to a subspace of Y.*
In particular, if $X \overset{Lip}{\hookrightarrow} \ell_2$ and X is infinite dimensional, then X is linearly isomorphic to ℓ_2 (we denote $X \simeq \ell_2$).

By using weak*- Gâteaux differentiability and the notion of Gauss-null sets, Heinrich and Mankiewicz [19] were then able to prove the following.

Theorem 3.5. (Heinrich, Mankiewicz 1982) *Let X be a separable Banach space and Z be a Banach space. Assume that $X \overset{Lip}{\hookrightarrow} Z^*$, then X is linearly isomorphic to a subspace of Z^*. In particular, if $X \overset{Lip}{\hookrightarrow} Y$, then X is linearly isomorphic to a subspace of Y^{**}.*

Remark. As we will recall later, a theorem by I. Aharoni [1] asserts that every separable metric space Lipschitz embeds into c_0. However, not every separable Banach space is linearly isomorphic to a subspace of c_0.

3.2. **The use of complemented subspaces.** For most of the Banach spaces besides ℓ_2, the structure of their subspaces is extremely complicated and the above differentiability results are clearly not sufficient. However, their complemented subspaces are often much simpler. Therefore, the next natural idea is to find conditions under which two Lipschitz isomorphic spaces are complemented in each other. This was initiated by Lindenstrauss in [32] and developed by Heinrich and Mankiewicz in [19]. Here are the statements that we shall use.

Theorem 3.6. (Lindenstrauss 1964) *Let Y be a closed subspace of a Banach space X. Assume that Y is complemented in Y^{**} and that there is a Lipschitz retraction from X onto Y. Then Y is complemented in X.*

Theorem 3.7. (Heinrich, Mankiewicz 1982)
(i) Let X and Y two Banach spaces and Y_0 be a separable complemented subspace of Y. Assume that X has (RNP), Y is complemented in Y^{**} and that $X \overset{Lip}{\sim} Y$. Then Y_0 is isomorphic to a complemented subspace of X (we denote $Y_0 \subseteq_c X$).

In particular

(ii) Let X, Y be two separable Banach spaces with (RNP) such that $X \overset{Lip}{\sim} Y$ and Y is complemented in Y^{**}. Then $Y \subseteq_c X$.

and

(iii) Let X, Y be two separable reflexive Banach spaces such that $X \overset{Lip}{\sim} Y$. Then $Y \subseteq_c X$ and $X \subseteq_c Y$.

It is now quite easy to deduce the following

Theorem 3.8. *Let X be a Banach space and $1 < p < +\infty$.*
(i) If $X \overset{Lip}{\sim} \ell_p$, then $X \simeq \ell_p$.
(ii) If $X \overset{Lip}{\sim} L_p = L_p([0,1])$, then $X \simeq L_p$.

Proof. (i) It is clear that X is infinite dimensional and it follows from the previous theorem that $X \subseteq_c \ell_p$. Then a classical result of Pełczyński [37] insures that $X \simeq \ell_p$.

(ii) It is not true that a complemented subspace of L_p is isomorphic to L_p. So let use that there are Banach spaces X_1 and Y_1 such that $L_p \simeq X \oplus X_1$ and $X \simeq L_p \oplus Y_1$. First notice that $L_p \oplus L_p \simeq L_p$. So $L_p \oplus X \simeq L_p \oplus L_p \oplus Y_1 \simeq L_p \oplus Y_1 \simeq X$.
On the other hand, $L_p \simeq \ell_p(L_p)$.
So $L_p \oplus X \simeq \ell_p(X \oplus X_1) \oplus X \simeq X \oplus \ell_p(X) \oplus \ell_p(X_1) \simeq \ell_p(X) \oplus \ell_p(X_1) \simeq L_p$. The proof is finished. This last trick, is known as the "Pełczyński" decomposition method. □

3.3. **Other results and main open questions.**

First, we need to say that it is not true that two Lipschitz isomorphic Banach spaces are always linearly isomorphic. Indeed

Theorem 3.9. (Aharoni, Lindenstrauss [2] 1978) *There exist an uncountable set Γ and a Banach space X such that $X \overset{Lip}{\sim} c_0(\Gamma)$ but X is not linearly isomorphic to any subspace of $c_0(\Gamma)$.*

However, the following was proved in [16]

Theorem 3.10. *Let X be a Banach space such that $X \overset{Lip}{\sim} c_0$. Then $X \simeq c_0$.*

We end this section with two major open questions.

(1) Assume that X, Y are two separable Banach spaces such that $X \overset{Lip}{\sim} Y$. Does this imply that $X \simeq Y$?

(2) Assume that X is a Banach space and $X \overset{Lip}{\sim} \ell_1$. Does this imply that $X \simeq \ell_1$?

4. Uniform homeomorphisms, coarse Lipschitz embeddings and asymptotic structure of Banach spaces

4.1. **Notation - Introduction.**

Definition 4.1. Let (M,d) and (N,δ) be two metric spaces and $f: M \to N$.
(a) f is a *uniform homeomorphism* if f is a bijection and f and f^{-1} are uniformly continuous (we denote $M \overset{UH}{\sim} N$).
(b) If (M,d) is unbounded, we define

$$\forall s > 0 \quad Lip_s(f) = \sup\{\frac{\delta((f(x),f(y))}{d(x,y)},\ d(x,y) \geq s\} \text{ and } Lip_\infty(f) = \inf_{s>0} Lip_s(f).$$

f is said to be *coarse Lipschitz* if $Lip_\infty(f) < \infty$.
(c) f is a *coarse Lipschitz embedding* if there exist $\theta, A, B > 0$ such that

$$\forall x,y \in M \quad d(x,y) \geq \theta \Rightarrow Ad(x,y) \leq \delta(f(x),f(y)) \leq Bd(x,y).$$

We denote $M \overset{CL}{\hookrightarrow} N$.

Remark. If M is a normed space and the following implication is satisfied:

$$\|x-y\| \leq \eta \Rightarrow \delta(f(x),f(y)) \leq \varepsilon.$$

Then

$$\|x-y\| \geq \eta \Rightarrow \delta(f(x),f(y)) \leq \varepsilon(\frac{\|x-y\|}{\eta}+1) \leq \frac{2\varepsilon}{\eta}\|x-y\|.$$

Therefore a uniformly continuous map defined on a normed space is coarse Lipschitz and a uniform homeomorphism between normed spaces is a bi-coarse Lipschitz bijection.

In the sequel we will study the uniform classification of Banach spaces. The main question being whether $X \overset{UH}{\sim} Y$ implies $X \simeq Y$? The general answer is negative, even in the separable case, as shown by the following result.

Theorem 4.2. (Ribe [39] **1984)** *Let $(p_n)_{n=1}^\infty$ in $(1,+\infty)$ be a strictly decreasing sequence such that $\lim p_n = 1$. Denote $X = (\sum_{n=1}^\infty \ell_{p_n})_{\ell_2}$. Then $X \overset{UH}{\sim} X \oplus \ell_1$.*

Note that X is reflexive, while $X \oplus \ell_1$ is not. Therefore reflexivity is not preserved under uniform homeomorphisms or coarse Lipschitz embeddings.

The local properties of a Banach space can be roughly described as the properties determined by its finite dimensional subspaces. The type and cotype of a Banach space or the super-reflexivity are typical examples of local properties of Banach spaces. The fact that local properties are preserved under coarse Lipschitz embeddings is essentially due to Ribe [40]. This is made precise in the following statement.

Theorem 4.3. *Let X and Y be two Banach spaces such that $X \overset{CL}{\hookrightarrow} Y$. Then there exists a constant $K \geq 1$ such that for any finite dimensional subspace E of X there is a finite dimensional subspace F of Y which is K-isomorphic to E.*

Proof. Instead of Ribe's original proof, we propose the modern argument using ultra-products as in [7].
So assume that $f : X \to Y$ is a coarse Lipschitz embedding and E is a finite dimensional subspace of X. Let $\mathcal{U}$ be a non principal ultrafilter on $\mathbb{N}$. To a bounded sequence $(x_n)_{n=1}^{\infty}$ in X, we associate the sequence $(f(x_n)/n)_{n=1}^{\infty}$ which is bounded in Y. This induces a Lipschitz embedding between the ultra-products $\Phi : X_{\mathcal{U}} \overset{C}{\hookrightarrow} Y_{\mathcal{U}}$. Then by Heinrich and Mankiewicz differentiation theorem, E is C-isomorphic to a subspace of $Y_{\mathcal{U}}^{**}$. We conclude by using the principle of local reflexivity and the fact that $Y_{\mathcal{U}}$ and Y have the same finite dimensional subspaces. □

Remark. By Kwapien's theorem, a Banach space is isomorphic to a Hilbert space if and only if it is of type 2 and cotype 2. Therefore a Banach space which coarse Lipschitz embeds in ℓ_2 is either finite dimensional or isomorphic to ℓ_2.

The next important result is the following.

Theorem 4.4. (Johnson, Lindenstrauss and Schechtman [22] **1996)**
Let $1 < p < \infty$ and X a Banach space such that $X \overset{UH}{\sim} \ell_p$. Then $X \simeq \ell_p$.

