CONTINUUM OF SOLUTIONS FOR AN ELLIPTIC PROBLEM WITH CRITICAL GROWTH IN THE GRADIENT

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Abstract. We consider the boundary value problem

$$(P_\lambda) \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = \lambda c(x) u + \mu(x)|\nabla u|^2 + h(x),$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$ is a bounded domain with smooth boundary. It is assumed that $c \gneqq 0$, c, h belong to $L^p(\Omega)$ for some $p > N/2$ and that $\mu \in L^\infty(\Omega)$. We explicit a condition which guarantees the existence of a unique solution of (P_λ) when $\lambda < 0$ and we show that these solutions belong to a continuum. The behaviour of the continuum depends in an essential way on the existence of a solution of (P_0). It crosses the axis $\lambda = 0$ if (P_0) has a solution, otherwise if bifurcates from infinity at the left of the axis $\lambda = 0$. Assuming that (P_0) has a solution and strenghtening our assumptions to $\mu(x) \geq \mu_1 > 0$ and $h \gneqq 0$, we show that the continuum bifurcates from infinity on the right of the axis $\lambda = 0$ and this implies, in particular, the existence of two solutions for any $\lambda > 0$ sufficiently small.

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^N, N \geq 3$, with smooth boundary $\partial\Omega$ in the sense of condition (A) of [19, p.6] (a sufficient condition for (A) is that $\partial\Omega$ satisfies the exterior uniform cone condition), we study, depending on the parameter $\lambda \in \mathbb{R}$, the existence and multiplicity of solutions of the boundary value problem

$$(P_\lambda) \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = \lambda c(x) u + \mu(x)|\nabla u|^2 + h(x),$$

where we assume

$$(\mathbf{A1}) \qquad \begin{cases} c \text{ and } h \text{ belong to } L^p(\Omega) \quad \text{for some } p > \frac{N}{2}, \\ c \gneqq 0 \text{ and } \mu \in L^\infty(\Omega). \end{cases}$$

Most of the results presented in this paper hold when $-\Delta$ is replace by a more general differential operator L in divergence form, see Remark 7.1. However for the simplicity of exposition we deal here with $L(u) = -\Delta u$.

2000 *Mathematics Subject Classification.* 35J50, 35Q41, 35Q55, 37K45.

Key words and phrases. Elliptic equations, quadratic growth in the gradient, continuum of solutions, topological degree.

D. A. is supported by Ministerio de Economía y Competitividad (Spain) MTM2012-31799 and Junta de Andalucía FQM-116, covered in part by EU-FEDER funds.

Elliptic quasilinear equations with a gradient dependence up to the critical growth $|\nabla u|^2$ were first studied by Boccardo, Murat and Puel in the 80's and it has been an active field of research until now. Under the condition that $c(x) \geq \alpha_0$ a.e. in Ω for some $\alpha_0 > 0$, the existence of a solution of (P_λ) when $\lambda < 0$ is a special case of the results of [9, 11]. Also in the case $\lambda = 0$ (or equivalently when $c \equiv 0$), Ferone and Murat [13] obtained the existence of a solution for (P_0), under the smallness assumption

$$\|\mu\|_\infty \|h\|_{\frac{N}{2}} < \mathcal{S}_N^2, \tag{1.1}$$

where $\mathcal{S}_N > 0$ is the best constant in Sobolev's inequality. This result was the first one assuming that $h(x) \in L^{N/2}(\Omega)$ but previous results, in the case $\lambda = 0$, were obtained under stronger regularity assumptions on $h(x)$ and assuming that a suitable norm of $h(x)$ is small (see for example [2, 14, 20]). In the particular case $\mu(x) \equiv \mu > 0$ and $h(x) \geq 0$, this existence result of [13] can be improved using Theorem 2.3 of Abdellaoui, Dall'Aglio and Peral in [1] who show that a sufficient condition for the existence of a solution for (P_0) is

$$\mu < \inf\left\{ \frac{\int_\Omega |\nabla \phi|^2 \, dx}{\int_\Omega h(x)\phi^2 \, dx} \, : \, \phi \in H_0^1(\Omega), \ \int_\Omega h(x)\phi^2 \, dx > 0 \right\}.$$

Concerning the uniqueness, a general theory for problems having quadratic growth in the gradient was developed in [5, 6]. When $c(x) \geq \alpha_0$ a.e. in Ω for some $\alpha_0 > 0$, the results of [6] imply the uniqueness of the solutions of (P_λ) for $\lambda < 0$.

In our first result we handle functions $c(x)$ that can vanish in some part of Ω. This does not seem to have been considered in the literature. Specifically, for the nonnegative and nonzero function $c(x)$ we set

$$W_c = \{w \in H_0^1(\Omega) : c(x)w(x) = 0, \ \text{a.e. } x \in \Omega\},$$

and, if $\text{meas}(\Omega \backslash \text{Supp}\, c) > 0$, we assume that the following condition holds

$$\textbf{(Hc)} \qquad \begin{cases} \displaystyle\inf_{\{u \in W_c,\, \|u\|_{H_0^1(\Omega)} = 1\}} \int_\Omega \left(|\nabla u|^2 - \|\mu^+\|_\infty h^+(x)u^2\right) dx > 0, \\ \displaystyle\inf_{\{u \in W_c,\, \|u\|_{H_0^1(\Omega)} = 1\}} \int_\Omega \left(|\nabla u|^2 - \|\mu^-\|_\infty h^-(x)u^2\right) dx > 0. \end{cases}$$

Here $\mu^+ = \max(\mu, 0)$, $\mu^- = \max(-\mu, 0)$, $h^+ = \max(h, 0)$ and $h^- = \max(-h, 0)$. As we shall see condition (Hc), along with (A1), suffices to guarantee the existence of a solution of (P_λ) for $\lambda < 0$. In Remark 1.1 we give some simple examples where the condition (Hc) holds. We also prove that, under the only condition (A1), the problem (P_λ) for $\lambda \leq 0$ has at most one solution. To obtain this uniqueness result it does not seems possible to extend the approach of [5, 6] and we follow a different strategy. As a first step we establish a regularity result inspired by [15]

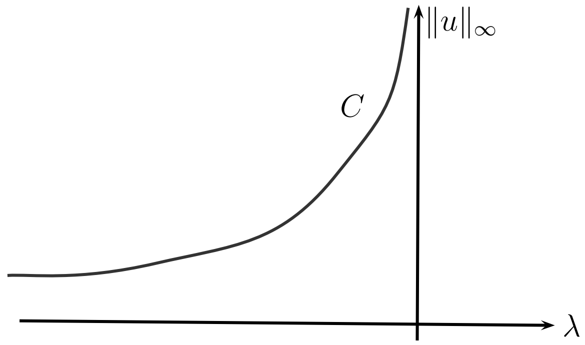

FIGURE 1. Bifurcation diagram when (P_0) has no solution

for the solutions of (P_λ). Then, using this regularity we derive our uniqueness result.

Our aim is also to point out that the unique solution of (P_λ) for $\lambda < 0$ belongs to a continuum C whose behavior at $\lambda = 0$ depends in an essential way on the existence of solution of (P_0). Throughout the paper we assume that the boundary of Ω is smooth in the sense of condition (A) of [19, p.6]. Under this assumption it is known, [19, Theorem IX.2.2] that any solution of (P_λ) belong to $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Denoting the solutions set

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : (\lambda, u) \text{ solves } (P_\lambda)\},$$

we prove the following result.

Theorem 1.1. *Assume that* (A1) *holds. If in addition, in the case that* $\text{meas}(\Omega \backslash \operatorname{Supp} c) > 0$, *we also assume that* (Hc) *holds, then*

1) *For $\lambda < 0$, (P_λ) has a unique solution u_λ.*
2) *There exists an unbounded continuum C of solutions in Σ whose projection* $\text{Proj}_{\mathbb{R}}C$ *on the λ-axis contains the interval* $]-\infty, 0[$.
3) *Moreover,* $\limsup_{\lambda \to 0^-} \|u_\lambda\|_\infty < \infty$ *if and only if (P_0) has a solution. In case (P_0) has a solution u_0, it is unique and*
$$\lim_{\lambda \to 0^-} \|u_\lambda - u_0\|_\infty = 0.$$
If (P_0) has no solution, then $\lim_{\lambda \to 0^-} \|u_\lambda\|_\infty = \infty$ *and $\lambda = 0$ is a bifurcation point from infinity for (P_λ) (see Figure 1).*

Remark 1.1. Condition (Hc) connects the two limit cases: $c(x) \geq \alpha_0 > 0$ and $c \equiv 0$ $(\lambda = 0)$. If $c(x) > 0$ a.e. on Ω we have $\text{meas}(\Omega \backslash \operatorname{Supp} c) = 0$. Thus, under (A1), a solution of (P_λ) exists for any $\lambda < 0$. If $\text{meas}(\Omega \backslash \operatorname{Supp} c) > 0$, the situation is more delicate. When both $\mu(x) \geq 0$ and $h(x) \geq 0$, (Hc) relates the *size* of $\mu(x)h(x)$ to the *size* of $\Omega \backslash \operatorname{Supp} c$, showing that the signs of $\mu(x)$ and $h(x)$

with respect to one another strongly influence the existence of solution of (P_λ) when $\lambda < 0$. Indeed, (Hc) holds if either $\mu(x) \geq 0$ and $h(x) \leq 0$ a.e. in Ω, or $\mu(x) \leq 0$ and $h(x) \geq 0$ a.e. in Ω. Moreover, it holds true under condition (1.1) since, from the Sobolev embedding, it follows that

$$\int_\Omega h(x)v^2 dx \leq \|h\|_{N/2}\|v\|_{2^*}^2 \leq \frac{1}{\mathcal{S}_N^2}\|h\|_{N/2}\|\nabla v\|_2^2.$$

Hence we obtain the above refered results as a corollary. In Remark 3.2 we show that (Hc) is somehow sharp for the existence of solution of (P_λ).

In connection with Theorem 1.1 we remark the interesting result by Porretta [22] for the case $c(x) \equiv 1$, $\mu(x) \equiv 1$ and $h \in L^\infty(\Omega)$. He has proved that when the problem (P_0) has no solution then the solutions of (P_λ) for $\lambda < 0$ blows-up completely, this behaviour being described in terms of the so-called ergodic problem.

Remark 1.2. We prove, in Corollary 3.2, that a sufficient condition for the existence of solution of (P_0) is that condition (Hc) is satisfied with $c(x) \equiv 0$. Then W_c is just $H_0^1(\Omega)$ and we write (H0) instead of (Hc).

Our next result show that the existence of a solution of (P_0) suffices to guarantee the existence of a continuum of solutions $C \subset \Sigma$ such that $\mathrm{Proj}_{\mathbb{R}} C$ contains $]-\infty, a]$ for some $a > 0$.

Theorem 1.2. *Assume* (A1) *and suppose that* (P_0) *has a solution. Then*

1) *For all* $\lambda \leq 0$, (P_λ) *has a unique solution* u_λ.
2) *There exists a continuum* $C \subset \Sigma$ *such that*
   (a) $\{(\lambda, u_\lambda) :\ \lambda \in]-\infty, 0]\} \subset C$.
   (b) $C \cap ([0, \infty[\times C(\overline{\Omega}))$ *is a unbounded set in* $\mathbb{R} \times C(\overline{\Omega})$.

   In particular, $\mathrm{Proj}_{\mathbb{R}} C$ *contains* $]-\infty, a]$ *for some* $a > 0$.

Finally, in the last part of the paper and under stronger assumptions, we study the behaviour in the half space $\{\lambda > 0\} \times C(\overline{\Omega})$ of the branch $C \subset \Sigma$ obtained in Theorem 1.2 and we obtain a multiplicity result.

First we note that, in case $\mu \equiv 0$, we cannot have multiplicity results except when λ is an eigenvalue of the problem

$$\varphi_1 \in H_0^1(\Omega) : -\Delta\varphi_1 = \gamma c(x)\varphi_1, \tag{1.2}$$

and $h(x)$ satisfies the *"good"* orthogonality condition. Hence, there is no hope to obtain multiplicity results just under our assumption (A1).

Multiplicity results have been considered by Abdellaoui, Dall'Aglio and Peral [1] for (P_λ) in the case $\lambda = 0$ and when $\mu(x)$ is replaced by some $g(u)$ satisfying $ug(u) < 0$. In a recent paper, Jeanjean and Sirakov [17] study the case $\lambda > 0$ when $\mu(x)$ is a positive constant but $h(x)$ may change sign and satisfy a condition related to (1.1). Using Theorem 2 of [17] an explicit $\lambda_0 > 0$ can be derived under which (P_λ) has two solutions whenever $\lambda \in]0, \lambda_0[$.

The above quoted multiplicity results have the common property that the coefficient of $|\nabla u|^2$ (either $g(u)$ or the constant μ) does not depend on x. This allows the authors to make a change a variable, similar to the one used in [18], in order to transform the problem in a semilinear one (i.e. without gradient dependence). Then variational methods are used to prove multiplicity results on the transformed problem. In our case, we consider problem (P_λ) with a non constant function coefficient $\mu(x)$, which implies that this change of variable is no more possible.

