COMMENTS ON THE PAPER: N. J. KALTON, SPACES OF LIPSCHITZ AND HÖLDER FUNCTIONS AND THEIR APPLICATIONS, COLLECT. MATH. 55, 2 (2004), 171-217.

GILLES LANCIEN

We will try in these few pages to give an account of the ideas developed by Nigel Kalton in his paper on "Spaces of Lipschitz and Hölder functions and their applications".

Let M be a pointed metric space, that is, a metric space equipped with a distinguished point denoted 0. The space $\mathrm{Lip}_0(M)$ is the space of real-valued Lipschitz functions on M which vanish at 0, equipped with the norm

$$\|f\|_{Lip} = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)},\ x,y \in M,\ x \neq y\right\}.$$

The Dirac map $\delta_M : M \to \mathrm{Lip}_0(M)^*$ defined by $\langle f, \delta(x)\rangle = f(x)$ for $f \in \mathrm{Lip}_0(M)$ and $x \in M$ is an isometric embedding from M into $\mathrm{Lip}_0(M)^*$. The closed linear span of $\{\delta_M(x),\ x \in M\}$ is denoted $\mathcal{F}(M)$ and called the *Lipschitz-free space over* M (or free space in short). It follows from the compactness of the unit ball of $\mathrm{Lip}_0(M)$ with respect to the topology of pointwise convergence, that $\mathcal{F}(M)$ can be seen as a canonical predual of $\mathrm{Lip}_0(M)$ such that the weak*-topology induced by $\mathcal{F}(M)$ on $\mathrm{Lip}_0(M)$ coincides with the topology of pointwise convergence on the bounded subsets of $\mathrm{Lip}_0(M)$. One of the crucial properties of the free spaces is that, if we identify through the Dirac map a metric space with a subset of its free space, then any Lipschitz map between metric spaces extends to a linear map between the corresponding free spaces. Indeed if $L : M \to N$ is Lipschitz, then $f \mapsto f \circ L$ is the adjoint of a linear extension $\hat{L} : \mathcal{F}(M) \to \mathcal{F}(N)$ of L such that $\|\hat{L}\| = Lip(L)$.

The deep study of the Banach space geometry of $\mathcal{F}(M)$ had been initiated in a previous paper by Godefroy and Kalton ([9]). We need to start with a few words on this seminal work in which the authors concentrated on the study of $\mathcal{F}(X)$, when X is a Banach space. In that case, there exists a canonical contractive quotient map $\beta_X : \mathcal{F}(X) \to X$ such that $\beta_X \delta_X = Id_X$. One of the main results of [9] is that if X is a separable Banach space, then there exists a linear isometry

$T : X \to \mathcal{F}(X)$ such that $\beta_X T = Id_X$. Consequently, any linear quotient from a separable Banach space with a Lipschitz lifting admits a continuous linear lifting. Then, they were able to combine this result with a work of Figiel [5], to prove that, whenever a separable Banach space X isometrically embeds into another Banach space Y, there exists a linear isometric embedding from X into Y. Let us also mention that a completely elementary approach for the construction of this lifting operator can be found in [7]. Another important but probably less advertised feature of this paper is their study of the bounded approximation property (BAP for short) of free spaces. They obtained the striking result that a Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the λ-BAP. It follows that the BAP is stable under Lipschitz homeomorphisms.

In his paper "Spaces of Lipschitz and Hölder functions and their applications", the goal of Nigel Kalton is to develop the tools from [9] in order to study the uniform homeomorphisms between Banach spaces. We need to recall the main notation. The starting idea is to use a change of distance on a given metric space M. More precisely, a map $\omega : [0,\infty) \to [0,\infty)$ is a *gauge* if ω is sub-additive and $\lim_{t\to 0} \omega(t) = \omega(0) = 0$. It is said to be *non-trivial* if moreover $\lim_{t\to 0} \frac{\omega(t)}{t} = \infty$. Starting with a given metric space (M, d), it will be helpful to consider the associated metric space $(M, \omega \circ d)$. We denote $\mathrm{Lip}_\omega(M) = \mathrm{Lip}_0(M, \omega \circ d))$ and $\mathcal{F}_\omega(M) = \mathcal{F}(M, \omega \circ d)$. The example $\omega(t) = t^\alpha$ with $0 < \alpha < 1$ will be crucial in this paper. Then $\mathrm{Lip}^{(\alpha)}(M) = \mathrm{Lip}_0(M, d^\alpha))$ and $\mathcal{F}^{(\alpha)}(M) = \mathcal{F}(M, d^\alpha)$. The little Lipschitz space $\mathrm{lip}(M)$, which is defined to be the set of all $f \in \mathrm{Lip}_0(M)$ such that

