# Stationary layered solutions in $\mathbf{R}^2$ for a class of non autonomous Allen-Cahn equations

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**Abstract.** We consider a class of non autonomous Allen-Cahn equations

$$-\Delta u(x,y) + a(x)W'(u(x,y)) = 0, \quad (x,y) \in \mathbf{R}^2, \tag{0.1}$$

where $W \in \mathcal{C}^2(\mathbf{R},\mathbf{R})$ is a multiple-well potential and $a \in \mathcal{C}(\mathbf{R},\mathbf{R})$ is a periodic, positive, non-constant function. We look for solutions to (0.1) having uniform limits as $x \to \pm\infty$ corresponding to minima of $W$. We show, via variational methods, that under a nondegeneracy condition on the set of heteroclinic solutions of the associated ordinary differential equation $-\ddot{q}(x) + a(x)W'(q(x)) = 0$, $x \in \mathbf{R}$, the equation (0.1) has solutions which depends on both the variables $x$ and $y$. In contrast, when $a$ is constant such nondegeneracy condition is not satisfied and all such solutions are known to depend only on $x$.

*Key Words:* Heteroclinic Solutions, Entire Solutions, Elliptic Equations, Variational Methods.

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## 1 Introduction

In this paper we deal with a class of semilinear elliptic equations of the form

$$-\Delta u(x,y) + a(x)W'(u(x,y)) = 0, \qquad (x,y) \in \mathbf{R}^2, \tag{1.1}$$

where we assume

$(H_1)$ $a \in \mathcal{C}(\mathbf{R})$ is periodic and positive,

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$(H_2)$ $W \in \mathcal{C}^2(\mathbf{R})$ satisfies

$(i)$ there exist $m \geq 2$ points $\sigma_1, \dots, \sigma_m \in \mathbf{R}$ such that $W(\sigma_i) = 0$, $W''(\sigma_i) > 0$ for any $i = 1 \dots m$ and $W(s) > 0$ for any $s \in \mathbf{R} \setminus \{\sigma_1, \dots, \sigma_m\}$,

$(ii)$ there exists $R_0 > 0$ such that $W'(s)s \geq 0$ for any $|s| \geq R_0$.

This kind of equation arises in various fields of Mathematical Physics and our assumptions on $W$ are modeled on the classical two well Ginzburg-Landau potential $W(s) = (s^2 - 1)^2$. In fact (1.1) can be viewed as a generalization of the stationary Allen-Cahn equation introduced in 1979 by S.M. Allen and J.W. Cahn (see [5]). We recall that the (parabolic) Allen-Cahn equation is a model for phase transitions in binary metallic alloys which corresponds to taking a constant function $a$ and the above double well potential in (1.1). In these models the function $u$ is an order parameter representing pointwise the state of the material. The global minima $\sigma_1, \dots, \sigma_m$ of $W$ are called the *pure phases* and different values of $u$ represent a mixed configuration. Formally, (1.1) is the Euler-Lagrange equation of the action functional $\Phi(u) = \frac{1}{2} \int |\nabla u|^2 dx\, dy + \int a(x) W(u) dx\, dy$. Its first term is the interfacial energy and it penalizes sharp transitions while the second one is associated to the volume energy density and penalizes the states far away from the equilibria. Note in fact that the global minima of $\Phi$ are exactly the pure states $u(x, y) = \sigma_i$ $(i = 1, \dots, m)$.

In this paper we look for two phase layered solutions of (1.1). Namely, given $\sigma_-$, $\sigma_+ \in \{\sigma_1, \dots, \sigma_m\}$, $\sigma_- \neq \sigma_+$, we look for solutions of (1.1) asymptotic as $x \to \pm\infty$ to the pure states $\sigma_\pm$, i.e., solutions of the boundary value problem

$$\begin{cases} -\Delta u(x, y) + a(x) W'(u(x, y)) = 0, & (x, y) \in \mathbf{R}^2 \\ \lim\limits_{x \to \pm\infty} u(x, y) = \sigma_\pm, & \text{uniformly w.r.t. } y \in \mathbf{R}. \end{cases} \tag{1.2}$$

Apart from its physical aspects, problem (1.2) presents interesting mathematical features. In a recent paper, [11], N. Ghoussoub and C. Gui proved a conjecture of De Giorgi (see [9]) related to (1.2). They obtained, in particular, the following result.