We replace (A1) by the stronger assumption

$$(\mathbf{A2}) \qquad \begin{cases} \Omega \text{ has a } C^{1,1} \text{ boundary } \partial\Omega, \\ c \text{ and } h \text{ belongs to } L^p(\Omega) \quad \text{for some } p > \frac{N}{2}, \\ c \gneqq 0,\ h \gneqq 0 \text{ and } \mu_2 \geq \mu(x) \geq \mu_1 \text{ for some } \mu_2 \geq \mu_1 > 0. \end{cases}$$

Let $\gamma_1 > 0$ denote the first eigenvalue of the problem (1.2). We prove the following theorem.

Theorem 1.3. *Assume* (A2) *and suppose that* (P_0) *has a solution. Then the continuum* $C \subset \Sigma$ *obtained in Theorem 1.2 consists of non negative functions, its projection* $\mathrm{Proj}_{\mathbb{R}} C$ *on the* λ*-axis is an unbounded interval* $]-\infty, \overline{\lambda}] \subset]-\infty, \gamma_1[$ *containing* $\lambda = 0$ *and* $C \subset \Sigma$ *bifurcates from infinity to the right of the axis* $\lambda = 0$. *Moreover, there exists* $\lambda_0 \in]0, \overline{\lambda}]$ *such that for all* $\lambda \in]0, \lambda_0[$*, the section* $C \cap (\{\lambda\} \times C(\overline{\Omega}))$ *contains two distinct non negative solutions of* (P_λ) *in* Σ *(see Figure 2).*

In order to prove Theorem 1.3 the key points are the observation that the continuum cannot cross the line $\lambda = \gamma_1$ and the derivation of a priori bounds, for any $a > 0$, on the (positive) solutions of (P_λ) for $\lambda \in]a, \gamma_1]$. These a priori bounds are obtained by an extension of the classical approach of Brezis and Turner [12].

The paper is organized as follows. In Section 2 we recall some results concerning the method of lower and upper solutions as well as a continuation theorem. In Section 3 we derive various existence results for problems of the type of (P_λ) when $\lambda \leq 0$. Section 4 deals with the uniqueness issue. In Section 5 we establish the existence of a continuum of solutions. Section 6 is devoted to the study of the branch in the half space $\{\lambda > 0\} \times C(\overline{\Omega})$ and in particular to the derivation of a

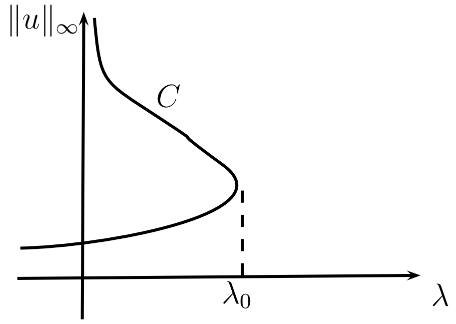

FIGURE 2. Bifurcation diagram when (P_0) has a solution.

priori bounds, see Proposition 6.1. The proofs of our three theorems are given in Section 7. Finally a technical result, Lemma 5.2, is proved in Section 8.

Acknowledgments The authors thank the referee for his comments which help to improve the initial version of this paper.

Notation.

1) For any measurable set $\omega \subset \mathbb{R}^N$ we denote by $\text{meas}(\omega)$ its Lebesgue measure.
2) For $p \in [1, +\infty[$, the norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We denote by p' the conjugate exponent of p, namely $p' = p/(p-1)$. The norm in $L^\infty(\Omega)$ is $\|u\|_\infty = \text{esssup}_{x\in\Omega}|u(x)|$.
3) For $v \in L^1(\Omega)$ we define $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.
4) For $h \in L^1(\Omega)$ we denote $h \gneqq 0$ if $h(x) \geq 0$ for a.e. $x \in \Omega$ and $\text{meas}(\{x \in \Omega : h(x) > 0\}) > 0$.
5) The space $H_0^1(\Omega)$ is equipped with the Poincaré norm $\|u\| := \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}$.
6) We denote by $C, D > 0$ any positive constants which are not essential in the problem and may vary from one line to another.

2. Preliminaries

In our proofs we shall use the method of lower and upper solutions. We present here Theorem 3.1 of [10] adapted to our setting. We consider the boundary value problem

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u + H(x, u, \nabla u) = f, \tag{2.1}$$

where $f \in L^1(\Omega)$ and H is a Carathéodory function from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ with a natural growth, i.e., for which there exist a nondecreasing function b from $[0, +\infty[$ into $[0, +\infty[$ and $k \in L^1(\Omega)$ such that, for a.e. $x \in \Omega$ and all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$|H(x, u, \xi)| \leq b(|u|)[k(x) + |\xi|^2].$$

We also recall (see [10]) that a *lower solution* (respectively, an *upper solution*) of (2.1) is a function α (respectively, β) $\in H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$-\Delta\alpha + H(x, \alpha, \nabla\alpha) \leq f(x) \text{ in } \Omega, \quad \alpha \leq 0 \text{ on } \partial\Omega,$$

(respectively,

$$-\Delta\beta + H(x, \beta, \nabla\beta) \geq f(x) \text{ in } \Omega, \quad \beta \geq 0 \text{ on } \partial\Omega).$$

This has to be understood in the sense that $\alpha^+ \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla\alpha\nabla v \, dx + \int_\Omega H(x, \alpha, \nabla\alpha)v \, dx \leq \int_\Omega f(x)v \, dx,$$

(respectively, $\beta^- \in H_0^1(\Omega)$ and $\int_\Omega \nabla\beta\nabla v \, dx + \int_\Omega H(x, \beta, \nabla\beta)v \, dx \geq \int_\Omega f(x)v \, dx$), for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω.

Theorem 2.1 (Boccardo-Murat-Puel [10])**.** *If there exist a lower solution α and an upper solution β of* (2.1) *with $\alpha \leq \beta$ a.e. in Ω, then there exists a solution u of* (2.1) *with $\alpha \leq u \leq \beta$ a.e. in Ω.*

We also need a continuation theorem. Let E be a real Banach space with norm $\|\cdot\|_E$ and $T : \mathbb{R} \times E \to E$ a completely continuous map, i.e. it is continuous and maps bounded sets to relatively compact sets. For $\lambda \in \mathbb{R}$, we consider the problem of finding the zeroes of $\Phi(\lambda, u) := u - T(\lambda, u)$, i.e.

$$(Q_\lambda) \qquad u \in E : \Phi(\lambda, u) = u - T(\lambda, u) = 0,$$

and we define

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times E : \Phi(\lambda, u) = 0\}.$$

Let $\lambda_0 \in \mathbb{R}$ be arbitrary but fixed and for $v \in E$ and $r > 0$, let $B(v, r) := \{u \in E : \|v - u\|_E < r\}$.

If we assume that u_{λ_0} is an isolated solution of (Q_{λ_0}), then the Leray-Schauder degree $\deg(\Phi(\lambda_0, \cdot), B(u_{\lambda_0}, r), 0)$ is well defined and is constant for $r > 0$ small enough. Thus it is possible to define the index

$$i(\Phi(\lambda_0, \cdot), u_{\lambda_0}) := \lim_{r \to 0} \deg(\Phi(\lambda_0, \cdot), B(u_{\lambda_0}, r), 0).$$

Theorem 2.2. *If (Q_{λ_0}) has a unique solution u_{λ_0} and $i(\Phi(\lambda_0, \cdot), u_{\lambda_0}) \neq 0$ then Σ possesses two unbounded components C^+, C^- in $[\lambda_0, +\infty[\times E$ and $]-\infty, \lambda_0] \times E$ respectively which meet at $(\lambda_0, u_{\lambda_0})$.*

Theorem 2.2 is essentially Theorem 3.2 of [23] (stated assuming that $\lambda_0 = 0$).

3. Some existence results

In this section we establish some existence results for the boundary value problem

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = d(x)u + \mu(x)|\nabla u|^2 + h(x), \tag{3.1}$$

under the assumption that

$$(\mathbf{A3}) \qquad \begin{cases} d \text{ and } h \text{ belong to } L^p(\Omega) \quad \text{for some } p > \frac{N}{2}, \\ \mu(x) \equiv \mu > 0 \text{ is a constant,} \\ d \leq 0 \text{ and } h \geq 0. \end{cases}$$

If $\text{meas}(\Omega \backslash \text{Supp}\, d) > 0$ we also set

$$W_d = \{w \in H_0^1(\Omega) : d(x)w(x) = 0, \ a.e.\, x \in \Omega\}$$

and we impose condition (Hc) for $c = d$, i.e., we require

$$(\mathbf{H}) \qquad m_2 := \inf_{\{u \in W_d,\, \|u\|=1\}} \int_\Omega (|\nabla u|^2 - \mu h(x) u^2)\, dx > 0.$$

Proposition 3.1. *Assume* (A3) *and, if* $\text{meas}(\Omega \backslash \text{Supp}\, d) > 0$*, also that* (H) *holds. Then* (3.1) *has a non negative solution.*

Remark 3.1. Observe that, under condition (A3), every solution u of (3.1) is non negative. In fact, using u^- as test function we obtain, as $d \leq 0$, $\mu > 0$ and $h \geq 0$,

$$0 \geq -\int_\Omega |\nabla u^-|^2 + \int_\Omega d(x)|u^-|^2 = \int_\Omega \left[\mu|\nabla u|^2 + h(x)\right] u^- \geq 0,$$

which implies that $u^- = 0$ i.e. $u \geq 0$.

To prove Proposition 3.1 we introduce the boundary value problem

$$v \in H_0^1(\Omega) : -\Delta v - \mu h(x) v = d(x) g(v) + h(x), \tag{3.2}$$

where

$$g(s) = \begin{cases} \frac{1}{\mu}(1+\mu s)\ln(1+\mu s), & \text{if} \quad s \geq 0, \\ -\frac{1}{\mu}(1-\mu s)\ln(1-\mu s), & \text{if} \quad s < 0. \end{cases} \tag{3.3}$$

Let us denote

$$G(s) = \int_0^s g(\xi)\, d\xi = \begin{cases} \dfrac{(1+\mu s)^2}{4\mu^2}[2\ln(1+\mu s) - 1] + \dfrac{1}{4\mu^2} & \text{if } s \geq 0, \\ G(-s), & \text{if } s < 0. \end{cases}$$

The properties of g that are useful to us are gathered in the following lemma.

Lemma 3.1.

1) *The function g is odd and continuous on* $\mathbb{R}$.

2) *$g(s)s > 0$ for $s \in \mathbb{R} \setminus \{0\}$, $G(s) \geq 0$ on $\mathbb{R}$.*
3) *For any $r \in]0,1[$, there exists $C = C(r,\mu) > 0$ such that, for all $|s| > \frac{1}{\mu}$, we have $|g(s)| \leq C|s|^{1+r}$.*
4) *We have $G(s)/s^2 \to +\infty$ as $|s| \to \infty$.* □

The idea of modifying the problem to obtain problem (3.2) is not new. It appears already in [18] in another context. It permits to obtain a non negative solution of (3.1).

Lemma 3.2. *Assume that* (A3) *hold.*

1) *Any solution of* (3.2) *belongs to $W^{2,p}(\Omega)$ and thus to $L^\infty(\Omega)$;*
2) *If $v \in H_0^1(\Omega)$ is a non negative solution of* (3.2) *then $u = (1/\mu)\ln(1 + \mu v) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a (non negative) solution of* (3.1).

Proof. 1) Let $v \in H_0^1(\Omega)$ be a solution of (3.2), that we write as

$$v \in H_0^1(\Omega) : -\Delta v = \left[\mu h(x) + d(x)\frac{g(v)}{v}\right] v + h(x).$$

By classical arguments, see for example [19, Theorem III-14.1], as $\partial\Omega$ satisfies the condition (A) of [19], the first part of the lemma will be proved if we can show that

$$\left[\mu h(x) + d(x)\frac{g(v)}{v}\right] \in L^{p_1}(\Omega) \quad \text{with } p_1 > N/2.$$

But by assumption d and μh belong to $L^p(\Omega)$, for some $p > N/2$ and we shall prove that the term $d(x)\frac{g(v)}{v}$ has the same property. This is the case because of the slow growth of $g(s)/s$ as $|s| \to \infty$, see Lemma 3.1-3). Specifically, for any $r \in]0,1[$, there exists a $C > 0$ such that, for all $|s| > \frac{1}{\mu}$,

$$|g(s)/s| \leq C|s|^r.$$

Thus, since $d \in L^p(\Omega)$ with $p > \frac{N}{2}$ and $v \in L^{\frac{2N}{N-2}}(\Omega)$, taking $r > 0$ sufficiently small (for example $r < \frac{4p-2N}{p(N-2)}$) we see, using Hölder inequality, that $d(x)g(v)/v \in L^{p_1}(\Omega)$, for some $p_1 > N/2$. This ends the proof of 1).