$$\forall \varepsilon > 0 \; \exists \delta > 0 \;\; d(x, y) \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon d(x, y)$$

will also be important. We shall use the notation $\mathrm{lip}_\omega(M)$ for $\mathrm{lip}(M, \omega \circ d)$ and $\mathrm{lip}^{(\alpha)}(M)$ for $\mathrm{lip}(M, d^\alpha)$. Note that $\mathrm{lip}(M)$ is often trivial, but if $\mathrm{lip}(M)$ id a-norming for some $a \ge 1$ and M is compact, then $\mathrm{lip}(M)^*$ is isometric to $\mathcal{F}(M)$ (Theorem 6.1). This statement can be found in the book of N. Weaver [17]. As it is noticed in [8], this is a particular case of a result of Petunin and Plichko [16] asserting that if X is a separable Banach space and S is a closed separating subspace of X^* included in the set of norm attaining functionals, then S^* is isometric to X. In particular if M is compact and ω is a non-trivial gauge, then $\mathrm{lip}_\omega(M)^*$ is isometric to $\mathcal{F}_\omega(M)$.

We are now ready to describe some of the important results obtained by Nigel Kalton in [11].

Bonic, Frampton and Tromba proved in [1] that if M is a finite dimensional compact, then $\text{lip}^{(\alpha)}(M)$ is isomorphic to c_0. One of the main questions adressed by Nigel Kalton in this paper is to study the converse of this statement. After showing that for every compact metric space M, $\text{lip}(M)$ almost isometrically embeds into c_0 (Theorem 6.6), Kalton provides a clever example of a non doubling (and therefore not finite dimensional) compact metric space M such that $\text{lip}^{(\alpha)}(M)$ is isomorphic to c_0 (Proposition 6.8). However, the most impressive results are obtained in section 8, where it is shown that if K is a compact convex subset of a Hilbert space and $\alpha \in (0,1)$, then $\text{lip}^{(\alpha)}(K)$ is isomorphic to c_0 if and only if K is finite dimensional (Theorem 8.3). This result is even extended to compact convex subsets of a Banach space with non trivial type and to compact convex subsets of a general Banach space if $\alpha \leq \frac{1}{2}$. Let us try to give an overview of the tools used by Nigel Kalton in these proofs. We believe that it is a nice example of Nigel's powerful insight on these problems and of his mastery of the deepest tools in analysis. For a Banach space X, let us denote by $\gamma_1(X)$ the supremum over all finite dimensional subspaces E of X of $\|u\|\,\|v\|$, where $u : E \to \ell_1$ and $v : \ell_1 \to X$ are linear operators such that $vu = I_E$. The key estimate is the following:

$$\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_2^n})) \geq c\Big(\frac{n}{\log n}\Big)^{\alpha/2}.$$

This is based on an estimate of the 2-summing constant, $\pi_2(\beta)$, where $\beta : \mathcal{F}^{(\alpha)}(B_{\ell_2^n}) \to \ell_2^n$ is the quotient map introduced above admitting $\delta : B_{\ell_2^n} \to \mathcal{F}^{(\alpha)}(B_{\ell_2^n})$ as a uniformly continuous lifting. This estimate is obtained by using the Pietsch factorization Theorem for β and the concentration of measure phenomenon satisfied by the usual normalized measure on the sphere of ℓ_2^n. Then the estimate on $\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_2^n}))$ follows from Grothendieck's inequality. Consider now a compact convex subset K of ℓ_2 which is not finite dimensional. We may as well assume that the linear span of K is dense in ℓ_2. Kalton shows that there exists $c' > 0$ such that

$$\forall n \in \mathbb{N} \ \ \gamma_1(\mathcal{F}^{(\alpha)}(K)) \geq c'\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_2^n})),$$

and deduces from the above estimates that $\gamma_1(\mathcal{F}^{(\alpha)}(K)) = \infty$ and therefore that $\mathcal{F}^{(\alpha)}(K)$ is not a $\mathcal{L}^1$-space. In particular, $\text{lip}^{(\alpha)}(K)$ is not isomorphic to c_0. We leave it to the reader to discover the refinements used by Kalton in order to obtain this result for a general infinite dimensional compact convex set.