**Theorem 1.1** *If $a(x) = a_0 > 0$ for all $x \in \mathbf{R}$ and if $u \in \mathcal{C}^2(\mathbf{R}^2)$ is a solution to (1.2) then $u(x, y) = q(x)$ for all $(x, y) \in \mathbf{R}^2$ where $q \in \mathcal{C}^2(\mathbf{R})$ is a solution of the problem*

$$\begin{cases} -\ddot{q}(x) + a_0 W'(q(x)) = 0, & x \in \mathbf{R} \\ \lim\limits_{x \to \pm\infty} q(x) = \sigma_\pm. \end{cases}$$

In other words, by Theorem 1.1, if $a$ is constant, then any solution of (1.2) depends only on the variable $x$ and it is a solution of the corresponding ordinary differential equation. Briefly, we say that when $a$ is constant, any solution of (1.2) is *one dimensional*.

The aim of the present paper is to show that this is in fact a particular feature of the autonomous Allen-Cahn stationary equation. Indeed, we will prove that for a set of nonconstant functions $a$ for which the associated set of one dimensional solutions is not "too degenerate", problem (1.2) possesses multiple two dimensional solutions.

To be more precise, we need to discuss first some features of the one dimensional problem associated to (1.2), i.e. the problem

$$\begin{cases} -\ddot{q}(x) + a(x)W'(q(x)) = 0, & x \in \mathbf{R} \\ \lim_{x\to\pm\infty} q(x) = \sigma_\pm. \end{cases} \tag{1.3}$$

Following some arguments developed in a series of papers on the heteroclinic problem (see e.g. [2], [12] and the references therein) we study the set of *minimal* solutions of (1.3). We consider the functional

$$F(q) = \int_{\mathbf{R}} \tfrac{1}{2}|\dot{q}(x)|^2 + a(x)W(q(x))\,dx$$

on the Hilbert space

$$E = \{q \in H^1_{loc}(\mathbf{R}) \mid \int_{\mathbf{R}} |\dot{q}(x)|^2\,dx < +\infty\}$$

endowed with the norm $\|q\|^2 = |q(0)|^2 + \int_{\mathbf{R}} |\dot{q}(x)|^2\,dx$.
We prove that given any $i \in \{1,\ldots,m\}$, there exists $j(i) \in \{1,\ldots,m\} \setminus \{i\}$ such that the functional $F$ attains a minimum on the set

$$\Gamma^i = \{q \in \{F < +\infty\} \mid \lim_{t\to-\infty} q(t) = \sigma_i,\ \lim_{t\to+\infty} q(t) = \sigma_{j(i)}\}.$$

Setting $c(i) = \min_{\Gamma^i} F(q)$, we consider the set of minimizers of $F$ on $\Gamma^i$

$$\mathcal{K}^i = \{q \in \Gamma^i \mid F(q) = c(i)\}.$$

Our result establishes that if the set $\mathcal{K}^i$ is discrete in a suitable sense, then (1.2) admits several two dimensional solutions distinct up to translations. Precisely, letting $T$ be the period of $a$, we assume that for some $i \in \{1,\ldots,m\}$ there results

$(*)_i$ there exists $\mathcal{K}^i_0 \subset \mathcal{K}^i$ such that, setting $\mathcal{K}^i_\xi = \{q(\cdot - T\xi) \mid q \in \mathcal{K}_{i,0}\}$ for $\xi \in \mathbf{Z}$, there results $\mathcal{K}^i = \cup_{\xi\in\mathbf{Z}}\mathcal{K}^i_\xi$ and moreover

- $(i)$ if $(q_n) \subset \mathcal{K}^i_0$, there exists $q \in \mathcal{K}^i_0$ such that, along a subsequence, $\|q_n - q\|_{H^1(\mathbf{R})} \to 0$,
- $(ii)$ there exists $\alpha > 0$ such that if $\xi \neq \xi'$ then $\mathsf{dist}(\mathcal{K}^i_\xi, \mathcal{K}^i_{\xi'}) \geq \alpha$.