2) Since $v \geq 0$ the problem (3.2) can be rewritten as

$$v \in H_0^1(\Omega) : -\Delta v = \frac{d(x)}{\mu}(1 + \mu v)\ln(1 + \mu v) + (1 + \mu v)h(x). \tag{3.4}$$

Let $v \in H_0^1(\Omega)$ be a non negative solution of (3.4), we want to show that $u = \frac{1}{\mu}\ln(1 + \mu v)$ is a solution of (3.1), namely that, for $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \big(\nabla u \nabla \phi - \mu|\nabla u|^2\phi - d(x)u\phi\big)\,dx = \int_\Omega h(x)\phi\,dx. \tag{3.5}$$

First observe that, as $v \in L^\infty(\Omega)$ and satisfies $v \geq 0$ in Ω we have $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Let $\psi = \phi/(1+\mu v)$. Clearly $\psi \in H_0^1(\Omega)$ and thus it can be used as test function in (3.4). Hence, we get

$$\begin{aligned} \int_\Omega \nabla v \nabla \psi \, dx &= \int_\Omega \frac{d(x)}{\mu} \ln(1+\mu v)\phi \, dx + \int_\Omega h(x)\phi \, dx \\ &= \int_\Omega d(x) u \phi \, dx + \int_\Omega h(x)\phi \, dx. \end{aligned} \tag{3.6}$$

Moreover, we have

$$\begin{aligned} \int_\Omega \nabla v \nabla \psi \, dx &= \int_\Omega \nabla \left(\frac{1}{\mu}(e^{\mu u} - 1) \right) \nabla \left(\frac{\phi}{1+\mu v} \right) dx \\ &= \int_\Omega e^{\mu u} \nabla u \left(\frac{\nabla \phi}{1+\mu v} - \frac{\mu \phi \nabla v}{(1+\mu v)^2} \right) dx \\ &= \int_\Omega \nabla u \left(\nabla \phi - \frac{\mu \phi \nabla (\frac{1}{\mu}(e^{\mu u} - 1))}{(1+\mu v)} \right) dx \\ &= \int_\Omega \nabla u (\nabla \phi - \mu \phi \nabla u) \, dx = \int_\Omega \left(\nabla u \nabla \phi - \mu |\nabla u|^2 \phi \right) dx. \end{aligned}$$

Combining this equality with (3.6) we see that u satisfies (3.5). This ends the proof of 2). $\square$

In order to find a solution of (3.2) we shall look to a critical point of the functional I defined on $H_0^1(\Omega)$ by

$$I(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 - \mu h(x) v^2) \, dx - \int_\Omega d(x) G(v) \, dx - \int_\Omega h(x) v \, dx.$$

As g has a subcritical growth at infinity, see Lemma 3.13), it is standard to show that $I \in C^1(H_0^1(\Omega), \mathbb{R})$ and that a critical point of I corresponds to a solution in $H_0^1(\Omega)$ of (3.2). To obtain a critical point of I we shall prove the existence of a global minimum of I. We define

$$m := \inf_{u \in H_0^1(\Omega)} I(u) \in \mathbb{R} \cup \{-\infty\}. \tag{3.7}$$

Lemma 3.3. *Assume* (A3) *and, if* $\operatorname{meas}(\Omega \backslash \operatorname{Supp} d) > 0$*, assume also that* (H) *holds. Then the infimum* m *defined by* (3.7) *is finite and it is reached by a non negative function in* $H_0^1(\Omega)$*. Consequently,* (3.2) *has a non negative solution.*

Proof. We divide the proof into two steps :

Step 1. *I is coercive.*

We assume by contradiction the existence of a sequence $\{v_n\} \subset H_0^1(\Omega)$ such that $\|v_n\| \to \infty$ and $I(v_n)$ is bounded from above. We define

$$w_n = \frac{v_n}{\|v_n\|}.$$

Clearly $\|w_n\| \equiv 1$ and we can assume that $w_n \rightharpoonup w$ weakly in $H_0^1(\Omega)$ and $w_n \to w$ strongly in $L^q(\Omega)$ for $q \in [2, \frac{2N}{N-2}[$. Since $I(v_n)$ is bounded from above, we have

$$\limsup_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} \leq 0. \tag{3.8}$$

We shall treat separately the two cases :

$$(1)\ \ w \in W_d \qquad \text{and} \qquad (2)\ \ w \notin W_d.$$

Case (1)*:* $w \in W_d$. In this case, taking (H) into account, it follows that

$$\int_\Omega (|\nabla w|^2 - \mu h(x) w^2) dx \geq m_2 \|w\|^2.$$

Thus, and since $G(s) \geq 0$ on $\mathbb{R}$ and $d(x) \leq 0$ in Ω, using the weak lower semi-continuity of $\int_\Omega |\nabla u|^2 dx$ and the weak convergence of w_n, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} &= \liminf_{n\to\infty} \left[\frac{1}{2}\int_\Omega (|\nabla w_n|^2 - \mu h(x) w_n^2) dx - \int_\Omega \frac{d(x)G(v_n)}{\|v_n\|^2} dx\right] \\
&\geq \frac{1}{2}\int_\Omega (|\nabla w|^2 - \mu h(x) w^2) dx \geq \frac{1}{2} m_2 \|w\|^2 \geq 0 \geq \limsup_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2},
\end{aligned} \tag{3.9}$$

i.e., $\lim_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} = 0$ and $w \equiv 0$. However, using that $2p/(p-1) < 2N/(N-2)$ and w_n is weakly convergent to $w = 0$ in $H_0^1(\Omega)$, we deduce the strong convergence of w_n to $w = 0$ in $L^{2p/(p-1)}(\Omega)$, which by the assumptions $d(x) \leq 0$ on Ω and $G(s) \geq 0$ on $\mathbb{R}$ implies that

$$\lim_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} \geq \frac{1}{2} - \lim_{n\to\infty} \frac{\mu}{2} \int_\Omega h(x) w_n^2 dx - \lim_{n\to\infty} \int_\Omega \frac{h(x) w_n}{\|v_n\|} dx \geq \frac{1}{2}.$$

This is a contradiction showing that case (1) cannot occurs.

Case (2)*:* $w \notin W_d$. Since $w \notin W_d$, necessarily $\Omega_0 = \{x \in \Omega : d(x)w(x) \neq 0\}$ has non zero measure and thus $|v_n(x)| = |w_n(x)|\,\|v_n\| \to \infty$ a.e. in Ω_0. Using the assumptions $d(x) \leq 0$ in Ω and $G(s) \geq 0$ on $\mathbb{R}$ we deduce from Lemma 3.1-4) and Fatou's lemma that

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega \frac{d(x)G(v_n)}{v_n^2} w_n^2 dx &\leq \limsup_{n\to\infty} \int_{\Omega_0} \frac{d(x)G(v_n)}{v_n^2} w_n^2 dx \\
&\leq \int_{\Omega_0} \limsup_{n\to\infty} \frac{d(x)G(v_n)}{v_n^2} w_n^2 dx = -\infty.
\end{aligned}$$

On the other hand, using that w_n is weakly convergent in $H_0^1(\Omega)$ and that, by Sobolev's embedding, $\|w_n\|_{\frac{2p}{p-1}}$ is bounded, it follows that

$$0 \geq \limsup_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} \geq \liminf_{n\to\infty} \frac{I(v_n)}{\|v_n\|^2} \geq -C - \limsup_{n\to\infty} \int_\Omega \frac{d(x)G(v_n)}{\|v_n\|^2} dx = +\infty,$$

a contradiction proving that case (2) does not occur and this ends the proof of Step 1.

Step 2. *Existence of a minimum of I.*

To show that I admits a global minimizer it now suffices to show that I is weakly lower semicontinuous i.e., if $\{v_n\} \subset H_0^1(\Omega)$ is a sequence such that $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$, and then $v_n \to v$ strongly in $L^q(\Omega)$ for $q \in [2, \frac{2N}{N-2}[$, we have

$$I(v) \leq \liminf_{n\to\infty} I(v_n). \tag{3.10}$$

Using the weak convergence of the sequence $\{v_n\}$ and the weak lower semicontinuity of $\int_\Omega |\nabla v|^2 dx$, we have

$$\frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega h(x) v dx \leq \liminf_{n\to\infty} \left[\frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \int_\Omega h(x) v_n dx \right]. \tag{3.11}$$

Also, the strong convergence in $L^{\frac{2p}{p-1}}(\Omega)$ implies that

$$\int_\Omega \mu h(x) v_n^2 dx \to \int_\Omega \mu h(x) v^2 dx. \tag{3.12}$$

Finally, since $-d(x)G(v_n) \geq 0$ on Ω, as a consequence of Fatou's lemma, we obtain

$$\int_\Omega -d(x)G(v) dx \leq \liminf_{n\to\infty} \int_\Omega -d(x)G(v_n) dx. \tag{3.13}$$

At this point (3.10) follows from (3.11)-(3.13).

Step 3. *Conclusion.*

To conclude the existence of a non negative minimum, observe that, as $h(x) \geq 0$ in Ω and $G(s)$ is even we have, for every $u \in H_0^1(\Omega)$,

$$I(|u|) \leq I(u),$$

and hence if $v \in H_0^1(\Omega)$ is a minimum of I then $|v|$ is also a minimum. Then we conclude that the infimum m is reached by a non negative function. □

Proof of Proposition 3.1. By Lemma 3.3, (3.2) admits a non negative solution $v \in H_0^1(\Omega)$ and thus, using Lemma 3.2, we deduce that (3.1) has a non negative solution. □

We now consider the problem.

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = d(x)u + W(x,u,\nabla u), \tag{3.14}$$

where we assume

$$(\mathbf{A4}) \begin{cases} d \le 0 \text{ with } d \in L^p(\Omega) \text{ for some } p > \frac{N}{2} \\ \text{and there exist } \mu_\pm \in]0,+\infty[\text{ and } h_\pm \in L^p(\Omega) \text{ with } h_\pm \ge 0, \text{ such that} \\ -\mu_-|\xi|^2 - h_-(x) \le W(x,u,\xi) \le \mu_+|\xi|^2 + h_+(x) \text{ on } \Omega \times \mathbb{R} \times \mathbb{R}^N. \end{cases}$$

Proposition 3.2. *Assume that* (A4) *holds and, if* $\operatorname{meas}(\Omega \backslash \operatorname{Supp} d) > 0$*, in addition, assume*

$$\begin{cases} \inf_{\{u \in W_d,\, \|u\|=1\}} \int_\Omega \left(|\nabla u|^2 - \mu_+ h_+(x) u^2\right) dx > 0, \\ \inf_{\{u \in W_d,\, \|u\|=1\}} \int_\Omega \left(|\nabla u|^2 - \mu_- h_-(x) u^2\right) dx > 0. \end{cases}$$

Then (3.14) *has a solution.*

Proof. To prove Proposition 3.2 we use Theorem 2.1. Thus we need to find a couple of lower and upper solutions (α, β) of (3.14), with $\alpha \le \beta$. Clearly, by (A4), any solution of

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = d(x)u + \mu_+|\nabla u|^2 + h_+(x), \tag{3.15}$$

is an upper solution of (3.14). Moreover, a solution of

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = d(x)u - \mu_-|\nabla u|^2 - h_-(x), \tag{3.16}$$

is a lower solution of (3.14). Now if $w \in X$ is a solution of

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = d(x)u + \mu_-|\nabla u|^2 + h_-(x), \tag{3.17}$$

then $u = -w$ satisfies (3.16). Thus if we find a non negative solution $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.15) and a non negative solution $u_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.17) then, setting $\beta = u_1$ and $\alpha = -u_2$, we have the required couple of lower and upper solutions for Theorem 2.1. By Proposition 3.1, we know that such non negative solutions of (3.15) and (3.17) exist and this concludes the proof. $\square$

As a direct consequence of the previous proposition, we obtain

Corollary 3.1. *Assume* (A1) *and, if* $\operatorname{meas}(\Omega \backslash \operatorname{Supp} c) > 0$*, assume also that* (Hc) *holds. Then* (P_λ) *has a solution for any* $\lambda < 0$.

As another direct consequence of Proposition 3.2, just noting that $W_d = H_0^1(\Omega)$ in case $d(x) \equiv 0$, we have

Corollary 3.2. *Assume* (A1) *and* (H0) *hold. Then* (P_0) *has a solution.*

Remark 3.2. Assume that c and h belong to $L^p(\Omega)$ for some $p > \frac{N}{2}$, and that $\mu \in L^\infty(\Omega)$. Assume that there exists an open subset $O(c)$ in Ω with C^1 boundary $\partial O(c)$ such that $c(x) = 0$ a.e. in $\overline{O(c)}$, $c(x) > 0$ a.e. in $\Omega \backslash \overline{O(c)}$ and $\mu(x) \geq \mu_1 > 0$, in $\overline{O(c)}$. Then $W_c = H_0^1(O(c))$ and a necessary condition for the existence of a solution of (P_λ) is that the first eigenvalue of the elliptic eigenvalue problem

$$(\lambda, \phi) \in \mathbb{R} \times H_0^1(O(c)) \ : \ -\mathrm{div}\left(\frac{\nabla \phi}{\mu(x)}\right) - h(x)\phi = \lambda \phi\,, \tag{3.18}$$

be positive, i.e. that

$$\inf_{\{\phi \in W_c, \|\phi\|=1\}} \int_\Omega \left(\frac{1}{\mu(x)}|\nabla \phi|^2 - h(x)\phi^2\right)\, dx > 0. \tag{3.19}$$

Indeed, to show (3.19), we use an argument inspired by [1, 13]. Suppose that (P_λ) has a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then for any $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \left(\nabla u \nabla(\phi^2) - \lambda c(x) u \phi^2 - \mu(x)|\nabla u|^2 \phi^2 - h(x)\phi^2\right)\, dx = 0. \tag{3.20}$$

and hence, for every $\phi \in C_0^\infty(\Omega) \cap W_c$ we obtain

$$\int_{O(c)} \left(\nabla u \nabla(\phi^2) - \mu(x)|\nabla u|^2 \phi^2 - h(x)\phi^2\right) dx = 0. \tag{3.21}$$

But, for $\phi \in C_0^\infty(\Omega) \cap W_c$, by Young inequality,

$$\begin{aligned} \int_{O(c)} \nabla u \nabla(\phi^2)\, dx &= \int_{O(c)} 2\phi \nabla u \nabla \phi\, dx \\ &\leq \int_{O(c)} \left(\frac{1}{\mu(x)}|\nabla \phi|^2 + \mu(x)|\nabla u|^2\phi^2\right) dx \end{aligned} \tag{3.22}$$

and thus by density

$$\int_{O(c)} \left(\frac{1}{\mu(x)}|\nabla \phi|^2 - h(x)\phi^2\right) dx \geq 0 \quad \text{for all } \phi \in W_c.$$

Thus, the infimum in (3.19) is non negative. If it is zero then, by Poincaré inequality, we also have that

$$\inf_{\{\phi \in W_c : \|\phi\|_2 = 1\}} \int_{O(c)} \left(\frac{1}{\mu(x)}|\nabla \phi|^2 - h(x)\phi^2\right) dx = 0. \tag{3.23}$$

Let us show that it cannot take place. Arguing by contradiction we assume that (3.23) hold. Then, by standard arguments, there exists a $\phi_0 \in W_c \setminus \{0\}$ such that

$$\int_{O(c)} \left(\frac{1}{\mu(x)}|\nabla \phi_0|^2 - h(x)\phi_0^2\right) dx = 0. \tag{3.24}$$

In addition, ϕ_0 is an eigenfunction associated to the first eigenvalue (which we are assuming equal to zero) of the eigenvalue problem (3.18). As a consequence, we may assume that $\phi_0(x) > 0$ in $O(c)$.