The original proof of this theorem was based on the so-called "Gorelik principle", that we shall not see in this course. However, we will show in detail how to obtain this result and improve it with other tools. The first one known as the "metric midpoints principle" is very classical and close in spirit to the proof of the Mazur-Ulam theorem. The other technique is due to N.J Kalton and based on the use of special graph metrics on the k-subsets of $\mathbb{N}$. The main objective of this course is to describe these recent tools and some of their implications. The end of this section is essentially taken from a paper by N.J. Kalton and N.L. Randrianarivony [31].

4.2. **The approximate midpoints principle.** Given a metric space X, two points $x, y \in X$, and $\delta > 0$, the approximate metric midpoint set between x and y with error δ is the set:

$$Mid(x, y, \delta) = \left\{ z \in X : \ \max\{d(x, z), d(y, z)\} \le (1+\delta)\frac{d(x, y)}{2} \right\}.$$

The use of metric midpoints in the study of nonlinear geometry is due to Enflo in an unpublished paper and has since been used elsewhere, e.g. [8], [18] and [22].

The following version of the midpoint Lemma was formulated in [31] (see also [7] Lemma 10.11).

Proposition 4.5. *Let X be a normed space and suppose M is a metric space. Let $f : X \to M$ be a coarse Lipschitz map. If $Lip_\infty(f) > 0$ then for any $t, \varepsilon > 0$ and any $0 < \delta < 1$ there exist $x, y \in X$ with $\|x - y\| > t$ and*

$$f(\mathrm{Mid}(x, y, \delta)) \subset \mathrm{Mid}(f(x), f(y), (1+\varepsilon)\delta).$$

Proof. Let $\varepsilon' > 0$. There exist $s > t$ such that $Lip_s(f) \le (1+\varepsilon')Lip_\infty(f)$. Then we can find $x, y \in X$ such that

$$\|x - y\| \ge \frac{2s}{1-\delta} \text{ and } \|f(x) - f(y)\| \ge \frac{1}{1+\varepsilon'} Lip_\infty(f)\|x-y\| \ge \frac{1}{(1+\varepsilon')^2} Lip_s(f)\|x-y\|.$$

Let now $u \in Mid(x,y,\delta)$. We have that $\|y-u\| \geq \frac{1-\delta}{2}\|x-y\| \geq s$. Therefore

$$\|f(y)-f(u)\| \leq Lip_s(f)\|y-u\| \leq Lip_s(f)\frac{1+\delta}{2}\|x-y\| \leq (1+\varepsilon')^2\frac{1+\delta}{2}\|f(x)-f(y)\|.$$

The same is true for $\|f(x)-f(u)\|$ and a choice of ε' small enough yields the conclusion. $\square$

In view of this proposition, it is natural to study the approximate metric midpoints in ℓ_p. This is done in the next lemma, which can be found in [31].

Lemma 4.6. *Let $1 \leq p < \infty$. We denote $(e_i)_{i=1}^\infty$ the canonical basis of ℓ_p and for $N \in \mathbb{N}$, E_N is the closed linear span of $\{e_i,\ i > n\}$. Let now $x, y \in \ell_p$, $\delta \in (0,1)$, $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$. Then*

(i) There exists $N \in \mathbb{N}$ such that $u + \delta^{1/p}\|v\|B_{E_N} \subset Mid(x,y,\delta)$.

(ii) There is a compact subset K of ℓ_p such that $Mid(x,y,\delta) \subset K + 2\delta^{1/p}\|v\|B_{\ell_p}$.

Proof. Fix $\nu > 0$. Let $N \in \mathbb{N}$ such that $\sum_{i=1}^N |v_i|^p \geq (1-\nu^p)\|v\|_p^p$.
(i) We may clearly assume that $p > 1$. Let now $z \in E_N$ so that $\|z\|^p \leq \delta\|v\|^p$. Then $\|x-(u+z)\|^p = \|v-z\|^p \leq \|v\|^p + (\|z\| + \nu\|v\|)^p \leq (1+\delta)^p\|v\|^p$, if ν was chosen small enough. The computation is the same for $\|y-(u+z)\| = \|v+z\|$. So $u+z \in Mid(x,y,\delta)$.
(ii) Assume that $u+z \in Mid(x,y,\delta)$ and write $z = z' + z''$ with $z' \in F_N = sp\{e_i,\ i \leq N\}$ and $z'' \in E_N$.
Since $\|v-z\|, \|v+z\| \leq (1+\delta)\|v\|$, we get, by convexity that $\|z'\| \leq \|z\| \leq (1+\delta)\|v\|$. Therefore, $u+z'$ belongs to the compact set $K = u + (1+\delta)\|v\|B_{F_N}$.
It follows also from convexity that

$$\forall i \geq 1 \quad \max\{|v_i|^p, |z_i|^p\} \leq \frac{1}{2}(|v_i - z_i|^p + |v_i + z_i|^p).$$

Summing over all i's yields: $(1-\nu^p)\|v\|^p + \|z''\|^p \leq \frac{1}{2}(\|v-z\|^p + \|v+z\|^p)$. Therefore $\|z''\|^p \leq [(1+\delta)^p - (1-\nu^p)]\|v\|^p \leq 2^p\delta\|v\|^p$, if ν was carefully chosen small enough. $\square$

We can now combine Proposition 4.5 and Lemma 4.6 to obtain

Proposition 4.7. *Let $1 \leq p < q < \infty$ and $f : \ell_q \to \ell_p$ a coarse Lipschitz map. Then for any $t > 0$ and any $\varepsilon > 0$ there exist $u \in \ell_q$, $\tau > t$, $N \in \mathbb{N}$ and K a compact subset of ℓ_p such that*

$$f(u + \tau B_{E_N}) \subset K + \varepsilon\tau B_{\ell_p}.$$

Proof. If $Lip_\infty(f) = 0$, the conclusion is clear. So we assume that $Lip_\infty(f) > 0$. We choose a small $\delta > 0$ (to be detailed later). Then we choose s large enough so that $Lip_s(f) \leq 2Lip_\infty(f)$ (the actual choice of s will also be specified later). Then, by Proposition 4.5,

$$\exists\, x, y \in \ell_q \ \ \|x-y\| \geq s \ \text{ and } \ f(Mid(x,y,\delta)) \subset Mid(f(x), f(y), 2\delta).$$

Denote $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$ and $\tau = \delta^{1/q}\|v\|$. By Lemma 4.6, there exists $N \in \mathbb{N}$ such that $u + \tau B_{E_N} \subset Mid(x, y, \delta)$ and there exists a compact subset K of ℓ_p so that $Mid(f(x), f(y), 2\delta) \subset K + (2\delta)^{1/p}\|f(x) - f(y)\|B_{\ell_p}$. But

$$(2\delta)^{1/p}\|f(x) - f(y)\| \leq 2Lip_\infty(f)(2\delta)^{1/p}\|x - y\| = 4Lip_\infty(f)2^{1/p}\delta^{1/p-1/q}\tau \leq \varepsilon\tau,$$

if δ was chosen initially chosen small enough. Then an appropriate choice of a large s will ensure that $\tau \geq \frac{1}{2}\delta^{1/q}s > t$. This finishes the proof. □

As a simple consequence, we have

Corollary 4.8. *Let $1 \leq p < q < \infty$.*
Then ℓ_q does not coarse Lipschitz embed into ℓ_p.

Proof. Let $f : \ell_q \to \ell_p$ be a coarse Lipschitz map. With the notation of the previous Proposition, we can find a sequence $(u_n)_{n=1}^\infty$ in $u + \tau B_{E_N}$ such that $\|u_n - u_m\| \geq \tau$ for $n \neq m$. Then $f(u_n) = k_n + \varepsilon\tau v_n$ with $k_n \in K$ and $v_n \in B_{\ell_p}$. Since K is compact, by extracting a subsequence, we may assume that $\|f(u_n) - f(u_m)\| \leq 3\varepsilon\tau$. Since ε can be chosen arbitrarily small and τ arbitrarily large, it implies that f cannot be a coarse Lipschitz embedding. □

4.3. **Kalton-Randriarivony's graphs.** Our next step will be to prove the conclusion of Corollary 4.8 for $1 \leq q < p < \infty$. This will be less elementary and require the introduction of special metric graphs. The use of these graphs in this setting is due to N.J. Kalton and N.L. Randriarivony [31]

Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$. We denote

$$G_k(\mathbb{N}) = \{\overline{n} = (n_1, .., n_k),\ n_i \in \mathbb{M}\ \ n_1 < .. < n_k\}.$$

Then we equip $G_k(\mathbb{M})$ with the distance $d(\overline{n}, \overline{m}) = |\{j,\ n_j \neq m_j\}|$. The fundamental result of the whole section is an estimate of the minimal distortion of any Lipschitz embedding of $(G_k(\mathbb{N}), d)$ in an ℓ_p-like Banach space.