Here the distance function $\mathsf{dist}$ is defined by

$$\mathsf{dist}(A,B) = \inf\{(\int_{\mathbf{R}} |q_1(x) - q_2(x)|^2 dx)^{\frac{1}{2}} \mid q_1 \in A,\ q_2 \in B\}, \quad A,\ B \subset \Gamma^i.$$

We remark that the assumption $(*)_i$ cannot hold if the function $a$ is constant since in this case the problem is invariant under the continuous group of translations. In fact, in the last section of this paper, we prove that $(*)_i$ is satisfied if and only if $\mathcal{K}^i$ is not a continuum in $\Gamma^i$, that is

$$(*)_i \text{ does not hold } \iff \mathcal{K}^i \text{ is homeomorphic to } \mathbf{R} \tag{1.4}$$

It is known that when $a$ is a small $L^\infty$ periodic perturbation of a positive constant, then the assumption $(*)_i$ can be studied by using Melnikov-Poincaré methods and we refer to [6] and the references therein for such arguments. Here, in Section 4, following [3], [4], we prove that $(*)_i$ is satisfied whenever $a$ is a *slowly oscillating* function. In particular, given any $T$-periodic non constant, continuous function $a > 0$, $(*)_i$ is satisfied for the equation

$$-\varepsilon^2 \Delta u + a(x) W'(u) = 0$$

whenever $\varepsilon > 0$ is small enough. In fact, we prove that given any $T$-periodic continuous function $a \geq 0$ and any $T$-periodic nonconstant, continuous function $b > 0$, setting $a_n(x) = a(x) + b(\frac{x}{n})$, condition $(*)_i$ is satisfied for any $i \in \{1, \dots, m\}$ whenever $n \in \mathbf{N}$ is sufficiently large. This shows in particular that $(*)_i$ holds if $a$ belongs to an $L^\infty$ dense subset of the set of periodic, positive and continuous functions. Finally, we mention that following [3], this result could be further refined proving that $(*)_i$ is "stable" under small $L^\infty$ perturbations of the function $a$.

The main result of the paper can be now stated in the following form.

**Theorem 1.2** *Let $(H_1)$-$(H_2)$ be satisfied. Then for any $i \in \{1, \dots, m\}$ for which $(*)_i$ holds there exist $\xi_1, \dots, \xi_l \in \mathbf{Z} \setminus \{0\}$ such that $\{\sum_{\iota=1}^{l} n_\iota \xi_\iota \,|\, n_\iota \in \mathbf{N} \cup \{0\}\} = \mathbf{Z}$ and for which for any $\iota \in \{1, \dots, l\}$ there exists a solution $u_\iota \in \mathcal{C}^2(\mathbf{R}^2)$ to (1.2) with $\sigma_- = \sigma_i$, $\sigma_+ = \sigma_{j(i)}$, satisfying*

$$\lim_{y \to -\infty} \mathsf{dist}(u_\iota(x,y), \mathcal{K}_0^i) = \lim_{y \to +\infty} \mathsf{dist}(u_\iota(x,y), \mathcal{K}_{\xi_\iota}^i) = 0. \tag{1.5}$$

We remark that by (1.5), since $\xi_\iota \neq 0$, the solution $u_\iota$ is truly two dimensional and since $\{\sum_{\iota=1}^{l} n_\iota \xi_\iota \,|\, n_\iota \in \mathbf{N} \cup \{0\}\} = \mathbf{Z}$, Theorem 1.2 guarantees the existence of at least two of such solutions.

Moreover, we point out that, by (1.4), the following alternative holds: the set of one dimensional solutions of (1.2) is homeomorphic to $\mathbf{R}$ (as in the autonomous case) or problem (1.2) has several two dimensional solutions.