This short description was not intended to explain in a few lines Nigel's argument. We just wanted to illustrate his amazing ability to see the right tools to use in a context where they seem quite unexpected even for most specialists.

Let us now comment on sections 4 and 5 of this paper. The starting idea is to build an elementary decomposition of the free space of a general metric space M. In Lemma 4.2, Kalton builds a family of operators $(T_n)_{n\in\mathbb{Z}}$ acting on $\mathcal{F}(M)$ with the following properties: for x outside the annulus $B(0,2^{n+1})\setminus B(0,2^{n-1})$, $T_n(\delta(x))=0$ and for all $\gamma\in\mathcal{F}(M)$, $\sum_{n\in\mathbb{Z}}\|T_n\gamma\|_{\mathcal{F}(M)}\leq 72\|\gamma\|_{\mathcal{F}(M)}$. This family can be seen as a "wavelet -like" decomposition adapted to free spaces. Before addressing the main applications of this decomposition method that are presented in this paper, let us mention some other consequences. In Proposition 4.4, Kalton deduces that if M is a uniformly discrete metric space, then $\mathcal{F}(M)$ has the approximation property and the Radon-Nikodym property. It is still unknown whether every free space of a uniformly discrete metric space has the BAP. In [12], Nigel Kalton studied this question more deeply and developed the notion of approximable metric spaces. This is the following non-linear analogue of the BAP: a metric space M is *approximable* if the identity on M is the pointwise limit of an equi-uniformly continuous sequence of maps with relatively compact range. He showed that a Banach space X is approximable if and only if for any net N of X, $\mathcal{F}(N)$ has the BAP. Then he was able to prove that if X is a Banach space with separable dual, then X and X^* are both approximable. The following question is still open: does the free space of any uniformly discrete metric space have the BAP? A positive answer would imply that every separable Banach space is approximable. A negative answer would be at least as important since it would imply the existence of an equivalent norm on ℓ_1 without the metric approximation property. We invite the interested reader to study these developments in [12]. We wish also to mention that Kalton's simple and elegant decomposition given by Lemma 4.2 was one of the tools that Aude Dalet [3] recently used to prove that for any countable compact metric space M, $\mathcal{F}(M)$ has the metric approximation property.

Let us now come back to the main objective of this paper, namely the study of uniform homeomorphisms between Banach spaces. For that purpose, this decomposition is used in Theorem 4.6 to show that for any metric space M and any ω non-trivial gauge, $\mathcal{F}_\omega(M)$ is a Schur space. This will provide a general technique for producing pairs of Banach spaces (Y,Z) so that Y and Z are uniformly homeomorphic but not linearly isomorphic. Indeed, it is not difficult to show that $Y=\mathrm{Ker}\beta_X\oplus X$ is uniformly homemorphic to $Z=\mathcal{F}_\omega(X)$ if ω is such that $\omega(t)=t$ for $t\geq 1$. Then, as soon as X is not a Schur space and ω is non-trivial, Y and Z cannot be linearly isomorphic. Later in [13], Nigel Kalton pushed the idea of building a general machinery for constructing such counterexamples much further. Again, anyone interested in this subject should read this paper. It contains astonishing examples such as two subspaces of ℓ_p with $1 < p \neq 2 < \infty$ or two subspaces of c_0 that are uniformly homeomorphic but not linearly isomorphic!

We will not try to describe in details the last two sections of the paper, where Nigel Kalton obtains various important improvements of previous results on the existence of liftings (or selections) of linear quotients that are uniformly continuous on the unit ball of the target space. It was known (see [2], Corollary 1.25) that a linear quotient defined on a uniformly convex Banach space (or more generally on a space with uniform normal structure) admits a uniformly continuous lifting. Kalton shows for instance (Theorems 10.1 and 10.8), that if $Q : X \to Y$ is a linear quotient and $Y = X/E$ with E super-reflexive subspace of X, or Y is a quotient of a separable $\mathcal{L}^1$-space by a reflexive subspace, then Q admits a lifting which is uniformly continuous on B_Y. On the other hand, any linear quotient from a $\mathcal{L}^\infty$ or $\mathcal{L}^1$-space onto ℓ_2 fails to have a uniformly continuous lifting on B_{ℓ_2} (Theorem 7.6).

Of course, it was hopeless to give in these few pages a fair description of the contents and importance of this paper and of [9]. This is only an invitation to read the details of these two seminal papers. We also wish to mention that Section 11 of [11] is a rich and fascinating program of research.