Theorem 4.9. (Kalton-Randriarivony 2008) *Let Y be a reflexive Banach space so that there exists $p \in (1, \infty)$ with the following property. If $y \in Y$ and $(y_n)_{n=1}^\infty$ is a weakly null sequence in Y, then*

$$\limsup \|y + y_n\|^p \leq \|y\|^p + \limsup \|y_n\|^p.$$

Assume now that $\mathbb{M}$ is an infinite subset of $\mathbb{N}$ and $f : G_k(\mathbb{M}) \to Y$ is a Lipschitz map. Then for any $\varepsilon > 0$, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ such that:

$$diam\ f(G_k(\mathbb{M}')) \leq 2Lip(f)k^{1/p} + \varepsilon.$$

Proof. We will prove by induction on k the following statement (H_k):
for any $f : G_k(\mathbb{M}) \to Y$ Lipschitz and any $\varepsilon > 0$, there exist an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ and $u \in Y$ so that:

$$\forall \overline{n} \in G_k(\mathbb{M}')\ \ \|f(\overline{n}) - u\| \leq Lip(f)k^{1/p} + \varepsilon.$$

Assume $k = 1$. By weak compactness, there is an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ and $u \in Y$ such that $f(n)$ tends weakly to u, as $n \to \infty$, $n \in \mathbb{M}_0$. It follows that

$$\forall n \in \mathbb{M}_0\ \ \|u - f(n)\| \leq \limsup_{m \in \mathbb{M}_0} \|f(m) - f(n)\| \leq Lip(f).$$

We then obtain (H_1) by taking a further subset $\mathbb{M}'$ of $\mathbb{M}_0$.

Assume that (H_{k-1}) is true and $f : G_k(\mathbb{M}) \to Y$ is Lipschitz and let $\varepsilon > 0$. Using again weak compactness, we can find an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ such that

$$\forall \overline{n} \in G_{k-1}(\mathbb{M}_0) \quad w - \lim_{n_k \in \mathbb{M}_0} f(\overline{n}, n_k) = g(\overline{n}) \in Y.$$

Clearly, the map $g : G_{k-1}(\mathbb{M}_0) \to Y$ satisfies $Lip(g) \leq Lip(f)$. Let $\eta > 0$, by the induction hypothesis we can find an infinite subset $\mathbb{M}_1$ of $\mathbb{M}_0$ and $u \in Y$ so that

$$\forall \overline{n} \in G_{k-1}(\mathbb{M}_1) \quad \|g(\overline{n}) - u\| \leq Lip(f)(k-1)^{1/p} + \eta.$$

Now,

$$\limsup_{n_k \in \mathbb{M}_1} \|u - f(\overline{n}, n_k)\|^p \leq \|u - g(\overline{n})\|^p + \limsup_{n_k \in \mathbb{M}_1} \|g(\overline{n}) - f(\overline{n}, n_k)\|^p$$

$$\leq (Lip(f)(k-1)^{1/p} + \eta)^p + Lip(f)^p.$$

It follows that

$$\limsup_{n_k \in \mathbb{M}_1} \|u - f(\overline{n}, n_k)\| \leq Lip(f)k^{1/p} + \frac{\varepsilon}{2},$$

if η was chosen small enough.

Finally we can use Ramsey's Theorem to obtain an infinite subset $\mathbb{M}'$ of $\mathbb{M}_1$ such that

$$\forall \overline{n}, \overline{m} \in G_k(\mathbb{M}') \quad \big|\|u - f(\overline{n})\| - \|u - f(\overline{m})\|\big| \leq \frac{\varepsilon}{2}.$$

This concludes the inductive proof of (H_k). □

Remarks.

(1) $Y = (\sum_{n=1}^{\infty} F_n)_{\ell_p}$, where the F_n's are finite dimensional is a typical example of a space satisfying the assumptions of Theorem 4.9.

(2) Reflexivity is an important assumption. Indeed, c_0 fulfills the other condition (actually for any p finite), but it is not difficult to check that all $(G_k(\mathbb{N}), d)$ Lipschitz embed into c_0 with a constant independent of k. This last fact can also be deduced from Aharoni's theorem (see [1]).

We are now in position to deduce the following.

Corollary 4.10. *Let $1 \leq q < p < \infty$.*
Then ℓ_q does not coarse Lipschitz embed into ℓ_p.

Proof. Suppose that $\ell_q \overset{CL}{\hookrightarrow} \ell_p$. Then, using homogeneity, we get the existence of $f : \ell_q \to \ell_p$ and $C \geq 1$ such that

$$\forall x, y \in \ell_q \quad \|x - y\|_q \geq 1 \Rightarrow \|x - y\|_q \leq \|f(x) - f(y)\|_p \leq C\|x - y\|_p \qquad (*)$$

Denote $(e_n)_{n=1}^{\infty}$ the canonical basis of ℓ_q. Consider the map $\varphi : G_k(\mathbb{N}) \to \ell_q$ defined by $\varphi(\overline{n}) = e_{n_1} + .. + e_{n_k}$. It is clear that $Lip(\varphi) \leq 2$. Besides, $\|\varphi(\overline{n}) - \varphi(\overline{m})\|_q \geq 1$ whenever $\overline{n} \neq \overline{m}$, so $Lip(f \circ \varphi) \leq 2C$. Then, by Theorem 4.9, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that diam $(f \circ \varphi)(G_k(\mathbb{M})) \leq 6Ck^{1/p}$. But diam $(\varphi(G_k(\mathbb{M})) = (2k)^{1/q}$. This is in contradiction with $(*)$, when k is large enough. □

Let us now indicate how to finish the proof of Theorem 4.4 on the uniqueness of the uniform structure of ℓ_p, for $1 < p < \infty$. We shall just reproduce the linear argument of [22].

Proof. Suppose that $X \overset{UH}{\sim} \ell_p$, with $1 < p < \infty$. We may assume that $p \neq 2$. Then the ultra-products $X_{\mathcal{U}}$ and $(\ell_p)_{\mathcal{U}}$ are Lipschitz isomorphic. In fact, $(\ell_p)_{\mathcal{U}}$ is isometric to some huge $L_p(\mu)$ space. since X is separable and $L_p(\mu)$ has (RNP), X is isomorphic to a subspace of $L_p(\mu)$ and therefore reflexive. Thus X is complemented in $X_{\mathcal{U}}$ and Theorem 3.7 implies that X is isomorphic to a complemented subspace of $L_p(\mu)$. Since X is separable, a classical argument yields that X is isomorphic to a complemented subspace of $L_p = L_p([0,1])$.
Now, it follows from corollaries 4.8 and 4.10 that X does not contain any isomorphic copy of ℓ_2. Then we can conclude with a classical result of Johnson and Odell [23] which asserts that any infinite dimensional complemented subspace of L_p that does not contain any isomorphic copy of ℓ_2 is isomorphic to ℓ_p. □

In fact, a lot more can be deduced from Theorem 4.9. The aim of the paper [31] was to prove the uniqueness of the uniform structure of $\ell_p \oplus \ell_p$. We will now try to explain this result. We start with the following improvement of corollaries 4.8 and 4.10.

Corollary 4.11. *Let $1 \leq p < q < \infty$. and $r \geq 1$ such that $r \notin \{p,q\}$.*
Then ℓ_r does not coarse Lipschitz embed into $\ell_p \oplus \ell_q$.

Proof. When $r > q$, the argument is based on a midpoint technique like in the proof of Corollary 4.8. If $r < p$, we mimic the proof of Corollary 4.10. So we assume that $1 \leq p < r < q < \infty$. This is the situation where the graph technique will provide the answer that the Gorelik principle did not.
So assume that $f : \ell_r \to \ell_p \oplus_\infty \ell_q$ is a map such that there exists $C \geq 1$ satisfying

$$\forall x, y \in \ell_r \quad \|x-y\|_r \geq 1 \Rightarrow \|x-y\|_r \leq \|f(x)-f(y)\| \leq C\|x-y\|_r \qquad (**)$$

The map f has two components: $f = (g,h)$. Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Denote $(e_n)_n$ the canonical basis of ℓ_r.
We start by applying the midpoint technique to the coarse Lipschitz map g and deduce from Proposition 4.7 that there exist $\tau > k$, $u \in \ell_r$, $N \in \mathbb{N}$ and K a compact subset of ℓ_p such that

$$g(u + \tau B_{E_N}) \subset K + \varepsilon\tau B_{\ell_p}.$$

Let $\mathbb{M} = \{n \in \mathbb{N},\ n > N\}$ and define $\varphi(\overline{n}) = u + \tau k^{-1/r}(e_{n_1} + .. + e_{n_k})$ for $\overline{n} \in G_k(\mathbb{M})$. Then we have that $(g \circ \varphi)(G_k(\mathbb{M})) \subset K + \varepsilon\tau B_{\ell_p}$. Thus, by Ramsey's theorem, there is an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ such that diam $(g \circ \varphi)(G_k(\mathbb{M}')) \leq 3\varepsilon\tau$.
We now turn to the graph technique in order to deal with the map h. We have that $\|\varphi(\overline{n}) - \varphi(\overline{m})\|_q \geq \tau k^{-1/r} \geq 1$ whenever $\overline{n} \neq \overline{m}$. So $Lip(h \circ \varphi) \leq Lip(f \circ \varphi) \leq 2k^{-1/r}\tau C$. Then, by Theorem 4.9, there exists an infinite subset $\mathbb{M}''$ of $\mathbb{M}'$ such that diam $(h \circ \varphi)(G_k(\mathbb{M}'')) \leq 6Ck^{1/q-1/r}\tau \leq \varepsilon\tau$, if k is big enough.
Finally we have that diam $(f \circ \varphi)(G_k(\mathbb{M}'')) \leq 3\varepsilon\tau$, while diam $\varphi(G_k(\mathbb{M}'')) \geq \tau$. This in contradiction with $(**)$ if $\varepsilon < 1/3$. □

We can now state and prove the main result of [31].