To prove Theorem 1.2, we use a global variational procedure looking for solutions to (1.2) as local minima of a suitable functional. We point out that the natural action functional $\Phi$ considered above is not useful to study solutions of (1.1) different from the pure states $u(x,y) = \sigma_i$ $(i = 1, \dots, m)$. Here, following a renormalization procedure in the spirit of the one introduced by P.H. Rabinowitz in [13], [14], we look for solutions of (1.2) as minima of the functional

$$\varphi(u) = \int_{\mathbf{R}} [\int_{\mathbf{R}} \tfrac{1}{2} |\nabla u(x,y)|^2 + a(x) W(u(x,y))\, dx - c(i)]\, dy$$

on the set

$$\mathcal{M}^i_\xi = \{u \in X^i \mid \lim_{y\to-\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_0) = \lim_{y\to+\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_\xi) = 0\}, \quad \xi \in \mathbf{Z}\setminus\{0\}$$

where

$$X^i = \{u \in H^1_{loc}(\mathbf{R}^2) \mid u(\cdot,y) \in \Gamma^i \text{ for a.e. } y \in \mathbf{R}\}.$$

Actually we show that a minimum $u$ of $\varphi$ on $\mathcal{M}^i_\xi$ is a classical solution of (1.2) which satisfies $u(x,y) \to \sigma_i$ as $x \to -\infty$ and $u(x,y) \to \sigma_{j(i)}$ as $x \to +\infty$, uniformly with respect to $y \in \mathbf{R}$. Not surprisingly, because of possible loss of compactness, the minimization procedure cannot be carried out successfully for every value of $\xi \in \mathbf{Z}$. Nevertheless following an argument, originally due to S.V. Bolotin and V.V. Kozlov [8], we manage to find a special set of generators $\xi_1,\ldots,\xi_l \in \mathbf{Z}\setminus\{0\}$ for which the infimum of $\varphi$ on $\mathcal{M}^i_{\xi_\iota}$, $1 \le \iota \le l$, is reached.

We end this introduction by mentioning a recent paper by S. Alama, L. Bronsard and C. Gui [1] which motivated and inspired our work. In [1] the authors proved that Theorem 1.1 fails in general when instead of a single equation one considers systems of autonomous Allen-Cahn equations. Indeed, under symmetry conditions on the potential and a suitable non degeneracy assumption on the set of minimizers to the associated one dimensional problem, in [1] the existence of a two dimensional solution is proved. We point out that in contrast to our direct minimization approach, in [1] an approximation procedure, using bounded domains in $\mathbf{R}^2$ was developed.

We also note some very recent papers by H. Berestycki, F. Hamel and R. Monneau, [7], and by A. Farina, [10], where Theorem 1.1 is extended to higher dimensions and to more general elliptic operators. See also the references therein for other results in this direction.

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## 2 The one dimensional problem

In this section we study the one dimensional problem associated to (1.1), namely, given $\sigma_\pm \in \{\sigma_1,\ldots,\sigma_m\}$ we look for solutions $q \in \mathcal{C}^2(\mathbf{R})$ to the problem

$$\begin{cases} -\ddot{q}(x) + a(x)W'(q(x)) = 0, \quad x \in \mathbf{R} \\ \lim_{x\to\pm\infty} q(x) = \sigma_\pm. \end{cases} \tag{2.1}$$

Following earlier works on the existence of heteroclinic solutions, see e.g. [2] and [12], we consider the action functional

$$F(q) = \int_{\mathbf{R}} \tfrac{1}{2}|\dot{q}(x)|^2 + a(x)W(q(x))\,dx$$

on the space

$$E = \{q \in H^1_{loc}(\mathbf{R}) \mid \int_{\mathbf{R}} |\dot{q}(x)|^2\,dx < +\infty\}$$

endowed with the Hilbertian norm $\|q\| = (|q(0)|^2 + \int_{\mathbf{R}} |\dot{q}(x)|^2\,dx)^{\frac{1}{2}}$.
It is standard to show that $F$ is weakly lower semicontinuous on $E$ and, plainly by $(H_1)$, $(H_2)$, that $F(q) \geq \frac{1}{2}\int_{\mathbf{R}} |\dot{q}(x)|^2\,dx$ for any $q \in E$.
By $(H_2)$, each $\sigma_i$ is a non degenerate minimum of $W$ and around each of these points $W$ behaves quadratically. Thus there exist $w_0 > 0$ and $\rho_0 \in (0, \frac{1}{6}\inf_{i\neq j}|\sigma_i - \sigma_j|)$ such that

$$\text{if } |s - \sigma_i| \leq 2\rho_0 \text{ then } W''(s) \geq 2w_0 \text{ for any } i \in \{1,\ldots,m\}. \tag{2.2}$$