Setting $\phi = \phi_0$ in (3.21), we have by (3.24) that

$$\int_{O(c)} \left(2\phi_0 \nabla u \nabla \phi_0 - \mu(x)|\nabla u|^2 \phi_0^2 - \frac{1}{\mu(x)} |\nabla \phi_0|^2 \right) dx = 0.$$

That is,

$$\int_{O(c)} \Big| \frac{1}{\sqrt{\mu(x)}} \nabla \phi_0 - \sqrt{\mu(x)} \phi_0 \nabla u \Big|^2 dx = 0$$

from which we deduce that

$$\nabla u = \mu(x) \frac{\nabla \phi_0}{\phi_0} \quad \text{in } O(c).$$

Taking into account that for every $\phi_0 \in H_0^1(O(c))$, $\phi_0 > 0$, we have $\frac{\nabla \phi_0}{\phi_0} \notin L^2(O(c))$ (by Poincaré inequality and Fatou lemma as ε tends to zero in the inequality $\frac{|\nabla \phi_0|^2}{\phi_0^2} \geq \frac{|\nabla \phi_0|^2}{(\phi_0+\varepsilon)^2} = |\nabla \left(\log(\frac{\phi_0}{\varepsilon} + 1)\right] |^2)$ and that $\mu(x) \geq \mu_1$ in $O(c)$, we obtain $\nabla u \notin L^2(O(c))$, a contradiction with $u \in H_0^1(\Omega)$ proving that (3.19) holds.

Now if in addition to the above assumptions we assume that $\mu(x) \equiv \mu > 0$ is a constant and $h(x) \geq 0$ it follows from (3.19) that, if (P_λ) has a solution, we have

$$\inf_{\{\phi \in W_c : \|\phi\| = 1\}} \int_\Omega \left(|\nabla \phi|^2 - \mu h(x) \phi^2 \right) dx > 0. \tag{3.25}$$

Note that under these assumptions, (Hc) coincides with (3.25) and thus (P_λ) when $\lambda < 0$ has a solution if and only if (Hc) holds. Finally when $\lambda = 0$ (equivalently when $c \equiv 0$), we have $O(c) = \Omega$, $W_c = H_0^1(\Omega)$ and (3.25) reduces to (H0). Thus (P_0) has a solution if and only if (H0) holds.

4. Uniqueness results

As in the previous section, we consider the boundary value problem (3.1). Here we assume

(A5)
$$\begin{cases} d \text{ and } h \text{ belong to } L^p(\Omega) \quad \text{for some } p > \frac{N}{2}, \\ d(x) \leq 0 \text{ in } \Omega \text{ and } \mu \in L^\infty(\Omega). \end{cases}$$

Our main result is

Proposition 4.1. *Assume that* (A5) *hold. Then* (3.1) *has at most one solution.*

To prove Proposition 4.1 we shall first prove that the solutions of (3.1) belong to $C(\overline{\Omega}) \cap W^{1,N}_{loc}(\Omega)$. Then, using this additional regularity, we prove the uniqueness.

Remark 4.1. Proposition 4.1 implies that (P_λ) for $\lambda \leq 0$ has at most one solution.

Remark 4.2. As we mention in the Introduction a general theory of uniqueness for problems with quadratic growth in the gradient was developed in [6] and extended in [5]. The uniqueness results closer to our setting are Theorems 2.1 and 2.2 of [5]. Unfortunately it is not possible to use directly these results to derive Proposition 4.1. Indeed, since $d(x)$ may vanish on some part of Ω, [5, Theorem 2.1] is not applicable. Also, to use [5, Theorem 2.2] which corresponds to the case $\lambda = 0$, we need either $h(x)$ to have a sign or to be sufficiently small.

Lemma 4.1. *Assume that* (A5) *hold. Then any solution of* (3.1) *belongs to* $C(\overline{\Omega}) \cap W^{1,N}_{loc}(\Omega)$.

Proof. Let $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be an arbitrary solution of (3.1). We divide the proof that $u \in W^{1,N}_{loc}(\Omega)$ into three steps.

Step 1. $u \in C(\overline{\Omega})$.

Since condition (A) holds the result follows directly from [19, Theorem IX.2.2]. Indeed, (3.1) is of the form of equation (1.1) of Section IV.1 of [19]. In addition, under (A5) the assumptions (1.2)-(1.3) considered in [19, Section IV.1] are satisfied. Hence, $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and in particular $u \in C(\overline{\Omega})$.

Step 2. $u \in W^{1,q}_{loc}(\Omega)$ for some $q > 2$.

It directly follows from [15, Theorem 2.5 p.155] that $u \in W^{1,q}_{loc}(\Omega)$ for some $q > 2$.

Step 3. Conclusion.

We follows some arguments of [7]. First note that without restriction we can assume that $q < N$. Since $u \in W^{1,q}_{loc}(\Omega)$ we have,

$$-\Delta u = \xi(x) \quad \text{where} \quad \xi(x) = d(x)u + \mu(x)|\nabla u|^2 + h(x) \in L^{\frac{q}{2}}_{loc}(\Omega). \tag{4.1}$$

By standard regularity argument, see for example [16, Theorem 9.11], we deduce that $u \in W^{2,\frac{q}{2}}_{loc}(\Omega)$. Now using Miranda's interpolation Theorem [21, Teorema IV] between $C^{0,\alpha}(\overline{\Omega})$ and $W^{2,\frac{q}{2}}_{loc}(\Omega)$ it follows, since $u \in C^{0,\alpha}(\overline{\Omega})$, that

$$u \in W^{1,t_1}_{loc}(\Omega) \quad \text{where} \quad t_1 = \frac{\frac{q}{2}(2-\alpha) - \alpha}{1-\alpha} > q.$$

If $t_1 \geq N$ we are done. Otherwise from (4.1) and classical regularity $u \in W^{2,\frac{t_1}{2}}_{loc}(\Omega)$. Denoting

$$t_n = \frac{\frac{t_{n-1}}{2}(2-\alpha) - \alpha}{1-\alpha} > t_{n-1} > q > 2 \tag{4.2}$$

by a bootstrap argument we get $u \in W^{2,\frac{t_n}{2}}_{loc}(\Omega)$ for all $n \in \mathbb{N}$ as long as $t_{n-1} \leq N$. We now claim that the sequence $\{t_n\}$ does not converge before reaching N. Indeed if we assume that $\{t_n\}$ has a finite limite l we deduce from (4.2) that $l = 2$ which contradicts $t_n > q > 2$. At this point the proof of the lemma is completed. $\square$

Using the fact that, under (A5), the solutions of (3.1) belong to $C(\overline{\Omega}) \cap W^{1,N}_{loc}(\Omega)$ we can now obtain our uniqueness result. Here we adapt an argument from [8].

Lemma 4.2. *Assume that* (A5) *hold. Then* (3.1) *has at most one solution in* $H^1_0(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap C(\overline{\Omega})$.

Proof. Let us assume the existence of two solutions u_1, u_2 of (3.1) in $H^1_0(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap C(\overline{\Omega})$. Then $v = u_1 - u_2$ is a solution of

$$v \in H^1_0(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap C(\overline{\Omega}) : -\Delta v = \mu(x)(\nabla u_1 + \nabla u_2)\,\nabla v + d(x)v. \tag{4.3}$$

For every $c \in \mathbb{R}$, let us consider the set $\Omega_c = \{x \in \Omega \,:\, |v(x)| = c\}$ and

$$J = \{c \in \mathbb{R} \,:\, \text{meas}(\Omega_c) > 0\}.$$

As meas(Ω) is finite, J is at most countable and, since for all $c \in \mathbb{R}$, $\nabla v = 0$ a.e. on Ω_c, we also have

$$\nabla v = 0 \text{ a.e. in } \bigcup_{c \in J} \Omega_c. \tag{4.4}$$

Define $Z = \Omega \setminus \bigcup_{c \in J} \Omega_c$ and let $G_k : \mathbb{R} \to \mathbb{R}$ be given by

$$G_k(s) = \begin{cases} 0, & \text{if } |s| \leq k, \\ (|s| - k)\,\text{sgn}(s), & \text{if } |s| > k. \end{cases} \tag{4.5}$$

Now, using $\varphi = G_k(v)$ as test function in (4.3), we deduce for all $k \geq 0$ that

$$\begin{aligned} \|\nabla G_k(v)\|_2^2 &= \int_\Omega |\nabla v|^2 \chi_{\{|v| \geq k\}}\,dx \\ &= \int_\Omega \mu(x)(\nabla u_1 + \nabla u_2)\,\nabla v\,G_k(v)\,dx + \int_\Omega d(x)\,v\,G_k(v)\,dx. \end{aligned}$$

Since $v \in C(\overline{\Omega})$ we have that $G_k(v)$ has a compact support in Ω for all $k > 0$, which together to the fact that $d(x) \leq 0$ on Ω and (4.4) implies that

$$\begin{aligned} \|\nabla G_k(v)\|_2^2 &\leq \int_\Omega \mu(x)(\nabla u_1 + \nabla u_2)\,\chi_{\{|v| \geq k\} \cap Z}\,\nabla v\,G_k(v)\,dx \\ &= \int_\Omega \mu(x)(\nabla u_1 + \nabla u_2)\,\chi_{\{|v| \geq k\} \cap Z}\,\nabla G_k(v)\,G_k(v)\,dx \\ &\leq \|\mu\|_\infty \|\nabla u_1 + \nabla u_2\|_{L^N(\{|v| \geq k\} \cap Z)} \|\nabla G_k(v)\|_2 \|G_k(v)\|_{2^*} \\ &\leq \mathcal{S}_N^{-1} \|\mu\|_\infty \|\nabla u_1 + \nabla u_2\|_{L^N(\{|v| \geq k\} \cap Z)} \|\nabla G_k(v)\|_2^2, \end{aligned} \tag{4.6}$$

where we recall that $\mathcal{S}_N$ denotes the Sobolev constant.

Assume by contradiction that $v \not\equiv 0$ and consider the function $F :]0, \|v\|_\infty] \to \mathbb{R}$ defined by

$$F(k) = \mathcal{S}_N^{-1} \|\mu\|_\infty \|\nabla u_1 + \nabla u_2\|_{L^N(\{|v|\geq k\}\cap Z)}, \quad \forall\, 0 < k < \|v\|_\infty.$$

Observe that F is non-increasing with $F(\|v\|_\infty) = 0$. Moreover, by definition of Z we have that F is continuous and we can choose $0 < k_0 < \|v\|_\infty$ such that $F(k_0) < 1$. By (4.6), $\|\nabla G_{k_0}(v)\|_2^2 \leq F(k_0)\|\nabla G_{k_0}(v)\|_2^2$, which implies that $\|\nabla G_{k_0}(v)\|_2 = 0$, i.e. $|v| \leq k_0 < \|v\|_\infty$, a contradiction proving that necessarily $v = 0$ and hence $u_1 = u_2$ concluding the proof. □

Proof of Proposition 4.1. This follows directly from Lemmas 4.1 and 4.2. □

5. Uniform L^∞-estimates and existence of a continuum

As in the previous section, we consider the boundary value problem (3.1) under the condition (A5).

Lemma 5.1. *Assume that* (A5) *hold and that* (3.1) *has a solution* $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. *Then*

1) *For any* $\widetilde{d}(x) \in L^p(\Omega)$, $p > \frac{N}{2}$ *with* $\widetilde{d}(x) \leq d(x)$, *the problem*

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = \widetilde{d}(x)u + \mu(x)|\nabla u|^2 + h(x), \tag{5.1}$$

has a unique solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. *Moreover,* u *satisfies*

$$\|u\|_\infty \leq 2\|u_0\|_\infty.$$

2) *There exists* $M_1 > 0$ *such that for any* $t \in [0, 1]$ *any solution* u_t *of*

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = (d(x) - 1)u + (1 - t)\mu(x)|\nabla u|^2 + h(x), \tag{5.2}$$

satisfies $\|u_t\|_\infty \leq M_1$.