Undoubtedly these two papers opened the door of a very interesting field of research: the study of the the linear properties of free spaces. So, let us finish by giving a list of some of the results obtained afterwards in this direction. Dutrieux and Ferenczi showed in [4] that for any infinite metric compact space K, $\mathcal{F}(K)$ is isomorphic to $\mathcal{F}(c_0)$ but not to $C([0,1])$. In [15], Naor and Schechtman proved that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to a subspace of L^1. Godard characterized in [6] the metric spaces whose free space is isometric to a subspace of L^1 as being isometric to subsets of an $\mathbb{R}$-tree. Godefroy and Osawa showed in [10] the existence of a compact metric space whose free space fails the approximation property, while it is shown in [14] that the free space of a doubling metric space has the bounded approximation property. In [3], A. Dalet proves that if M is a countable metric space with relatively compact balls, then $\mathcal{F}(M)$ is isometric to $\text{lip}(M)^*$ and has the metric approximation property. Finally, E. Pernecká and P. Hájek showed that $\mathcal{F}(\ell_1)$ has a Schauder basis. As the reader can see, thanks to the impulse of Godefroy and Kalton, the knowledge in the field is growing.

However, still very little is known and the subject is full of fascinating open questions that are very often connected to important problems on the non-linear classification of Banach spaces. For instance it is not known if $\mathcal{F}(\ell_1)$ is complemented in its bidual. A positive answer would imply that a Banach space that is Lipschitz equivalent to ℓ_1 is linearly isomorphic to ℓ_1.

Anyone interested in this subject should start by reading [9] and then dive with delight in this paper on "Spaces of Lipschitz and Hölder functions and their applications" by Nigel Kalton, which contains, in our opinion, the most advanced information available on the structure of free spaces.

References

[1] R. Bonic, J. Frampton and A. Tromba, Λ-manifolds, *J. Funct. Anal.*, **3**, (1969), 310-320.

[2] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis, *A.M.S. Colloquium publications*, vol 48, American Mathematical Society, Providence, RI, (2000).

[3] A. Dalet, Free spaces over countable compact metric spaces, *arxiv:1307.0735.*

[4] Y. Dutrieux and V. Ferenczi, The Lipschitz free Banach spaces of $C(K)$-spaces, *Proc. Amer. Math. Soc.*, **134**(4), (2006), 1039-1044.

[5] T. Figiel, On non linear isometric embeddings of normed linear spaces, *Bull. Acad. Polon. Sci. Math. Astro. Phys.*, **16**, (1968), 185-188.

[6] A. Godard, Tree metrics and their Lipschitz-free spaces, *Proc. Amer. Math. Soc.* **138**(12), (2010) 4311-4320.

[7] G. Godefroy, Linearization of isometric embeddings between Banach spaces: an elementatry approach, *Operators and Matrices*, **6**(2), 339-345.

[8] G. Godefroy, The use of norm attainment, *Bull. Belgian Math. Soc.*, **20**, (2013), 417-423.

[9] G. Godefroy and N. J. Kalton, Lipschitz-free Banach spaces, *Studia Math.*, **159**, (2003), 121–141.

[10] G. Godefroy and N. Ozawa, Free Banach spaces and the approximation properties, *Proc. Amer. Math. Soc.*, to appear.

[11] N. J. Kalton, Spaces of Lipschitz and Hölder functions and their applications, *Collect. Math.*, **55**(2), (2004), 171–217.

[12] N. J. Kalton, The uniform structure of Banach spaces, *Math. Ann.*, **354** (2012), 1247-1288.

[13] N. J. Kalton, Examples of uniformly homeomorphic Banach spaces, *Israel J. of Math.*, **194**, (2013), 151-182.

[14] G. Lancien and E. Pernecká , Approximation properties and Schauder decompositions in Lipschitz-free spaces, *J. Funct. Anal.*, **264**, (2013), 2323-2334.

[15] A. Naor, G. Schechtman, Planar earthmover is not in L^1, *SIAM J. Comput.*, **37**(3), (2007), 804-826.

[16] J. I Petunin and A. N Plichko, Some properties of the set of functionals that attain a supremum on the unit sphere, *Ukrain. Mat. Z.*, **26**, (1974) 102-106.

[17] N. Weaver, Lipschitz algebras, *World scientific*, 1999.

Laboratoire de Mathématiques de Besançon, Université de Franche-Comté, 16 route de Gray, 25 030 Besançon-Cedex, FRANCE
E-mail address: gilles.lancien@univ-fcomte.fr