Theorem 4.12. *Let $1 < p < q < \infty$ such that $2 \notin \{p, q\}$.*
Assume that X is Banach space such that $X \overset{UH}{\sim} \ell_p \oplus \ell_q$. Then $X \simeq \ell_p \oplus \ell_q$.

Proof. The key point is to show that X does not contain any isomorphic copy of ℓ_2. This follows clearly from the above corollary. To conclude the proof, we need to use a few deep linear results. The cases $1 < p < q < 2$ and $2 < p < q$, were actually settled in [22], which is not a surprise after the remark made at the beginning of the proof of the previous corollary. So let us only explain the case $1 < p < 2 < q$.

Assume that $X \overset{UH}{\sim} \ell_p \oplus \ell_q = E$. Then $X_{\mathcal{U}} \overset{Lip}{\sim} E_{\mathcal{U}} = L_p(\mu) \oplus L_q(\mu)$. As in the proof of Theorem 4.4, we obtain that $X \subseteq_c L_p \oplus L_q$.
Since $\ell_2 \subsetneq X$ and $q > 2$, a theorem of W.B Johnson [21] insures that any bounded operator from X into L_q factors through ℓ_q. Then, it is not difficult to see that $X \subseteq_c L_p \oplus \ell_q$.
Then we notice that L_p and ℓ_q are totally incomparable, which means that they have no isomorphic infinite dimensional subspaces. We can now use a theorem of Èdelšteĭn and Wojtaszczyk [11] to obtain that $X \simeq F \oplus G$, with $F \subseteq_c L_p$ and $G \subseteq_c \ell_q$. First it follows from [37] that G is isomorphic to ℓ_q or is finite dimensional. On the other hand, we know that $\ell_2 \subsetneq F$, and by the Johnson-Odell theorem [23] F is isomorphic to ℓ_p or finite dimensional.
Summarizing, we have that X is isomorphic to ℓ_p, ℓ_q or $\ell_p \oplus \ell_q$. But we already know that ℓ_p and ℓ_q have unique uniform structure. Therefore X is isomorphic to $\ell_p \oplus \ell_q$.

□

Remarks.

(1) This result extends to finite sums of ℓ_p spaces. More precisely, if $1 < p_1 < .. < p_n < \infty$ are all different from 2, then $\ell_{p_1} \oplus ... \oplus \ell_{p_n}$ has a unique uniform structure.

(2) Let $1 < p < \infty$ and $p \neq 2$. It is unknown whether L_p or even $\ell_p \oplus \ell_2$ has a unique uniform structure.

In [31], other applications of Theorem 4.9 yield interesting partial results in this direction. Let us just state them without proof.

Theorem 4.13. *Let $1 < p < \infty$ and $p \neq 2$.*
Then $\ell_p(\ell_2)$ and therefore L_p do not coarse Lipschitz embed into $\ell_p \oplus \ell_2$.

4.4. **Asymptotic stucture of Banach spaces.** In this last paragraph of our central section, we will further explore the ideas and implications of Theorem 4.9. Our main purpose, will be to give an abstract version of it that will show some stability properties of the asymptotic structure of Banach spaces under coarse Lipschitz embeddings. First, we need a few definitions.

Definition 4.14. Let $(X, \| \ \|)$ be a Banach space and $t > 0$. We denote by B_X the closed unit ball of X and by S_X its unit sphere. For $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t,x,Y) = \sup_{y\in S_Y} \|x+ty\| - 1 \quad \text{and} \quad \overline{\delta}(t,x,Y) = \inf_{y\in S_Y} \|x+ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x\in S_X} \inf_{\dim(X/Y)<\infty} \overline{\rho}(t,x,Y) \quad \text{and} \quad \overline{\delta}_X(t) = \inf_{x\in S_X} \sup_{\dim(X/Y)<\infty} \overline{\delta}(t,x,Y).$$

The norm $\| \ \|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

It is said to be *asymptotically uniformly convex* (in short AUC) if

$$\forall t > 0 \quad \overline{\delta}_X(t) > 0.$$

These moduli have been first introduced by V. Milman in [35].

Remarks.

(1) If $X = (\sum_{n=1}^\infty F_n)_{\ell_p}$, $1 \le p < \infty$ and the F_n's are finite dimensional, then $\overline{\rho}_X(t) = \overline{\delta}_X(t) = (1+t^p)^{1/p} - 1$.

(2) For all $t \in (0,1)$, $\overline{\rho}_{c_0}(t) = 0$.

(3) The following consequence of the definition of $\overline{\rho}_X(t)$ will be useful: for any $x \in X \setminus \{0\}$ and any weakly null sequence in X

$$\limsup \|x + x_n\| \le \|x\|\Big(1 + \overline{\rho}_X\big(\frac{\limsup \|x_n\|}{\|x\|}\big)\Big).$$

This is clearly a general version of the assumption of Theorem 4.9.

We will start with a positive result on the stability of reflexivity under coarse Lipschitz embeddings. One should remember that Ribe's counterexample implies that this is not true in general. The following result appeared in [5].

Theorem 4.15. *Let X be a Banach space and Y be a reflexive Banach space with an equivalent AUS norm. Assume that X coarse Lipschitz embeds into Y. Then X is reflexive.*

Proof. We can clearly assume that X and Y are separable. Then it follows from the work of Odell and Schlumprecht (see [36]) that Y can be renormed in such a way that there exists $p \in (1,+\infty)$ so that

$$\limsup \|y + y_n\|^p \le \|y\|^p + \limsup \|y_n\|^p, \tag{4.1}$$

whenever $y \in Y$ and (y_n) is a weakly null sequence in Y.
Suppose now that X is a non reflexive Banach space and fix $\theta \in (0,1)$. Then, James' Theorem [21] insures the existence of a sequence $(x_n)_{n=1}^\infty$ in S_X and a sequence $(x_n^*)_{n=1}^\infty$ in S_{X^*} such that

$$x_n^*(x_i) = \theta \text{ if } n \le i \quad \text{and} \quad x_n^*(x_i) = 0 \text{ if } n > i.$$

In particular

$$\|x_{n_1} + .. + x_{n_k} - (x_{m_1} + .. + x_{m_k})\| \ge \theta k, \quad n_1 < .. < n_k < m_1 < .. < m_k. \tag{4.2}$$

Assume finally that $f : X \to Y$ is a map so that

$$\forall x, x' \in X \quad \|x - x'\| \geq \theta \Rightarrow \|x - x'\| \leq \|f(x) - f(x')\| \leq C\|x - x'\|. \tag{4.3}$$

For $k \in \mathbb{N}$ we consider $h : G_k(\mathbb{M}) \to X$ defined by $h(\overline{n}) = x_{n_1} + .. + x_{n_k}$. We have that $\|h(\overline{n}) - h(\overline{m})\| \geq \theta$, whenever $\overline{n} \neq \overline{m}$. Thus $Lip(f \circ h) \leq 2C$. Then Theorem 4.9 insures the existence of an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that diam $(f \circ h)(G_k(\mathbb{M})) \leq 6Ck^{1/p}$. This in contradiction with (4.2) and (4.3). Therefore X cannot coarse Lipschitz embed into Y. □

It is proved in [17] that the condition "having an equivalent AUS norm" is stable under uniform homeomorphisms. So we immediately deduce.

Corollary 4.16. *The class of all reflexive Banach spaces with an equivalent AUS norm is stable under uniform homeomorphisms.*

We shall now give a more abstract version of Theorem 4.9 as it is done in the last section of [31]. So let us consider a reflexive Banach space Y and denote by $\overline{\rho}_Y$ its modulus of asymptotic uniform smoothness. It is easily checked that $\overline{\rho}_Y$ is an Orlicz function. Then we define the Orlicz sequence space:

$$\ell_{\overline{\rho}_Y} = \{x \in \mathbb{R}^{\mathbb{N}},\ \exists r > 0 \ \ \sum_{n=1}^{\infty} \overline{\rho}_Y(\frac{|x_n|}{r}) < \infty\},$$

equipped with the norm

$$\|x\|_{\overline{\rho}_Y} = \inf\{r > 0,\ \sum_{n=1}^{\infty} \overline{\rho}_Y(\frac{|x_n|}{r}) \leq 1\}.$$

Fix now $a = (a_1, .., a_k)$ a sequence of non zero real numbers and define the following distance on $G_k(\mathbb{M})$, for $\mathbb{M}$ infinite subset of $\mathbb{N}$:

$$\forall \overline{n}, \overline{m} \in G_k(\mathbb{M}), \quad d_a(\overline{n}, \overline{m}) = \sum_{j,\ n_j \neq m_j} |a_j|.$$

At this stage, we should point out something that we have so far carefully hidden. The fact that the elements of $G_k(\mathbb{M})$ are ordered k-subsets of $\mathbb{M}$ has been totally useless until now. Indeed, the very same proof would work if we just consider the k-subsets of $\mathbb{M}$ with the distance $d'(A, B) = \frac{1}{2}|A \Delta B|$. Now, with the definition of d_a we clearly need to work with ordered k-subsets and we shall prove the following generalization of Theorem 4.9.