Proof. 1) Let $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (3.1) and set

$$\beta(x) = u_0(x) + \|u_0\|_\infty, \quad \alpha(x) = u_0(x) - \|u_0\|_\infty.$$

Then $\alpha \leq 0 \leq \beta$ and, using that $\widetilde{d}(x) \leq d(x) \leq 0$, we have

$$\begin{aligned} -\Delta\beta &= d(x)(\beta - \|u_0\|_\infty) + \mu(x)|\nabla\beta|^2 + h(x) \\ &= \widetilde{d}(x)\beta + \mu(x)|\nabla\beta|^2 + h(x) + (d(x) - \widetilde{d}(x))\beta - d(x)\|u_0\|_\infty \\ &\geq \widetilde{d}(x)\beta + \mu(x)|\nabla\beta|^2 + h(x). \end{aligned}$$

Thus β is an upper solution of (5.1). Similarly α is a lower solution of (5.1). By Theorem 2.1, (5.1) has a solution $u(x)$ satisfying

$$\alpha(x) \leq u(x) \leq \beta(x) \quad \text{in } \Omega.$$

Since uniqueness of solutions of (5.1) follows from Proposition 4.1, this concludes the proof of the 1).

2) Since $d(x) \leq 0$, then $\operatorname{Supp}(d(x)-1) = \Omega$ and thus, by Proposition 3.1, there exists a non negative solution β (resp. α) of

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = (d(x)-1)u + \|\mu^+\|_\infty |\nabla u|^2 + h^+$$

(resp. $u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = (d(x)-1)u + \|\mu^-\|_\infty |\nabla u|^2 + h^-$). For any $t \in [0,1]$, we can observe that β (resp. $-\alpha$) is an upper (resp. lower) solution of (5.2). Thus there exists a solution u_t of (5.2) satisfying $-\alpha \leq u_t \leq \beta$. By Proposition 4.1, uniqueness of solutions of (5.2) holds and thus 2) holds with $M_1 = \max(\|\beta\|_\infty, \|\alpha\|_\infty)$. □

We now transform (3.1) into a fixed point problem. By Corollary 3.1 used with $c(x) \equiv 1$ and $\lambda = -1$, or alternatively Theorem 2 of [11], we know that, for any $f \in L^p(\Omega)$ the problem

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u + u - \mu(x)|\nabla u|^2 = f(x), \tag{5.3}$$

has a solution. We also know from Proposition 4.1 that it is unique. Thus it is possible to define the operator $K^\mu : L^p(\Omega) \longrightarrow H_0^1(\Omega) \cap L^\infty(\Omega)$ by $K^\mu f = u$ where u is the unique solution of (5.3). The following lemma, which is proved in the Appendix, will be crucial.

Lemma 5.2. *If $\mu \in L^\infty(\Omega)$ then the operator K^μ is a completely continuous operator from $L^p(\Omega)$ into $C(\overline{\Omega})$.*

Next we define the continuous operator $N : C(\overline{\Omega}) \longrightarrow L^p(\Omega)$ by,

$$N(u) = (d(x)+1)\, u + h(x), \qquad \text{for any } u \in C(\overline{\Omega}).$$

With these notations, $u \in C(\overline{\Omega})$ is a solution of (3.1) if and only if u is a fixed point of $K^\mu \circ N$; i.e., if and only if

$$u = K^\mu(N(u)).$$

Now let $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ be given by $T = K^\mu \circ N$. The following result holds.

Proposition 5.1. *Assume that* (A5) *holds and that* (3.1) *has a solution $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then*

$$i(I - T, u_0) = 1.$$

Proof. To show the proposition, we use homotopy arguments. We consider two one-parameter problems, namely the problem (5.2) with $t \in [0,1]$ and the following one

$$u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = (d(x)-s)u + \mu(x)|\nabla u|^2 + h(x), \tag{5.4}$$

for $s \in [0,1]$. Applying Lemma 5.1 we deduce that

1) Any solution $u_s(x)$ of (5.4) with $s \in [0,1]$ satisfies $\|u_s\|_\infty \leq 2\|u_0\|_\infty$. (Case 1) with $\widetilde{d}(x) = d(x) - s$).

2) There exists $M_1 > 0$ such that for any $t \in [0,1]$ any solution $u_t(x)$ of (5.2) satisfies $\|u_t\|_\infty \le M_1$. (Case 2)).

Observe that, if we set

$$\widetilde{N}_s(u) = (d(x) + 1 - s)u + h(x),$$

then problem (5.4) (resp. problem (5.2)) is equivalent to $u - K^\mu(\widetilde{N}_s(u)) = 0$ (resp. $u - K^{(1-t)\mu}(\widetilde{N}_1(u)) = 0$). Thus setting $M = \max(2\|u_0\|_\infty, M_1)$, we have, for all s, $t \in [0,1]$ and all $u \in C(\overline{\Omega})$ with $\|u\|_\infty = M$,

$$u - K^\mu(\widetilde{N}_s(u)) \neq 0, \quad u - K^{(1-t)\mu}(\widetilde{N}_1(u)) \neq 0.$$

Therefore, by homotopy invariance of the degree, we obtain

$$\begin{aligned} \deg(I - T, B(0,M), 0) &= \deg(I - K^\mu \circ \widetilde{N}_0, B(0,M), 0) \\ &= \deg(I - K^\mu \circ \widetilde{N}_1, B(0,M), 0) \\ &= \deg(I - K^0 \circ \widetilde{N}_1, B(0,M), 0) = 1. \end{aligned}$$

By Proposition 4.1, u_0 is the unique solution of (3.1) and thus

$$i(I - T, u_0) = \deg(I - T, B(0,M), 0) = 1. \qquad \square$$

In the rest of the section, we apply the above results to the problem (P_λ). First, from Lemma 5.1 we directly obtain the following a priori estimates for (P_λ) with $\lambda < 0$.

Corollary 5.1. *Assume* (A1) *and, if* $\mathrm{meas}(\Omega \setminus \mathrm{Supp}\, c) > 0$, *assume also that* (Hc) *holds. Then for any* $\lambda_0 < 0$ *there exists* $R = R(\lambda_0) > 0$ *such that, for all* $\lambda \le \lambda_0$, *the unique solution* u_λ *of* (P_λ) *satisfies*

$$\|u_\lambda\|_\infty \le R.$$

Proof. The existence and uniqueness of solutions of (P_λ) when $\lambda < 0$, is already known from Corollary 3.1 and Proposition 4.1. Now the L^∞-bound is obtained from Lemma 5.1, Point 1) used with $d(x) = \lambda_0 c(x)$ and $\widetilde{d}(x) = \lambda c(x)$. That is, the conclusion holds with $R(\lambda_0) = 2\|u_{\lambda_0}\|_\infty$. $\square$

Remark 5.1. A direct consequence of Corollary 5.1 is that none of $\lambda \in]-\infty, 0[$ is a bifurcation point from infinity of (P_λ). (Recall that $\lambda \in \mathbb{R}$ is called a bifurcation point from infinity of (P_λ) if there exists a sequence $\{u_n\}$ of solutions of (P_{λ_n}) with $\lambda_n \to \lambda$ and $\|u_n\|_\infty \to \infty$).

6. Behaviour of the continuum in the half space $\{\lambda > 0\} \times C(\overline{\Omega})$

As a first consequence of (A2) we obtain the following result.

Lemma 6.1. *Assume that* (A2) *holds. For* $\gamma_1 > 0$, *the first eigenvalue of* (1.2), *we have*

1) *If* $\lambda < \gamma_1$, *any solution of problem* (P_λ) *is non negative.*
2) *If* $\lambda = \gamma_1$, *problem* (P_λ) *has no solution.*
3) *If* $\lambda > \gamma_1$, *problem* (P_λ) *has no non negative solutions.*

Proof. First we assume that $\lambda < \gamma_1$. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (P_λ). Using u^- as test function in (P_λ) we obtain

$$-\int_\Omega (|\nabla u^-|^2 - \lambda c(x)|u^-|^2)dx = \int_\Omega (\mu(x)|\nabla u|^2 u^- + h(x)u^-)dx.$$

Since $\lambda < \gamma_1$ the left hand side is non positive and since $\mu(x) \geq 0$ and $h(x) \geq 0$ the right hand side non negative. So necessarily $u^- \equiv 0$ i.e., $u \geq 0$. This proves 1).

Now let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (P_λ). Using $\varphi_1 > 0$, the first eigenfunction of (1.2), as test function in (P_λ) we obtain

$$\begin{aligned}(\gamma_1 - \lambda)\int_\Omega c(x)u\varphi_1 dx &= \int_\Omega \nabla u \nabla \varphi_1 dx - \int_\Omega \lambda c(x)u\varphi_1 dx \\ &= \int_\Omega \mu(x)|\nabla u|^2\varphi_1 dx + \int_\Omega h(x)\varphi_1 dx.\end{aligned} \tag{6.1}$$

Since $\mu(x) \geq 0$ and $h(x) \gneqq 0$, the right hand-side of the above identity is positive. Thus when $\lambda = \gamma_1$, (P_λ) has no solution and 2) is proved.

Finally, when $\lambda > \gamma_1$ and $u \in H_0^1(\Omega)$ is a non negative solution of (P_λ), the left hand-side of (6.1) is non positive which contradicts the positivity of the right hand side. This proves 3). □

To prove the second part of Theorem 1.3, the key point is the derivation of a priori bounds for solution of (P_λ) for $\lambda > 0$. Actually we derive these bounds under a slightly more general assumption than needed.

We consider the problem

$$(R_\lambda) \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta u = \lambda c(x)u + H(x, \nabla u),$$

where we assume

$$(\mathbf{A6}) \qquad \begin{cases} \Omega \text{ has a } C^{1,1} \text{ boundary } \partial\Omega, \\ c \gneqq 0 \text{ and } c \text{ belongs to } L^p(\Omega) \quad \text{for some } p > \frac{N}{2} \\ \mu_1[|\xi|^2 + h(x)] \leq H(x,\xi) \leq \mu_2[|\xi|^2 + h(x)] \\ \text{for some } 0 < \mu_1 \leq \mu_2 < \infty \text{ and } h \geq 0 \text{ with } h \in L^p(\Omega). \end{cases}$$

Adapting the approach of [12], we prove the following result.

Proposition 6.1. *Assume that* (A6) *holds. Then for any* $\Lambda_1 > 0$ *there exists a constant* $M > 0$ *such that, for each* $\lambda \geq \Lambda_1$*, any non negative solution* u *of* (R_λ) *satisfies*

$$\|u\|_\infty \leq M.$$

In the proof of Proposition 6.1 the following two technical lemmas will be useful.

Lemma 6.2. *Let* $p > \frac{N}{2}$ *and* $\theta \in]0,1[$. *There exist* $r \in]0,1[$ *and* $\alpha \in]0, \frac{p-1}{2p-1}[$ *such that, if we define*

$$q = 1 + r + \frac{1+\theta\alpha}{1-\alpha}, \qquad \tau = \frac{1}{q}\frac{\alpha}{1-\alpha} \tag{6.2}$$

then it holds

$$\frac{1}{p} \leq q \leq \frac{2N(p-1)}{p(N-2+2\tau)} \tag{6.3}$$

and

$$1 - \alpha < \frac{2}{q}. \tag{6.4}$$

Proof. First observe that for all $\alpha \in]0,1[$, there exists $r_0 > 0$ such that, for any $0 < r \leq r_0$, (6.4) holds true. Indeed, since $r > 0$, we have

$$q > 1 + \frac{1+\theta\alpha}{1-\alpha} = \frac{2-\alpha+\theta\alpha}{1-\alpha} \quad \text{ or equivalently } \quad \frac{2}{q} < \frac{2(1-\alpha)}{2-\alpha+\theta\alpha}.$$

Also letting $r \to 0^+$ we obtain

$$\frac{2}{q} \nearrow \frac{2(1-\alpha)}{2-\alpha+\theta\alpha}.$$

Thus if

$$1 - \alpha < \frac{2(1-\alpha)}{2-\alpha+\theta\alpha} \tag{6.5}$$

there exists $r_0 > 0$ such that, for all $0 < r \leq r_0$, (6.4) is satisfied. But (6.5) is equivalent to $\alpha(\theta - 1) < 0$ which is always true.

Now, observe that, from the definition of q, we have $q \searrow 2$ as $r \searrow 0$ and $\alpha \searrow 0$. Finally, we see from the definition of τ, that $\tau \searrow 0$ as $\alpha \searrow 0$. Thus as $\alpha \searrow 0$,

$$\frac{2N(p-1)}{p(N-2+2\tau)} \nearrow \frac{2N(p-1)}{p(N-2)} > 2,$$

where the inequality is obtained using the assumption that $p > \frac{N}{2}$. At this point it is clear that taking $r > 0$ sufficiently close to 0 and $\alpha > 0$ sufficiently close to 0, that (6.3) will also hold. $\square$

Lemma 6.3. *Let Ω be a bounded domain with a $C^{1,1}$-boundary and assume that $b, c \in L^p(\Omega)$ with $p > \frac{N}{2}$. For any p, $q \geq 1$ and $\tau \in [0,1]$ satisfying (6.3), there exists $C > 0$ such that, for all $w \in H_0^1(\Omega)$*

$$\left\| \frac{b^{1/q} w}{\varphi_1^\tau} \right\|_q \leq C \|b\|_p \|\nabla w\|_2,$$

where $\varphi_1 > 0$ denotes the first eigenfunction of (1.2).