Theorem 4.17. (Kalton-Randriarivony 2008) *Let Y be a reflexive Banach space, $\mathbb{M}$ an infinite subset of $\mathbb{N}$ and $f : (G_k(\mathbb{M}), d_a) \to Y$ a Lipschitz map. Then for any $\varepsilon > 0$, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ such that:*

$$diam\ f(G_k(\mathbb{M}')) \leq 2eLip(f)\|a\|_{\overline{\rho}_Y} + \varepsilon.$$

Proof. Define the following norm on $\mathbb{R}^2$:
$N(\xi, \eta) = |\eta|$ if $\xi = 0$ and $N_2(\xi, \eta) = |\xi|(1 + \overline{\rho}_Y(\frac{|\eta|}{|\xi|})$ if $\xi \neq 0$. Then define by induction $N_k(\xi_1, .., \xi_k) = N_2(N_{k-1}(\xi_1, .., \xi_{k-1}), \xi_k)$ on $\mathbb{R}^k$.

(a) The first step is to prove that for any $\varepsilon > 0$, there exist an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ and $u \in Y$ so that:

$$\forall \overline{n} \in G_k(\mathbb{M}') \ \ \|f(\overline{n}) - u\| \le N_k(a_1, .., a_k)Lip(f) + \varepsilon.$$

For the argument just notice that for any $y \in Y$ and any weakly null sequence (y_n) in Y: $\limsup \|y + y_n\| \le N_2(\|y\|, \limsup \|y_n\|)$. Then just mimic the proof of Theorem 4.9.

(b) The conclusion then follows from the inequality: $N_k(a) \le e\|a\|_{\overline{\rho}_Y}$.
Indeed, let a so that $\|a\|_{\overline{\rho}_Y} \le 1$ and assume as we may that $N_k(a) > 1$. Denote r the smallest integer in $\{1, .., k\}$ such that $N_r(a_1, .., a_r) > 1$. Then

$$\forall j > r \ \ N_j(a_1, .., a_j) \le N_{j-1}(a_1, .., a_{j-1})(1 + \overline{\rho}_Y(|a_j|)).$$

If $r > 1$, $N_r(a_1, .., a_r) \le N_2(1, a_r) = 1 + \overline{\rho}_Y(|a_r|)$ and if $r = 1$, $N_1(a_1) = |a_1| \le 1 + \overline{\rho}_Y(|a_1|)$. In both cases

$$N_k(a) \le \prod_{i=1}^{k}(1 + \overline{\rho}_Y(|a_i|),$$

which yields the conclusion. □

As it is described in [28] we can now derive the following.

Corollary 4.18. *Let X be a Banach space and Y be a reflexive Banach space. Assume that X coarse Lipschitz embeds into Y. Then there exists $C > 0$ such that for any normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X and any sequence $a = (a_1, .., a_k)$ of non zero real numbers, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that:*

$$\forall \overline{n} \in G_k(\mathbb{M}) \ \ \|\sum_{i=1}^{k} a_i x_{n_i}\| \le C\|a\|_{\overline{\rho}_Y}.$$

Proof. Let $f : X \to Y$ so that

$$\forall x, x' \in X \ \ \|x - x'\| \ge 1 \Rightarrow \|x - x'\| \le \|f(x) - f(x')\| \le C\|x - x'\|. \tag{4.4}$$

For $\lambda > 0$, we consider $h : (G_k(\mathbb{N}), d_a) \to X$ defined by $h(\overline{n}) = \lambda \sum_{i=1}^k a_i x_{n_i}$. We clearly have that $Lip(h) \le 2\lambda$. Notice that we may assume, by passing to a subsequence, that (x_n) is a basic sequence with basis constant at most 2. In particular, if $\overline{n} \neq \overline{m}$, $\|h(\overline{n}) - h(_m)\| \ge \frac{\lambda}{4} \min\{|a_i|, \ 1 \le i \le k\}$. So we can choose $\lambda > 0$ so that $\|h(\overline{n}) - h(_m)\| \ge 1$, whenever $\overline{n} \neq \overline{m}$. Then, $F = f \circ h$ is $2C\lambda$-Lipschitz. Then it follows from Theorem 4.17 that there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n}, \overline{m} \in \mathbb{M}, \text{ with } n_1 < .. < n_k < m_1 < .. < m_k \ \ \|F(\overline{n}) - F(\overline{m})\| \le 4Ce\lambda\|a\|_{\overline{\rho}_Y} + 1.$$

The left hand side of (4.4) yields

$$\forall \overline{n}, \overline{m} \in \mathbb{M} \text{ st } n_1 < .. < n_k < m_1 < .. < m_k \ \ \|\sum_{i=1}^{\infty} a_i x_{n_i} - \sum_{i=1}^{\infty} a_i x_{m_i}\| \le 4Ce\|a\|_{\overline{\rho}_Y} + \frac{1}{\lambda}.$$

Letting m_k,..,m_1 and then λ tend to ∞ we obtain

$$\forall n_1 < .. < n_k \in \mathbb{M} \quad \|\sum_{i=1}^{\infty} a_i x_{n_i}\| \le 4Ce\|a\|_{\overline{\rho}_Y}.$$

□

The above result can be rephrased in a more abstract way, by using the notion of spreading models. We shall not detail this generalization, but just mention this other statement (see [28] for details).

Corollary 4.19. *Let X be a Banach space and Y be a reflexive Banach space. Assume that X coarse Lipschitz embeds into Y. Then there exists $C > 0$ such that for any spreading model (e_i) of a normalized weakly null sequence in X and any finitely supported sequence $a = (a_i)$ in $\mathbb{R}$:*

$$\|\sum a_i e_i\|_S \le K\|a\|_{\overline{\rho}_Y}.$$

Remarks. In a recent preprint, N.J. Kalton [28] made a real breakthrough by proving some permanence properties of the asymptotic uniform convexity under coarse Lipschitz embeddings. It will be impossible for us to give in this course a fair idea of the proofs. Let us just mention the main result.

Theorem 4.20. (Kalton 2010) *Let X and Y be Banach spaces such that X coarse Lipschitz embeds into Y. Then there exists $C > 0$ such that for any spreading model (e_i) of a normalized weakly null sequence in X:*

$$\forall k \in \mathbb{N} \quad \|\sum_{i=1}^{k} e_i\|_{\overline{\delta}_Y} \le K\|\sum_{i=1}^{k} e_i\|_S.$$

In the same paper [28], this is used, together with new deep linear results to show the stability of the class of subspaces of ℓ_p ($1 < p < \infty$) under coarse Lipschitz embeddings, or of the class of quotients of ℓ_p ($1 < p < \infty$) under uniform homeomorphisms. It is also proved that for $1 < p, r < \infty$, the Banach space $(\sum_{n=1}^{\infty} \ell_r^n)_{\ell_p}$ has a unique uniform structure.

5. Universality questions

The general question addressed in this section is the following: given a class $\mathcal{M}$ of metric spaces and a type (E) of embedding, try to describe the Banach spaces X such that for any M in $\mathcal{M}$, there exists an embedding of type (E) from M into X. The classes of metric spaces that we shall consider are: $\mathcal{S}$ the class of all separable metric spaces, $\mathcal{K}$ the class of all compact metric spaces, $\mathcal{P}$ the class of all proper metric spaces (i.e. with relatively compact balls) and $\mathcal{LF}$ the class of all locally finite metric spaces (i.e. with finite balls). Concerning the embeddings, we will look at isometric, Lipschitz and coarse embeddings. We now need to define this last notion.

Definition 5.1. Let (M,d) and (N,δ) be two unbounded metric spaces.
A map $f: M \to N$ is said to be a *coarse embedding* if there exist two increasing functions $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ such that $\lim_\infty \rho_1 = +\infty$ and

$$\forall x,y \in M \quad \rho_1(d(x,y)) \leq \delta(f(x),f(y)) \leq \rho_2(d(x,y)).$$

We denote $M \overset{co}{\hookrightarrow} N$.

5.1. **Isometric embeddings.** This section will be very short. We first recall the fundamental result by S. Banach and S. Mazur.