Proof. For p, $q \geq 1$, $\tau \in [0,1]$ satisfying (6.3), define $s \geq 1$ by

$$\frac{1}{s} = \frac{1}{2} - \frac{1-\tau}{N}.$$

It follows from the second inequality of (6.3) that $\frac{1}{q} \geq (1 - \frac{1}{p})^{-1} \frac{1}{s}$, and this implies

$$\frac{1}{pq} \leq \frac{1}{q} - \frac{1}{s}.$$

From the first inequality of (6.3), we have $\frac{1}{pq} \leq 1$. Thus there exists $\nu \geq 1$ such that

$$\frac{1}{pq} \leq \frac{1}{\nu} \leq \frac{1}{q} - \frac{1}{s}.$$

That is $\nu \geq 1$ satisfies

$$\frac{\nu}{q} \leq p \quad \text{and} \quad \frac{1}{q} \geq \frac{1}{\nu} + \frac{1}{s}.$$

On the other hand, since $0 \leq \overline{c}(x) := \min\{c(x), 1\} \leq c(x)$ and $\varphi_1 \geq 0$, we deduce by the maximum principle that

$$\varphi_1 \geq \psi \text{ in } \Omega,$$

where $\psi \in C^1(\overline{\Omega})$ is the solution of

$$\psi \in H_0^1(\Omega) \ : \ -\Delta \psi = \lambda_1 \overline{c}(x) \varphi_1 \,.$$

By Hopf lemma [24, Lemma 3.26], if $d_\Omega(x)$ denotes the distance of $x \in \Omega$ to the boundary $\partial\Omega$, then there exists $C > 0$ such that

$$\varphi_1(x) \geq \psi(x) \geq C\, d_\Omega(x), \quad \forall x \in \Omega.$$

Now by the Sobolev's embedding and [12, Lemma 2.2], we have, for some constant $C > 0$,

$$\left\| \frac{b^{1/q} w}{\varphi_1^\tau} \right\|_q \leq C \|b^{1/q}\|_\nu \left\| \frac{w}{(d_\Omega)^\tau} \right\|_s \leq C' \|b\|_p^{1/q} \|\nabla w\|_2$$

and the lemma is proved. $\square$

Proof of Proposition 6.1. Fix $\lambda > \Lambda_1$ and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a non negative solution of (R_λ). By Points 2)-3) of Lemma 6.1 we deduce that $\lambda < \gamma_1$. Hence without loss of generality we suppose $\Lambda_1 < \gamma_1$ and $\lambda \in [\Lambda_1, \gamma_1]$.

We define

$$w_i(x) = \frac{1}{\mu_i}(e^{\mu_i u(x)} - 1) \ \text{ and } \ g_i(s) = \frac{1}{\mu_i} \ln(1 + \mu_i s) \text{ for } i = 1, 2.$$

Then we have

$$u = g_1(w_1) = g_2(w_2), \tag{6.6}$$

$$e^{\mu_i u} = 1 + \mu_i w_i, \quad i = 1, 2. \tag{6.7}$$

Direct calculations give us

$$\begin{aligned} -\Delta w_i &= \lambda e^{\mu_i u} c(x) u + e^{\mu_i u}[H(x, \nabla u) - \mu_i |\nabla u|^2] \\ &= \lambda(1 + \mu_i w_i) c(x) g_i(w_i) + (1 + \mu_i w_i)[H(x, \nabla u) - \mu_i |\nabla u|^2]. \end{aligned}$$

Since $\Lambda_1 \le \lambda \le \gamma_1$, we have by (A6)

$$\begin{aligned} -\Delta w_1 &\ge \Lambda_1(1 + \mu_1 w_1) c(x) g_1(w_1) + \mu_1(1 + \mu_1 w_1) h(x), \\ -\Delta w_2 &\le \gamma_1(1 + \mu_2 w_2) c(x) g_2(w_2) + \mu_2(1 + \mu_2 w_2) h(x). \end{aligned}$$

Setting $A_1 = \min(\Lambda_1, \mu_1)$, $A_2 = \max(\gamma_1, \mu_2)$, it becomes

$$-\Delta w_1 \ge A_1(1 + \mu_1 w_1)[c(x) g_1(w_1) + h(x)], \tag{6.8}$$

$$-\Delta w_2 \le A_2(1 + \mu_2 w_2)[c(x) g_2(w_2) + h(x)]. \tag{6.9}$$

From the inequalities (6.8) and (6.9), we shall deduce that w_2 is uniformly bounded in $H_0^1(\Omega)$. This will lead to the proof of the theorem by classical results relating the L^∞ norm of a lower solution to its $H_0^1(\Omega)$ norm. We divide the proof into three steps.

Step 1. *Let $\theta = (\mu_2 - \mu_1)\mu_2^{-1} \in]0, 1[$. Then there exists $C > 0$ independent of $\lambda \in [\Lambda_1, \gamma_1]$ such that*

$$\int_\Omega (1 + \mu_1 w_1)[c(x) g_1(w_1) + h(x)] \varphi_1 \, dx \le C, \tag{6.10}$$

$$\int_\Omega (1 + \mu_2 w_2)^{1-\theta}[c(x) g_2(w_2) + h(x)] \varphi_1 \, dx \le C. \tag{6.11}$$

Indeed, using $\varphi_1 > 0$ as a test function in (6.8), we have

$$\gamma_1 \int_\Omega c(x) w_1 \varphi_1 \, dx \ge A_1 \int_\Omega (1 + \mu_1 w_1)[c(x) g_1(w_1) + h(x)] \varphi_1 \, dx.$$

We note that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $t \le \varepsilon(1 + \mu_1 t) g_1(t) + C_\varepsilon$ for all $t \ge 0$. Thus

$$\gamma_1 \int_\Omega c(x) w_1 \varphi_1 \, dx \le \varepsilon \gamma_1 \int_\Omega (1 + \mu_1 w_1)[c(x) g_1(w_1) + h(x)] \varphi_1 \, dx + C'_\varepsilon$$

and choosing $\varepsilon = \frac{A_1}{2\gamma_1}$, we obtain (6.10). Now observe that by (6.7),

$$1 + \mu_1 w_1 = e^{\mu_1 u} = (e^{\mu_2 u})^{1-\theta} = (1 + \mu_2 w_2)^{1-\theta}.$$

Thus from (6.6) we see that (6.11) is nothing but (6.10).

Step 2. *There exists a constant $C > 0$ independent of $\lambda \in [\Lambda_1, \gamma_1]$ such that*

$$\|\nabla w_2\|_2 \leq C. \tag{6.12}$$

First we use Lemma 6.2 to choose α, $r \in]0, 1[$ such that q and τ given in (6.2) satisfy (6.3) and (6.4).

Using w_2 as a test function in (6.9) it follows that

$$\|\nabla w_2\|_2^2 \leq A_2 \int_\Omega (1 + \mu_2 w_2)[c(x) g_2(w_2) + h(x)] w_2 \, dx.$$

Now using Hölder's inequality, (6.11) and since $w_2 \leq (1 + \mu_2 w_2)\mu_2^{-1}$ we have

$$\begin{aligned}
\|\nabla w_2\|_2^2 &\leq \frac{A_2}{\mu_2} \int_\Omega (1 + \mu_2 w_2)[c(x) g_2(w_2) + h(x)] \frac{\varphi_1^\alpha}{(1 + \mu_2 w_2)^{\theta\alpha}} \frac{(1 + \mu_2 w_2)^{1+\theta\alpha}}{\varphi_1^\alpha} \, dx \\
&\leq \frac{A_2}{\mu_2} \left(\int_\Omega (1 + \mu_2 w_2)[c(x) g_2(w_2) + h(x)] \frac{\varphi_1}{(1 + \mu_2 w_2)^\theta} \, dx \right)^\alpha \\
&\qquad \times \left(\int_\Omega (1 + \mu_2 w_2)[c(x) g_2(w_2) + h(x)] \frac{(1 + \mu_2 w_2)^{\frac{1+\theta\alpha}{1-\alpha}}}{\varphi_1^{\frac{\alpha}{1-\alpha}}} \, dx \right)^{1-\alpha} \\
&\leq \frac{A_2}{\mu_2} C^\alpha \left(\int_\Omega (1 + \mu_2 w_2)[c(x) g_2(w_2) + h(x)] \frac{(1 + \mu_2 w_2)^{\frac{1+\theta\alpha}{1-\alpha}}}{\varphi_1^{\frac{\alpha}{1-\alpha}}} \, dx \right)^{1-\alpha}.
\end{aligned}$$

We note that for r given by Lemma 6.2, there exists $C_r > 0$

$$g_2(t) \leq t^r + C_r \quad \text{for all } t \geq 0.$$

Thus, direct calculations shows that

$$(1 + \mu_2 w_2)[c(x) g(w_2) + h(x)](1 + \mu_2 w_2)^{\frac{1+\theta\alpha}{1-\alpha}} \leq (c(x) + h(x))(w_2^q + C),$$

where q is given in (6.2). Therefore for some $C, C' > 0$ independent of $\lambda \in [\Lambda_1, \gamma_1]$

$$\|\nabla w_2\|_2^2 \leq C \left(\int_\Omega \left(\frac{(c(x) + h(x))^{1/q} w_2}{\varphi_1^\tau} \right)^q dx \right)^{1-\alpha} + C',$$

with q and τ given in (6.2). Here the fact that $\alpha < (p-1)/(2p-1)$ has been used. Applying Lemma 6.3, we then obtain

$$\|\nabla w_2\|_2^2 \leq C \|c + h\|_p^{q(1-\alpha)} \|\nabla w_2\|_2^{q(1-\alpha)} + C'.$$

By (6.4), we have $q(1-\alpha) < 2$ and this concludes the proof of Step 2.

Step 3. *Conclusion.*

We just have to show that the uniform estimate (6.12) derived in Step 2 gives an uniform estimate in the L^∞ norm. Recall that, as a consequence of Theorem 4.1 of [25] combined with Remark 1 on page 289 of that paper (see also Remark 2 p. 202 of [19]), we know that if $w \in H_1(\Omega)$ satisfies

$$\begin{aligned} -\Delta w &\leq d(x)w + f(x), && \text{in } \Omega,\\ w &\leq 0, && \text{on } \partial\Omega, \end{aligned}$$

with d, $f \in L^{p_1}(\Omega)$ for some $p_1 > \frac{N}{2}$, then w satisfies

$$\|w^+\|_\infty \leq C(\|w\|_1 + \|f\|_{p_1}),$$

where C depends on p_1, meas(Ω) and $\|d\|_{p_1}$.

Since w_2 satisfies (6.9), we apply the result of [25] with

$$d(x) = c(x)A_2(1+\mu_2 w_2(x))\frac{\ln(1+\mu_2 w_2(x))}{\mu_2 w_2(x)} + A_2^2 h(x) \quad \text{and} \quad f(x) = A_2 h(x).$$

Observe that, for any $r \in]0,1[$, there exists $C > 0$ such that, for all $x \in \Omega$,

$$c(x)A_2(1+\mu_2 w_2(x))\frac{\ln(1+\mu_2 w_2(x))}{\mu_2 w_2(x)} \leq C\, c(x)|w_2(x)|^r.$$

Thus, since $c(x) \in L^p(\Omega)$ with $p > \frac{N}{2}$ and w_2 is bounded in $L^{\frac{2N}{N-2}}(\Omega)$, taking $r > 0$ sufficiently small we see, using Hölder's inequality, that $c(x)|w_2(x)|^r \in L^{p_1}(\Omega)$ for some $p_1 > \frac{N}{2}$. Now as $h \in L^p(\Omega)$ for some $p > \frac{N}{2}$, clearly all the assumptions of Theorem 4.1 of [25] are satisfied. From (6.12) we then deduce that there exists a constant $C > 0$, independent of $\lambda \in [\Lambda_1, \gamma_1]$ such that

$$\|w_2\|_\infty \leq C.$$

Now since $u = g_2(w_2)$ we deduce that a similar estimate holds for the non negative solutions of (R_λ) and the proof of the proposition is completed. □

7. Proofs of the main results.

In this section we give the proofs of our three theorems.

Proof of Theorem 1.1. The uniqueness of the solution of (P_λ) for $\lambda \leq 0$ is a consequence of Remark 4.1. By Corollary 3.1, (P_λ) with $\lambda < 0$ has a solution $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$. This proves Point 1). To establish the existence of a continuum of solutions of (P_λ), we define $T_\lambda : C(\overline{\Omega}) \to C(\overline{\Omega})$ as

$$T_\lambda(u) = K^\mu((\lambda c(x) + 1)u + h(x)).$$

Hence, (P_λ) is transformed into the fixed point problem $u = T_\lambda(u)$. From Proposition 5.1 we immediately deduce that, for any $\lambda < 0$,

$$i(I - T_\lambda, u_\lambda) = 1.$$

Therefore, if we fix a $\lambda_0 < 0$, by Theorem 2.2 where $E = C(\overline{\Omega})$ and $\Phi(\lambda, u) = u - T_\lambda(u)$, there exists a continuum $C = C^+ \cup C^-$ of solutions of (P_λ) emanating from $(\lambda_0, u_{\lambda_0})$. Taking into account the unboundedness of C^+ and C^- and Corollary 5.1, necessarily $]-\infty, 0[\subset \mathrm{Proj}_{\mathbb{R}} C$ and the proof of Point 2) is concluded.

To prove Point 3), we apply Lemma 5.1 with $d(x) = \overline{\lambda} c(x)$, $\widetilde{d}(x) = \lambda c(x)$ and $\lambda \leq \overline{\lambda} < 0$, to deduce that

$$\|u_\lambda\|_\infty \leq 2\|u_{\overline{\lambda}}\|_\infty \quad \text{for all } \lambda \leq \overline{\lambda} < 0.$$

In particular, if $C_0 := \liminf_{\lambda \to 0^-} \|u_\lambda\|_\infty < \infty$, then there exists a sequence $\overline{\lambda}_n \to 0^-$ such that $C_0 = \lim_{n\to\infty} \|u_{\overline{\lambda}_n}\|_\infty < \infty$. Hence, for every sequence $\lambda_n \to 0^-$ we deduce by the above inequality that $\limsup_{n\to\infty} \|u_{\lambda_n}\|_\infty \leq 2C_0$, which implies that $\limsup_{\lambda\to 0^-} \|u_\lambda\|_\infty < \infty$. Therefore, we have either $\lim_{\lambda\to 0^-} \|u_\lambda\|_\infty = \infty$ or $\limsup_{\lambda\to 0^-} \|u_\lambda\|_\infty < \infty$.

In the first case, using Lemma 5.1 with $d(x) \equiv 0$ and $\widetilde{d}(x) = \lambda c(x)$, we see that (P_0) cannot have a solution. On the other hand, in the last case, for any sequence $\lambda_n \to 0^-$, (u_{λ_n}) is a bounded sequence in $L^\infty(\Omega)$. Thus by Lemma 5.2,

$$u_{\lambda_n} = K^\mu((\lambda_n c(x) + 1)u_{\lambda_n} + h(x))$$

is relatively compact in $C(\overline{\Omega})$. Taking a subsequence if necessary, we may assume $u_{\lambda_n} \to u_0$ in $L^\infty(\Omega)$ for some $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. It is clear that u_0 satisfies $u_0 = K^\mu(u_0 + h(x))$, that is, u_0 is a solution of (P_0). Since we have uniqueness of solutions of (P_0) by Remark 4.1, the limit u_0 does not depend on the choice of λ_n and thus we have $u_\lambda \to u_0$ in $L^\infty(\Omega)$ as $\lambda \to 0^-$. This ends the proof. $\square$

Proof of Theorem 1.2. If we assume that (P_0) has a solution u_0 then using Lemma 5.1 with $d(x) \equiv 0$ and $\widetilde{d}(x) = \lambda c(x)$ we obtain the existence of a solution u_λ of (P_λ) for any $\lambda < 0$. Using Remark 4.1 Point 1) follows.

Now by Proposition 5.1, we know that $i(I - T_0, u_0) = 1$. Thus by Theorem 2.2 there exists a continuum $C \subset \Sigma$ such that both

$$C \cap ([0, \infty[\times C(\overline{\Omega})) \quad \text{and} \quad C \cap (]-\infty, 0] \times C(\overline{\Omega}))$$

are unbounded. Clearly $\{(\lambda, u_\lambda) : \lambda \in]-\infty, 0]\} \subset C$ and Point 2) holds. $\square$

Proof of Theorem 1.3. Let $C \subset \Sigma$ be the continuum obtained in Theorem 1.2. By Lemma 6.1, Point 2) we know that $]-\infty, 0] \subset \mathrm{Proj}_{\mathbb{R}} C \subset]-\infty, \gamma_1[$. Lemma 6.1, Point 1) shows that it consists of non negative functions. In addition, by Theorem 1.2, Point 2), $C \cap ([0, \gamma_1[\times C(\overline{\Omega}))$ is unbounded and hence its projection on $C(\overline{\Omega})$ has to be unbounded. Now we know, by Proposition 6.1, that for every $\Lambda_1 \in]0, \gamma_1[$, there is an a priori bound on the non negative solutions for $\lambda \geq \Lambda_1$. This means that the projection of $C \cap ([\Lambda_1, \gamma_1[\times C(\overline{\Omega}))$ on $C(\overline{\Omega})$ is bounded. Thus C must emanate from infinity to the right of $\lambda = 0$. This proves the first part of the theorem.

Since C contains $(0, u_0)$ with u_0 the unique solution of (P_0), there exists a $\lambda_0 \in]0, \gamma_1[$ such that the problem (P_λ) has at least two solutions for $\lambda \in]0, \lambda_0[$. At this point the proof of the theorem is completed. □

Remark 7.1. The results presented in this paper extend to more general differential operator L in divergence form. Following [17], for our existence results not relying on our uniqueness results, we can handle

$$-\,div(A(x)\nabla u) = \lambda c(x)u + \mu < A(x)\nabla u, \nabla u > +h(x) \tag{7.1}$$

where we assume that $A \in L^\infty(\Omega)^{N\times N}$ with $\Lambda_1 I \geq A \geq \Lambda_2 I$ for some $\Lambda_1 \geq \Lambda_2 > 0$. To derive our uniqueness result we need in addition that $A(x) \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)^{N\times N}$ in (7.1). See [4] in that direction.

8. Appendix : Proof of Lemma 5.2.

To prove Lemma 5.2, we need some preliminary results.

Lemma 8.1. *Let $\{f_n\} \subset L^p(\Omega)$ be a bounded sequence. Then the sequence $\{u_n\} = \{K^\mu(f_n)\}$ is bounded in $L^\infty(\Omega)$ and in $H^1_0(\Omega)$.*

Proof. First we observe that the boundedness of $\{u_n\}$ in $L^\infty(\Omega)$ is a direct consequence of Theorem 1 of [11]. To show that $\{u_n\}$ is also bounded in $H^1_0(\Omega)$ we use a trick that can be found for example in [9]. Let $t = \|\mu\|^2_\infty/2$, $E_n = \exp(tu_n^2)$ and consider the functions $v_n = E_n u_n$. We have $v_n \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and

$$\nabla v_n = E_n(1 + 2tu_n^2)\nabla u_n.$$

Hence using v_n as test functions in

$$u_n \in H^1_0(\Omega) \cap L^\infty(\Omega) : -\Delta u_n + u_n = \mu(x)|\nabla u_n|^2 + f_n(x),$$

and the bound of $\{u_n\}$ in $L^\infty(\Omega)$, we obtain the existence of a constant $D > 0$ such that

$$\begin{aligned}
&\int_\Omega E_n(1 + 2tu_n^2)|\nabla u_n|^2 dx + \int_\Omega E_n u_n^2 dx \\
&\qquad = \int_\Omega f_n(x)E_n u_n dx + \int_\Omega \mu(x)|\nabla u_n|^2 E_n u_n dx \\
&\qquad \leq D + \|\mu\|_\infty \int_\Omega E_n^{1/2}|\nabla u_n||u_n||\nabla u_n|E_n^{1/2} dx \\
&\qquad \leq D + \|\mu\|_\infty \left[\frac{1}{2\|\mu\|_\infty}\int_\Omega E_n|\nabla u_n|^2 dx + \frac{1}{2}\|\mu\|_\infty \int_\Omega u_n^2|\nabla u_n|^2 E_n dx\right] \\
&\qquad \leq D + \frac{1}{2}\int_\Omega E_n(1 + 2tu_n^2)|\nabla u_n|^2 dx.
\end{aligned}$$

We then deduce that

$$\int_\Omega E_n|\nabla u_n|^2 dx + \int_\Omega E_n u_n^2 dx \leq 2D.$$

Recording that $E_n \geq 1$, this shows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. $\square$

Proof of Lemma 5.2. The proof we give is inspired by [11] combined with [3, Remark 2.6] (based in turn on ideas from [19]).

Step 1. *K^μ is a bounded operator from $L^p(\Omega)$ to $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1[$*

Assume that $\{f_n\}$ is a bounded sequence in $L^p(\Omega)$. By Lemma 8.1, $u_n = K^\mu(f_n)$ is bounded in $L^\infty(\Omega)$. We claim that u_n is also bounded in $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in]0,1[$. Indeed, consider a function $\zeta \in C^\infty(\Omega)$ with $0 \leq \zeta(x) \leq 1$, and compact support in a ball B_ρ of radius $\rho > 0$, and set $A_{k,\rho} = \{x \in B_\rho \cap \Omega : |u(x)| > k\}$.

Let us consider the function G_k given by (4.5). For $\varphi(s) = se^{\gamma s^2}$ with $\gamma > 0$ large (to be precised later) we take $\phi = \varphi(G_k(u_n))\zeta^2$ as test function in (5.3). Hence we have

$$\begin{aligned}\int_\Omega \nabla u_n \nabla(G_k(u_n))\varphi'(G_k(u_n))\zeta^2 dx &= \int_\Omega [-u_n + f_n(x)]\varphi(G_k(u_n))\zeta^2 dx \\ &\quad + \int_\Omega \mu(x)|\nabla u_n|^2 \varphi(G_k(u_n))\zeta^2 dx \\ &\quad -2\int_\Omega \zeta\varphi(G_k(u_n))\nabla u_n \nabla\zeta dx.\end{aligned}$$

Now observe that, for $\gamma > \frac{\|\mu\|_\infty^2}{4}$, we have $1 + 2\gamma s^2 - \|\mu\|_\infty |s| \geq 1/2$ and hence $\varphi'(s) - \|\mu\|_\infty |\varphi(s)| \geq \frac{1}{2}e^{\gamma s^2} \geq \frac{1}{2}$. Moreover, we have $G_k(u_n(x))\zeta^2(x) = 0$ for $x \notin A_{k,\rho}$ and $\nabla G_k(u_n) = \nabla u_n$ in $A_{k,\rho}$. This implies that

$$\begin{aligned}&\frac{1}{2}\int_{A_{k,\rho}} |\nabla G_k(u_n)|^2 \zeta^2 dx \\ &\quad \leq \int_{A_{k,\rho}} [\varphi'(G_k(u_n)) - \|\mu\|_\infty |\varphi(G_k(u_n))|]\, |\nabla G_k(u_n)|^2 \zeta^2 dx \\ &\quad \leq \int_{A_{k,\rho}} [-u_n + f_n(x)]\varphi(G_k(u_n))\zeta^2 dx \\ &\qquad\quad + \int_{A_{k,\rho}} (|\mu(x)| - \|\mu\|_\infty)|\nabla u_n|^2 |\varphi(G_k(u_n))|\zeta^2 \\ &\qquad\qquad -2\int_{A_{k,\rho}} \zeta\varphi(G_k(u_n))\nabla u_n \nabla\zeta dx \\ &\quad \leq \int_{A_{k,\rho}} [-u_n + f_n(x)]\varphi(G_k(u_n))\zeta^2 dx + 2\int_{A_{k,\rho}} |\zeta|\, |\varphi(G_k(u_n))||\nabla u_n|\, |\nabla\zeta| dx.\end{aligned}$$

Now recall the existence of C_1 and C_2 such that, for all $n \in \mathbb{N}$, $\|u_n\|_\infty \leq C_1$ and $\|f_n\|_p \leq C_2$. Let C_3 such that, for all $s \in [-C_1, C_1]$, $|\varphi(s)| \leq C_3|s|$ and recall that

$0 \le \zeta \le 1$. Hence we obtain $C = C(C_1, C_2, C_3)$ such that

$$\begin{aligned}\frac{1}{2}\int_{A_{k,\rho}} |\nabla G_k(u_n)|^2 \zeta^2 dx &\le C(\text{meas}(A_{k,\rho}))^{1-\frac{1}{p}} + 2C_3 \int_{A_{k,\rho}} |\zeta||\nabla u_n||\nabla\zeta||G_k(u_n)|dx \\ &\le C(\text{meas}(A_{k,\rho}))^{1-\frac{1}{p}} + \frac{1}{4}\int_{A_{k,\rho}} |\zeta|^2|\nabla u_n|^2 dx \\ &\qquad + 4C_3^2 \int_{A_{k,\rho}} |\nabla\zeta|^2 |G_k(u_n)|^2 dx,\end{aligned}$$

by using Young's inequality. Hence, recalling that, on $A_{k,\rho}$, we have $\nabla G_k(u_n) = \nabla u_n$, we conclude that

$$\frac{1}{4}\int_{A_{k,\rho}} |\nabla u_n|^2 \zeta^2 dx \le C\left((\text{meas}(A_{k,\rho}))^{1-\frac{1}{p}} + \int_{A_{k,\rho}} |\nabla\zeta|^2 |G_k(u_n)|^2 dx\right),$$

where $C = C(C_1, C_2, C_3)$ is a generic constant.