Theorem 5.2. (Banach - Mazur 1933) *Any separable Banach space is linearly isometric to a subspace of $C([0,1])$. As a consequence, any separable metric space is isometric to a subset of $C([0,1])$.*

Let us also mention, that using the tools developed by G. Godefroy and N.J. Kalton in [15], Y. Dutrieux and the author ([10]) were able to show the following.

Theorem 5.3. *There exists a compact metric space K such that any Banach space containing an isometric copy of K has a subspace which is linearly isometric to $C([0,1])$.*

Remarks.
(a) In [42], A. Szankowski constructed a separable reflexive Banach space containing an isometric copy of every finite dimensional normed space. We thank V. Zizler for pointing out this result to us.
(b) Assume that a Banach space contains an isometric copy of any locally finite metric space. We do not know if it necessarily contains an isometric copy of $C([0,1])$.

5.2. **Lipschitz embeddings.** We start with the most important result of this subsection, due to Aharoni [1], which states that c_0 is universal for separable metric spaces and Lipschitz embeddings. More precisely:

Theorem 5.4. (Aharoni 1974) *There exists a universal constant $K \geq 1$ such that for any separable metric space K, we have $M \overset{K}{\hookrightarrow} c_0$.*

In fact, Aharoni proved that K can be taken such that $K \leq 6+\varepsilon$, for any $\varepsilon > 0$. He also showed that K cannot be taken less than 2 for $M = \ell_1$. The optimal quantitative result was obtained by N.J. Kalton and the author in [30] who proved the following.

Theorem 5.5. *Let (M,d) be a separable metric space. Then there exists $f: M \to c_0$ such that*

$$\forall x \neq y \in M \quad d(x,y) \leq \|f(x)-f(y)\|_\infty < 2d(x,y).$$

Open questions.
(a) Is the converse of Aharoni's theorem true? Namely if c_0 Lipschitz embeds into a Banach space X, does X admit a subspace linearly isomorphic to c_0?
(b) Similarly, if a Banach space is universal for Lipschitz embeddings and compact metric spaces (or proper metric spaces) does it admit a subspace linearly isomorphic to c_0?

We finish this paragraph on Lipschitz embeddings with a recent characterization, due to F. Baudier and the author [6] and G. Schechtman [41] of the Banach spaces that are universal for locally finite metric spaces.

Theorem 5.6. *Let X be a Banach space. The the following assertions are equivalent.*
(i) X has a trivial cotype.
(ii) There is a universal constant $K \geq 1$ such that for any locally finite metric space M: $M \overset{K}{\hookrightarrow} X$.

Proof. We will only sketch the proof, and use the following characterization of Banach spaces without cotype due to B. Maurey and G. Pisier [33]: there is a constant $K \geq 1$ (which can actually always be taken less than $1 + \varepsilon$, for any $\varepsilon > 0$) such that for any $n \in \mathbb{N}$ there is a n-dimensional subspace X_n of X and an isomorphism $T_n : \ell_\infty^n \to X_n$ with $\|T_n\| \, \|T_n^{-1}\| \leq K$.
(i)⇒(ii). Let M be a locally finite metric space. Fix x_0 in M and denote $B_n = B(x_0, 2^n)$ for $n \in \mathbb{N}$. Then B_n is finite and the map $\Phi_n : B_n \to \ell_\infty^{|B_n|}$ defined by

$$\forall x \in B_n \quad \Phi_n(x) = (d(x,y) - d(x_0,y))_{y \in B_n}$$

is an isometric embedding of B_n into $\ell_\infty^{|B_n|}$. It is a classical embedding known as the Fréchet-embedding.
Then we use the assumption *(i)* and a classical gliding hump argument to build a subspace Z of X with a finite dimensional Schauder decomposition: $Z = Z_1 \oplus .. \oplus Z_n \oplus ....$ so that for any $n \geq 1$ there is an isomorphism $T_n : \ell_\infty^{|B_n|} \to Z_n$, with $\|T_n\| \leq 2$ and $\|T_n^{-1}\| \leq 1$. Denote $\psi_n = T_n \circ \Phi_n$. We finally define $\Phi : M \to Z$ by embedding each set $B_n \setminus B_{n-1}$ in $Z_n \oplus Z_{n+1}$ as follows:

$$\forall x \in B_n \setminus B_{n-1}, \ \Phi(x) = \lambda(x)\psi_n(x) + (1 - \lambda(x))\psi_{n+1} \text{ where } \lambda(x) = \frac{2^{n+1} - d(x,x_0)}{2^n}.$$

(ii)⇒(i). The proof of this converse relies on an argument due to G. Schechtman [41]. Let us fix $n \in \mathbb{N}$. Then for any $k \in \mathbb{N}$, there exists a map $f_k : (\frac{1}{k}\mathbb{Z}^n, \| \ \|_\infty) \to X$ such that $f_k(0) = 0$ and

$$\forall x, y \in \frac{1}{k}\mathbb{Z}^n \quad \|x - y\|_\infty \leq \|f_k(x) - f_k(y)\| \leq K\|x - y\|_\infty.$$

Then we can define a map $\lambda_k : \ell_\infty^n \to \frac{1}{k}\mathbb{Z}^n$ such that for all $x \in \ell_\infty^n$: $\|\lambda_k(x) - x\|_\infty = d(x, \frac{1}{k}\mathbb{Z}^n)$. We can now set $\varphi_k = f_k \circ \lambda_k$.
Let $\mathcal{U}$ be a non trivial ultrafilter. We define $\varphi : \ell_\infty^n \to X_\mathcal{U} \subseteq X_\mathcal{U}^{**}$ by $\varphi(x) = (\varphi_k(x))_\mathcal{U}$. It is easy to check that φ is a Lipschitz embedding. Then it follows from Theorem 3.7 that ℓ_∞^n is K-isomorphic to a linear subspace of $X_\mathcal{U}^{**}$. Finally, using the local reflexivity principle and properties of the ultra-product, we get that ℓ_∞^n is $(K+1)$-isomorphic to a linear subspace of X.

□

5.3. **Coarse embeddings.** Very little is known about the coarse embeddings of metric spaces into Banach spaces. We will start by mentioning without proof a result of F. Baudier [3].

Theorem 5.7. *Let X be a Banach space without cotype. Then every proper metric space embed coarsely and uniformly into X.*

For some time it was not known if a reflexive Banach space could be universal for separable metric spaces and coarse embeddings. This section will be essentially devoted to explaining Kalton's solution for this question. This will enable us to introduce a new graph distance on $G_k(\mathbb{N})$. This is taken from the article [26] by N.J. Kalton.

Theorem 5.8. (Kalton 2007) *Let X be a separable Banach space. Assume that c_0 coarsely embeds into X. Then one of the iterated duals of X has to be non separable. In particular, X cannot be reflexive.*

Before to proceed with the proof of this theorem, we need to introduce a new graph metric on $G_k(\mathbb{M})$, for $\mathbb{M}$ infinite subset of $\mathbb{N}$. We will say that $\overline{n} \neq \overline{m} \in G_k(\mathbb{M})$ are adjacent (or $d(\overline{n},\overline{m}) = 1$) if they interlace or more precisely if

$$m_1 \leq n_1 \leq .. \leq m_k \leq n_k \ \text{ or } \ n_1 \leq m_1 \leq .. \leq n_k \leq m_k.$$

For simplicity we will only show that X cannot be reflexive. So let us assume that X is a reflexive Banach space and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = w - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Note that for $1 \leq i \leq k$, $\partial^i f$ is a bounded map from $G_{k-i}(\mathbb{N})$ into X and that $\partial^k f$ is an element of X. We first need to prove a series of simple lemmas about the operation ∂.

Lemma 5.9. *Let $h : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that*

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad |h(\overline{n}) - \partial^k h| < \varepsilon.$$

Proof. The set $\mathbb{M} = \{m_1, .., m_i, ..\}$ is built by induction on i so that for any subset $\overline{n}$ of$\{m_1, .., m_i\}$ with $1 \leq |\overline{n}| \leq \min(i, k)$, we have $|\partial^{k-|\overline{n}|} h(\overline{n}) - \partial^k h| < \varepsilon$.
For i, we easily pick m_1 such that $|\partial^{k-1} h(m_1) - \partial^k h| < \varepsilon$.
Assume now that $m_1, .., m_i$ have been constructed. Then for every $\overline{n} \subset \{m_1, .., m_i\}$ with $|\overline{n}| \leq k-1$, there is $\mathbb{M}_{\overline{n}} \in \mathcal{U}$ such that

$$\forall \overline{n} \in \mathbb{M}_{\overline{n}} \quad m > m_i \ \text{ and } \ |\partial^{k-|\overline{n}|-1} h(\overline{n}, m) - \partial^k h| < \varepsilon.$$

Then pick $m_{i+1} \in \mathbb{M} = \bigcap_{\overline{n}} \mathbb{M}_{\overline{n}}$, where $\overline{n}$ runs through the subsets of $\{m_1, .., m_i\}$ satisfying $|\overline{n}| \leq k-1$. □

The proof of the next Lemma is clear.