Now we argue as in [19, Theorem IV-1.1, p.251]. For $\sigma \in]0,1[$, choose ζ such that $\zeta \equiv 1$ in the concentric ball $B_{\rho-\sigma\rho}$ (concentric to B_ρ) of radius $\rho - \sigma\rho$ and such that $|\nabla\zeta| < \frac{2}{\sigma\rho}$. Hence, we obtain

$$\begin{aligned}\int_{A_{k,\rho-\sigma\rho}} |\nabla u_n|^2 dx &\le C(1 + (\max_{A_{k,\rho}}(|u(x)| - k))^2 \||\nabla\zeta|^2\|_{L^p(A_{k,\rho})})(\text{meas}(A_{k,\rho}))^{1-\frac{1}{p}} \\ &\le C(1 + \frac{4}{\rho^2\sigma^2}(\rho^N\omega_N)^{1/p}(\max_{A_{k,\rho}}(|u(x)| - k))^2)(\text{meas}(A_{k,\rho}))^{1-\frac{1}{p}},\end{aligned}$$

where ω_N denotes the measure of the unit ball of $\mathbb{R}^N$. Hence, for $k \ge C_1 \ge \max_{B_\rho}|u_n| - \delta$, we have

$$\int_{A_{k,\rho-\sigma\rho}} |\nabla u_n|^2 dx \quad\le\quad \gamma\left(1 + \frac{1}{\sigma^2\rho^{2(1-\frac{N}{2p})}}(\max_{A_{k,\rho}}(|u(x)| - k))^2\right)(\text{meas}(A_{k,\rho}))^{1-\frac{1}{p}}.$$

This means that, for $\delta > 0$ small enough and every $M \ge C_1 \ge \|u_n\|_\infty$, we have $u_n \in B_2(\Omega, M, \gamma, \delta, \frac{1}{2p})$ (see [19, pag. 81]).

Applying [19, Theorem II-6.1 and Theorem II-7.1, p.90 and 91], we deduce that $u_n \in C^{0,\alpha}(\overline{\Omega})$ with $\|u_n\|_{C^{0,\alpha}}$ bounded by a constant C_4 which depends only on $\Omega, M, \gamma, \delta$ and the claim is proved.

Step 2. *K^μ maps bounded sets of $L^p(\Omega)$ to relatively compact sets of $C(\overline{\Omega})$.*

This can be easily deduced from Step 1 and the compact embedding of $C^{0,\alpha}(\overline{\Omega})$ into $C(\overline{\Omega})$.

Step 3. *K^μ is continuous from $L^p(\Omega)$ to $H_0^1(\Omega)$.*

Let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that $f_n \to f$ in $L^p(\Omega)$ and let $\{u_n\}$ be the corresponding solutions of (5.3). By Lemma 8.1, there exists $C > 0$ such that, for all $n \in \mathbb{N}$, $\|u_n\|_\infty \le C$ and $\|u_n\| \le C$. Hence for every subsequence $\{u_{n_k}\}$, there exists a subsubsequence $\{u_{n_{k_j}}\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ and $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u_{n_{k_j}} \rightharpoonup u$ weakly in H, $u_{n_{k_j}} \to u$ strongly in $L^{p'}(\Omega)$ and $u_{n_{k_j}} \to u$ almost everywhere.

Let us prove that $u_{n_{k_j}} \to u$ strongly in H and that u is the solution of (5.3). In that case we shall deduce that $u_n \to u$ in $H_0^1(\Omega)$, namely the continuity of K^μ from $L^p(\Omega)$ to $H_0^1(\Omega)$. Let us define $\tilde{u}_j = u_{n_{k_j}} - u$. Observe that $\tilde{u}_j$ satisfies

$$\tilde{u}_j \in H_0^1(\Omega) \cap L^\infty(\Omega) : -\Delta \tilde{u}_j + \tilde{u}_j = f_{n_{k_j}}(x) + \mu(x)|\nabla u_{n_{k_j}}|^2 + \Delta u - u.$$

Consider the test function $\tilde{v}_j = \widetilde{E}_j \tilde{u}_j$ where $\widetilde{E}_j = \exp(\tilde{t}\tilde{u}_j^2)$ and $\tilde{t} = 2\|\mu\|_\infty^2$. As $\tilde{u}_j \in H_0^1(\Omega) \cap L^\infty(\Omega)$ we have $\tilde{v}_j \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and using the inequality

$$|\nabla u_{n_{k_j}}|^2 \leq 2(|\nabla \tilde{u}_j|^2 + |\nabla u|^2),$$

we obtain

$$\begin{aligned}
&\int_\Omega \widetilde{E}_j(1 + 2\tilde{t}\tilde{u}_j^2)|\nabla \tilde{u}_j|^2\,dx + \int_\Omega \widetilde{E}_j \tilde{u}_j^2\,dx \\
&\quad = \int_\Omega \nabla \tilde{u}_j \nabla \tilde{v}_j\,dx + \int_\Omega \tilde{u}_j \tilde{v}_j\,dx \\
&\quad = \int_\Omega f_{n_{k_j}}(x)\tilde{v}_j\,dx + \int_\Omega \mu(x)|\nabla u_{n_{k_j}}|^2 \tilde{v}_j\,dx \\
&\qquad\qquad - \int_\Omega \widetilde{E}_j \nabla u \nabla \tilde{u}_j(1 + 2\tilde{t}\tilde{u}_j^2)\,dx - \int_\Omega u\tilde{v}_j\,dx \\
&\quad \leq \int_\Omega f_{n_{k_j}}(x)\tilde{E}_j\tilde{u}_j\,dx - \int_\Omega \tilde{E}_j \nabla u \nabla \tilde{u}_j(1 + 2\tilde{t}\tilde{u}_j^2)\,dx - \int_\Omega u\tilde{E}_j\tilde{u}_j\,dx \\
&\qquad +2\|\mu\|_\infty \left(\int_\Omega \tilde{E}_j^{1/2}|\tilde{u}_j||\nabla \tilde{u}_j||\nabla \tilde{u}_j|\tilde{E}_j^{1/2}\,dx + \int_\Omega |\nabla u|^2 \tilde{E}_j \tilde{u}_j\,dx \right) \\
&\quad \leq \int_\Omega f_{n_{k_j}}(x)\tilde{E}_j\tilde{u}_j dx - \int_\Omega \tilde{E}_j \nabla u \nabla \tilde{u}_j(1 + 2\tilde{t}\tilde{u}_j^2)\,dx - \int_\Omega u\tilde{E}_j\tilde{u}_j\,dx \\
&\qquad +2\|\mu\|_\infty \Bigg(\|\mu\|_\infty \int_\Omega \tilde{E}_j|\tilde{u}_j|^2|\nabla \tilde{u}_j|^2\,dx \\
&\qquad\qquad + \frac{1}{4\|\mu\|_\infty} \int_\Omega \tilde{E}_j|\nabla \tilde{u}_j|^2\,dx + \int_\Omega |\nabla u|^2 \tilde{E}_j\tilde{u}_j\,dx \Bigg) \\
&\quad \leq \int_\Omega f_{n_{k_j}}(x)\tilde{E}_j\tilde{u}_j\,dx - \int_\Omega \tilde{E}_j \nabla u \nabla \tilde{u}_j(1 + 2\tilde{t}\tilde{u}_j^2)\,dx - \int_\Omega u\tilde{E}_j\tilde{u}_j\,dx \\
&\qquad + \frac{1}{2}\int_\Omega \tilde{E}_j(1 + 2\tilde{t}\tilde{u}_j^2)|\nabla \tilde{u}_j|^2 dx + 2\|\mu\|_\infty \int_\Omega |\nabla u|^2 \tilde{E}_j\tilde{u}_j\,dx.
\end{aligned}$$

Hence we deduce that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \tilde{E}_j(1+2\tilde{t}\tilde{u}_j^2)|\nabla \tilde{u}_j|^2 dx + \int_\Omega \tilde{E}_j \tilde{u}_j^2 dx \\
&\qquad \le \int_\Omega (f_{n_{k_j}}(x) - f(x))\tilde{E}_j \tilde{u}_j dx + 2\|\mu\|_\infty \int_\Omega |\nabla u|^2 \tilde{E}_j \tilde{u}_j dx \\
&\qquad\quad - \int_\Omega \tilde{E}_j \nabla u \nabla \tilde{u}_j (1+2\tilde{t}\tilde{u}_j^2) dx - \int_\Omega u\tilde{E}_j \tilde{u}_j dx + \int_\Omega f(x)\tilde{E}_j\tilde{u}_j dx.
\end{aligned}
\tag{8.1}
$$

Let us prove that each of the terms on the right hand side converges to zero. For the first one, as the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$ there exists $C_1 > 0$ such that, for all $j \in \mathbb{N}$, $\|\widetilde{E}_j\|_\infty \le C_1$. This implies the existence of a constant $C > 0$ such that

$$
\lim_{j\to\infty} \left| \int_\Omega (f_{n_{k_j}}(x) - f(x))\widetilde{E}_j \tilde{u}_j dx \right| \le C \lim_{j\to\infty} \|f_{n_{k_j}} - f\|_p = 0. \tag{8.2}
$$

For the second term we have $|\nabla u|^2 \widetilde{E}_j \tilde{u}_j \to 0$ a.e. in Ω as $\tilde{u}_j \to 0$ a.e. in Ω and $\widetilde{E}_j$ is bounded. Moreover, for all $j \in \mathbb{N}$,

$$
\left| |\nabla u|^2 \widetilde{E}_j \tilde{u}_j \right| \le CC_1 |\nabla u|^2
$$

with $CC_1|\nabla u|^2 \in L^1(\Omega)$. Hence by Lebesgue's dominated convergence theorem we have that

$$
\int_\Omega |\nabla u|^2 \widetilde{E}_j \tilde{u}_j dx \to 0.
$$

To prove that the third term converges to zero, observe that $\nabla \tilde{u}_j \rightharpoonup 0$ weakly in $L^2(\Omega)$. Hence if we prove that $\widetilde{E}_j \nabla u(1+2\tilde{t}\tilde{u}_j^2)$ converges strongly in $L^2(\Omega)$, we shall obtain

$$
\int_\Omega \widetilde{E}_j \nabla u \nabla \tilde{u}_j (1+2\tilde{t}\tilde{u}_j^2) dx \to 0.
$$

Observe that $\widetilde{E}_j \nabla u(1+2\tilde{t}\tilde{u}_j^2) \to \nabla u$ a.e. in Ω. Moreover we have

$$
\left| \widetilde{E}_j \nabla u (1+2\tilde{t}\tilde{u}_j^2) \right| \le C_1(1+2\tilde{t}C^2)|\nabla u| \quad \text{with} \quad C_1(1+2\tilde{t}C^2)\nabla u \in L^2(\Omega).
$$

Hence, again by Lebesgue dominated convergence theorem, we have $\widetilde{E}_j \nabla u(1 + 2\tilde{t}\tilde{u}_j^2) \to \nabla u$ strongly in $L^2(\Omega)$. For the two last terms observe that

$$
u\widetilde{E}_j \tilde{u}_j \to 0 \text{ a.e. in } \Omega \quad \text{and} \quad |u\widetilde{E}_j\tilde{u}_j| \le CC_1|u|
$$

with $CC_1|u| \in L^1(\Omega)$. This holds true also for $f\widetilde{E}_j\tilde{u}_j$. Hence again we have

$$
\int_\Omega u\widetilde{E}_j\tilde{u}_j dx \to 0 \quad \text{and} \quad \int_\Omega f\widetilde{E}_j\tilde{u}_j dx \to 0.
$$

This implies, by (8.1), that

$$\lim_{j\to\infty} \|\tilde{u}_j\|^2 \le \lim_{j\to\infty} 2\left(\frac{1}{2}\int_\Omega \widetilde{E}_j(1+2\tilde{t}\tilde{u}_j^2)|\nabla \tilde{u}_j|^2 dx + \int_\Omega \widetilde{E}_j \tilde{u}_j^2 dx\right) = 0.$$

As $\tilde{u}_j \to 0$ weakly in $H_0^1(\Omega)$ we obtain $\tilde{u}_j \to 0$ strongly in $H_0^1(\Omega)$, namely $u_{n_{k_j}} \to u$ strongly in $H_0^1(\Omega)$. Hence we can pass to the limit in the equation and $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies

$$-\Delta u + u - \mu(x)|\nabla u|^2 = f, \qquad \text{in } \Omega,$$

At this point we have proved the continuity of K^μ from $L^p(\Omega)$ to $H_0^1(\Omega)$.

Step 4. *K^μ is continuous from $L^p(\Omega)$ to $C(\overline{\Omega})$.*

Let $\{f_n\} \subset L^p(\Omega)$ be a sequence such that $f_n \to f$ in $L^p(\Omega)$. In particular the sequence $\{f_n\}$ is bounded in $L^p(\Omega)$. Hence, by Step 1, for every subsequence $\{f_{n_k}\}_k$ the set $\{u_{n_k} = K^\mu(f_{n_k}) : k \in \mathbb{N}\}$ is relatively compact in $C(\overline{\Omega})$ i.e. there exists a subsequence $(u_{n_{k_j}})_j$ which converges in $C(\overline{\Omega})$ to $v \in C(\overline{\Omega})$. By Step 3, $u_{n_{k_j}} \to u = K^\mu(f)$ in $H_0^1(\Omega)$. In particular $u_{n_{k_j}} \to v$ in $C(\overline{\Omega})$ and $u_{n_{k_j}} \to u$ in $L^2(\Omega)$. By unicity of the limit, we conclude that $u = v$. As this is true for every subsequence, we have also that, if $f_n \to f$ in $L^p(\Omega)$ then $u_n = K^\mu(f_n) \to u = K^\mu(f)$ in $C(\overline{\Omega})$ which concludes the proof. □

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