Lemma 5.10. *Let $f : G_k(\mathbb{N}) \to X$ and $g : G_k(\mathbb{M}) \to X^*$ be two bounded maps. Define $f \otimes g : G_{2k}(\mathbb{N}) \to \mathbb{R}$ by*

$$(f \otimes g)(n_1, .., n_{2k}) = \langle f(n_2, n_4, .., n_{2k}), g(n_1, .., n_{2k-1}) \rangle.$$

Then $\partial^2(f \otimes g) = \partial f \otimes \partial g$.

Lemma 5.11. *Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that*

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

Proof. For all $\overline{n} \in G_k(\mathbb{N})$, we can find $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n}) \rangle = \|f(\overline{n})\|$. By an iterated application of the previous lemma we get that

$$|\partial^{2k}(f \otimes g)| = |\langle \partial^k f, \partial^k g \rangle| \leq \|\partial^k f\|.$$

Then, by Lemma 5.9, there is an infinite subset $\mathbb{M}_0$ of $\mathbb{N}$ such that for all $\overline{p} \in G_{2k}(\mathbb{M}_0)$: $|(f \otimes g)(\overline{p})| \leq \|\partial^k f\| + \varepsilon$. Then write $\mathbb{M}_0 = \{n_1 < m_1 < .. < n_i < m_i < ..\}$ and set $\mathbb{M} = \{n_1 < n_2 < .. < n_i < ..\}$. Thus for all $\overline{n} = (n_1, .., n_k) \in G_k(\mathbb{M})$,

$$\|f(\overline{n})\| = \langle f(\overline{n}), g(\overline{n}) \rangle \leq |\langle f(m_1, .., m_k), g(n_1, .., n_k) \rangle| + \omega_f(1) \leq \|\partial^k f\| + \varepsilon + \omega_f(1).$$

□

The last preparatory lemma is the following.

Lemma 5.12. *Let $\varepsilon > 0$, X be a separable reflexive Banach space and I be an uncountable set. Assume that for each $i \in I$, $f_i : G_k(\mathbb{N}) \to X$ is a bounded map. Then there exist $i \neq j \in I$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that*

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|f_i(\overline{n}) - f_j(\overline{n})\| \leq \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.$$

Proof. Since X is separable and I uncountable, there exist $i \neq j \in I$ such that $\|\partial^k f_i - \partial^k f_j\| < \varepsilon/2$. Then we can apply Lemma 5.11 to $(f_i - f_j)$ to conclude. □

We are now ready for the proof of the theorem. As we will see, the proof relies on the fact that c_0 contains uncountably many isometric copies of the $G_k(\mathbb{N})$'s with too many points far away from each other (which will be in contradiction with Lemma 5.12).

Proof of Theorem 5.8. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_{k=1}^\infty$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \quad s_A(n) = \sum_{k \leq n,\ k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \quad f_A(\overline{n}) = \sum_{i=1}^k s_A(n_i).$$

Then the $h \circ f_A$'s form an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X. It therefore follows from Lemma 5.12 that there are two distinct infinite subsets A and B of $\mathbb{N}$ and another infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that:

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|h \circ f_A(\overline{n}) - h \circ f_B(\overline{n})\| \leq \omega_{h \circ f_A}(1) + \omega_{h \circ f_B}(1) + 1 \leq 2\omega_h(1) + 1.$$

But, since $A \neq B$, there is $\overline{n} \in G_k(\mathbb{M})$ with $\|f_A(\overline{n}) - f_B(\overline{n})\| = k$. By taking arbitrarily large values of k we deduce that h cannot be a coarse embedding. $\square$

Remarks.

(a) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

(b) It is now easy to adapt this proof in order to obtain the stronger result stated in Theorem 5.8. Indeed, one just has to change the definition of the operator ∂ as follows. If $f : G_k(\mathbb{N}) \to X$ is bounded, define $\partial f : G_{k-1}(\mathbb{N}) \to X^{**}$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = w^* - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

We leave it to the reader to rewrite the argument.

(c) On the other hand, N.J. Kalton proved in [25] that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. In particular, X does not contain any subspace linearly isomorphic to c_0.

(d) We conclude by mentioning that N.J. Kalton recently used the same graph distance on $G_k(\omega_1)$, where ω_1 is the first uncountable ordinal (see [29]). As a consequence he showed that the unit balls of ℓ_∞/c_0 or $C([0, \omega_1])$ do not uniformly embed into ℓ_∞. He also built a (non separable) Banach space X such that there is no uniform retract from X^{**} onto X.

6. Metric invariants

In this last section, we will try to characterize some linear classes of Banach spaces by a purely metric condition. The conditions we will consider will be of the following type. Given a metric space M, what are the Banach spaces X so that $M \overset{Lip}{\hookrightarrow} X$. Or, given a family $\mathcal{M}$ of metric spaces, what are the Banach spaces X for which there is a constant $C \geq 1$ so that for all M in $\mathcal{M}$, $M \overset{C}{\hookrightarrow} N$.

If the linear class of Banach spaces that can be characterized in such a way is already known to be stable under Lipschitz or coarse-Lipschitz embeddings, this can be seen as an improvement of this previous result. We will also show one situation where this process yields new results about such stabilities.

We will only consider two examples. First we shall review (without proof) the results about hyperbolic trees. Then we will prove in detail a recent characterization of super-reflexivity through the embedding of "diamond graphs".

6.1. **Trees.** We start with J. Bourgain's metric characterization of super-reflexivity given in [9]. The metric invariant discovered by Bourgain is the collection of the hyperbolic dyadic trees of arbitrarily large height N. If we denote $\Delta_0 = \{\emptyset\}$, the root of the tree. Let $\Omega_i = \{-1, 1\}^i$, $\Delta_N = \bigcup_{i=0}^N \Omega_i$ and $\Delta_\infty = \bigcup_{i=0}^\infty \Omega_i$. Then we equip Δ_∞, and by restriction every Δ_N, with the hyperbolic distance ρ, which is defined as follows. Let s and s' be two elements of Δ_∞ and let $u \in \Delta_\infty$ be their greatest common ancestor. We set

$$\rho(s,s') = |s| + |s'| - 2|u| = \rho(s,u) + \rho(s',u).$$

Bourgain's characterization is the following:

Theorem 6.1. (Bourgain 1986) *Let X be a Banach space. Then X is not superreflexive if and only if there exists a constant $C \geq 1$ such that for all $N \in \mathbb{N}$, $(\Delta_N, \rho) \overset{C}{\hookrightarrow} X$.*

Remarks. It has been proved in [4] that this is also equivalent to the metric embeddability of the infinite hyperbolic dyadic tree (Δ_∞, ρ). It should also be noted that in [9] and [4], the embedding constants are bounded above by a universal constant. We also recall that it follows from the Enflo-Pisier renorming theorem ([12] and [38]) that super-reflexivity is equivalent to the existence of an equivalent uniformly convex and (or) uniformly smooth norm.

Similarly , one can define for a positive integer N, $T_N = \bigcup_{i=0}^N \mathbb{N}^i$, where $\mathbb{N}^0 := \{\emptyset\}$. Then $T_\infty = \bigcup_{N=1}^\infty T_N$ is the set of all finite sequences of positive integers. Then the hyperbolic distance ρ is defined on T_∞ as previously. The following asymptotic analogue of Bourgain's theorem has been obtained by F. Baudier, N.J. Kalton and the author in [5].

Theorem 6.2. *Let X be a reflexive Banach space. The following assertions are equivalent.*

(i) There exists $C \geq 1$ such that $T_\infty \overset{C}{\hookrightarrow} X$.

(ii) There exists $C \geq 1$ such that for any N in $\mathbb{N}$, $T_N \overset{C}{\hookrightarrow} X$.

(iii) X does not admit any equivalent asymptotically uniformly smooth norm <u>or</u> X does not admit any equivalent asymptotically uniformly convex norm.

We will only mention one application of this result.

Corollary 6.3. *The class of all reflexive Banach spaces that admit an equivalent AUS norm and an equivalent AUC norm is stable under coarse Lipschitz embeddings.*

Proof. Assume that X coarse Lipschitz embeds in a space Y which is reflexive, AUS renormable and AUC renormable. First, it follows from Theorem 4.15 that X is reflexive. Assume now that X is not AUS renormable or not AUC renormable. Then, we know from Theorem 6.2 ($(iii) \Rightarrow (i)$) that T_∞ Lipschitz embeds into X and therefore into Y. This is in contradiction with $(i) \Rightarrow (iii)$ in Theorem 6.2. $\square$

Open questions.

(a) We do not know if the class of all reflexive and AUS renormable Banach spaces is stable under coarse Lipschitz embeddings.

(b) We do not know if the class of all Banach spaces that AUS renormable and AUC renormable is stable under coarse Lipschitz embeddings or uniform homeomorphisms.

6.2. **Diamonds.** in this very last paragraph we will detail a nice result By W.B Johnson and G. Schechtman [24] who recently characterized the super-reflexivity through the non embeddability of the so-called "diamond graphs". Let us start with an intuitive description of these graphs. D_0 is made of two connected vertices (therefore at distance 1), that we shall call T (top) and B (bottom). D_1 is a diamond, therefore made of four vertices T, B, L (left) and R (right) and four edges : $[B,L]$, $[L,T]$, $[T,R]$ and $[R,B]$. Assume D_N is constructed, then D_{N+1} is obtained by replacing each edge of D_N by a diamond D_1. The distance on D_{N+1} is the path metric of this new discrete graph. Throughout this section the graph distance on a diamond D_N will be denoted by d.

The result is the following

Theorem 6.4. *Let X be a Banach space. Then X is not super-reflexive if and only if there is a constant $C \geq 1$ such that for all $N \in \mathbb{N}$, $(D_N, d) \overset{C}{\hookrightarrow} X$.*

Proof. $(\Leftarrow)$: Suppose that X is super-reflexive. Then we may assume that its norm is uniformly convex. Namely, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that $\|(x+y)/2\| \leq 1 - \delta(\varepsilon)$ whenever $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$. We start with the following simple Lemma.

Lemma 6.5. *Let $f : D_1 \to X$ be an injective map such that $Lip(f) \leq M$ and $Lip(f^{-1}) \leq 1$. Then*

$$\|f(T) - f(B)\| \leq 2M\big(1 - \delta(\frac{2}{M})\big).$$

Proof. Without loss of generality, we may assume that $f(B) = 0$. We have that

$$\|\frac{f(L)}{M} - \frac{f(R)}{M}\| \geq \frac{2}{M}.$$

On the other hand

$$\|\frac{f(L)}{M}\|,\ \|\frac{f(R)}{M}\|,\ \|\frac{f(T)-f(R)}{M}\|, \|\frac{f(T)-f(L)}{M}\| \leq 1.$$

Therefore, by uniform convexity

$$\|\frac{f(L)+f(R)}{2M}\|,\ \|\frac{f(T)}{M} - \frac{f(L)+f(R)}{2M}\| \leq 1 - \delta(\frac{2}{M}).$$

Hence

$$\|f(T)\| \leq 2M\big(1 - \delta(\frac{2}{M})\big).$$

□

We now denote by $M_N = \inf\{M \geq 1,\ D_N \overset{M}{\hookrightarrow} X\}$. So, for any $M > M_N$ there is $f : D_N \to X$ with $Lip(f) \leq M$ and $Lip(f^{-1}) \leq 1$. By the previous Lemma, the distance between the images of the top and bottom points of a sub-diamond D_1 of D_N is at most $2M\big(1 - \delta(\frac{2}{M})\big)$. By construction, the set of all the top and bottom points of the the copies of D_1 in D_N make a doubled copy of D_{N-1}. Therefore

$$\forall M > M_N \quad 2M_{N-1} \leq 2M\big(1 - \delta(\frac{2}{M})\big), \quad \text{thus} \quad M_{N-1} \leq M_N\big(1 - \delta(\frac{2}{M_N})\big).$$

Assume now that the increasing sequence $(M_N)_N$ is bounded and denote μ its limit. We get that $\mu \leq \mu\big(1-\delta(\frac{2}{\mu})\big)$, which is impossible. This finishes the proof of the first implication.

$(\Rightarrow)$: Our first step will be to describe how to build inductively an isometric copy of D_N in ℓ_1. More precisely, we shall see it as a subset of $(\{0,1\}^{2^N}, \|\ \|_1)$. This could actually be taken as our definition of D_N. So D_0 is simply $\{0,1\}$. Assume that D_{N-1} is constructed as a subset of $(\{0,1\}^{2^{N-1}}, \|\ \|_1)$. Then we define an operation $\delta : D_{N-1} \to \{0,1\}^{2^N}$, by $\delta(a_1,..,a_{2^{N-1}}) = (a_1,a_1,a_2,a_2,..,a_{2^{N-1}},a_{2^{N-1}})$. By applying δ, we are just "'doubling" D_{N-1} and constructing the top and bottom points of the copies of D_1 in D_N. Now we have to introduce the left and right points of those D_1's. This will be done by noting that for any a, a' in D_{N-1} with $\|a-a'\|_1 = 1$ there are exactly two points in $\{0,1\}^{2^N}$ that are at distance 1 from $\delta(a)$ and $\delta(a')$. We add those points to $\delta(D_{N-1})$ to finish the construction of D_N.
Let us make some remarks on the D_N's. Let $1 \leq i \leq N$. If a and a' are adjacent in D_i, then $x = \delta^{N-i}(a)$ and $x' = \delta^{N-i}(a')$ belong to D_N and differ exactly on one interval of the form $I =](j-1)2^{N-i}, j2^{N-i}]$, on which x is constantly 0 and x' is constantly 1 (for instance). The set of vertices between x and x' (or equal to x and x' outside I) is an isometric copy of D_{N-i}, that will be called a sub-diamond D of level i of D_N. Let us now denote $I_0 =](j-1)2^{N-i}, (j-1)2^{N-i} + 2^{N-i-1}]$ and $I_1 =](j-1)2^{N-i} + 2^{N-i-1}, j2^{N-i}]$. We denote $v_T = x'$ and $v_B = x$ the top and bottom vertices of D (remember that $v_B = 0$ on I, $v_T = 1$ on I and $v_B = v_T$ elsewhere). Similarly, we denote v_L and v_R the left and right vertices of D. They can be described by $v_L = 0$ on I_0, $v_L = 1$ on I_1, $v_R = 1$ on I_0, $v_R = 0$ on I_1 and $v_R = v_L = v_T = v_B$ elsewhere.
Let $y \in D$. We will say that y is below the diagonal of D if $d(y,v_B) \leq d(y,v_T)$. We will say that y is on the left of D if $d(y,v_L) \leq d(y,v_R)$.

Assume now that X is not super-reflexive. Fix $N \in \mathbb{N}$ and $\theta \in (0,1)$. James' criterion insures the existence of $(x_i)_{i=1}^{2^N}$ in S_X and $(x_i^*)_{i=1}^{2^N}$ in S_{X^*} such that

$$x_n^*(x_i) = \theta,\ i \geq n \quad \text{and} \quad x_n^*(x_i) = 0,\ i < n.$$

Note that it follows from the above that
(i) For any subset I of $\{1,..,2^N\}$,

$$\|\sum_{i\in I} x_i\| \geq \theta |I|.$$

(ii) for any sub-interval I of $\{1,..,2^N\}$ and any $(a_i)_{i\in I} \subset \{0,1\}$,

$$\|\sum_{i=1}^{2^N} a_i x_i\| \geq \frac{\theta}{2}\sum_{i\in I} a_i.$$

Let now $f : D_N \to X$ defined by $f(a) = \sum_{i=1}^{2^N} a_i x_i$. We will show that this is a C-embedding, with C being a universal constant. Notice, that we have just replaced the canonical basis of $\ell_1^{2^N}$ by the sequence $(x_i)_{i=1}^{2^N}$.
So fix u and v in D_N.

Case 1. If there exists a geodesic path joining $B = (0, 0, .., 0)$ and $T = (1, 1, .., 1)$ and passing through u and v, then $\|f(u) - f(v)\| = \|\sum_{i\in I} x_i\|$ for some subset I of $\{1, .., 2^N\}$ such that $|I| = d(u, v)$. So we have that

$$\theta d(u, v) \leq \|f(u) - f(v)\| \leq d(u, v).$$

Case 2. Otherwise, there is a sub-diamond D of D_N (say of level k) such that (for instance) u is on the left of D and v is on the right of D. Let I be the sub-interval of size 2^{N-k} corresponding to D in our previous description. Write $I = I_0 \cup I_1$ as above. We denote again v_T, v_B, v_L and v_R the top, bottom, left and right vertices of D.

Case 2.1. Assume that u and v are below (for instance) the diagonal of D. Then $d(u, v) = d(u, v_B) + d(v, v_B)$. Moreover $u = 0$ on I_0 and $v = 0$ on I_1. So we have

$$\begin{aligned} \|f(u) - f(v)\| &= \|\sum_{i\in I_1} u_i x_i - \sum_{i\in I_0} v_i x_i\| \leq \|\sum_{i\in I_1} u_i x_i\| + \|\sum_{i\in I_0} v_i x_i\| \\ &\leq \sum_{i\in I_1} u_i + \sum_{i\in I_0} v_i = d(u, v_B) + d(v, v_B) = d(u, v). \end{aligned}$$

On the other hand,

$$\|f(u) - f(v)\| \geq \frac{\theta}{2} \max(\sum_{i\in I_1} u_i, \sum_{i\in I_0} v_i) \geq \frac{\theta}{4} d(u, v).$$

Case 2.2. Assume (for instance) that u is above and v below the diagonal of D. Then $2^{k-1} \leq d(u, v) \leq 2^k$. We also have that

$$\|f(u) - f(v)\| \leq 2^k \leq 2d(u, v)$$

and, since u is above the diagonal of D, $u = 1$ on I_1 and

$$\|f(u) - f(v)\| \geq \frac{\theta}{2} \sum_{i\in I_1} |u_i| = \theta 2^{k-2} \geq \frac{\theta}{4} d(u, v).$$

□

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