Moreover, since $a$ is positive, we have that for any $r \in (0, \rho_0)$

$$\mu_r = \inf\{a(x)W(q) \mid x \in \mathbf{R},\ q \notin \cup_{i=1}^m B_r(\sigma_i)\} > 0.$$

Therefore, if $q \in E$ and $q(x) \notin \cup_{i=1}^m B_r(\sigma_i)$ for any $x \in (s,p)$ then

$$\int_s^p \tfrac{1}{2}|\dot{q}|^2 + a(x)W(q)\,dx \geq \tfrac{1}{2(p-s)}\Big(\int_s^p |\dot{q}|\,dx\Big)^2 + \mu_r(p-s) \geq \sqrt{2\mu_r}\,|q(p) - q(s)|. \tag{2.3}$$

By (2.3), we can characterize the asymptotic behavior of the trajectories in $E$.

**Lemma 2.1** *If $q \in \{F < +\infty\}$, then $\lim_{x\to\pm\infty} q(x) = q(\pm\infty) \in \{\sigma_1,\ldots,\sigma_m\}$. Moreover, for any $b > 0$ there exists $C(b) > 0$ such that if $F(q) \leq b$ then $\|q\|_{L^\infty(\mathbf{R})} \leq C(b)$.*

*Proof.* Since $F(q) < +\infty$, by (2.3), there exists $q(+\infty) \in \{\sigma_1,\ldots,\sigma_m\}$ such that $\liminf_{x\to+\infty}|q(x) - q(+\infty)| = 0$. Fixing an arbitrary $r \in (0, \frac{\rho_0}{2})$, we assume by contradiction that $\limsup_{x\to+\infty}|q(x) - q(+\infty)| > 2r$. Then there exists a sequence of disjoint intervals $(p_i, s_i)$, $i \in \mathbf{N}$, such that $|q(p_i) - q(+\infty)| = r$, $|q(s_i) - q(+\infty)| = 2r$ and $r < |q(x) - q(+\infty)| < 2r < \rho_0$ for any $x \in \cup_i(p_i, s_i)$. By (2.3) this implies that $F(q) = +\infty$, a contradiction. In the same way one shows that there exists $q(-\infty) \in \{\sigma_1,\ldots,\sigma_m\}$ such that $q(x) \to q(-\infty)$ as $x \to -\infty$.
The second part of the lemma is again a simple consequence of (2.3). Indeed, let $r \in (0, \rho_0)$ and fix $S > 0$ such that $|\sigma_i| \leq S$ for any $i \in \{1,\ldots,m\}$. If $F(q) \leq b$ and $|q(x)| \geq S + r$ for some $x \in \mathbf{R}$, then by (2.3) one infers that $b \geq \sqrt{2\mu_r}\,|q(x) - (S+r)|$. Therefore if $F(q) \leq b$ then $|q(x)| \leq \frac{b}{\sqrt{2\mu_r}} + S + r$ for any $x \in \mathbf{R}$ and the lemma follows. $\square$

Given $i, j \in \{1,\ldots,m\}$, we define the class

$$\Gamma_{i,j} = \{q \in \{F < +\infty\} \mid \lim_{x\to-\infty} q(x) = \sigma_i,\ \lim_{x\to+\infty} q(x) = \sigma_j\}.$$

Letting $c_{i,j} = \inf_{\Gamma_{i,j}} F(q)$, we observe that, by (2.3), $c(i) = \min_{j\neq i} c_{i,j} > 0$, for any $i \in \{1,\ldots,m\}$. We choose and fix $j(i) \in \{1,\ldots,m\}$ such that $c(i) = c_{i,j(i)}$ and we set $\Gamma^i = \Gamma_{i,j(i)}$.

We shall prove that $F$ has minima in each class $\Gamma^i$. To this aim we start noticing that the trajectories in $\Gamma^i$ with action close to the minimum satisfy suitable concentration properties.

**Lemma 2.2** *There exists $\bar\delta_0 \in (0, \rho_0)$ such that for any $\delta \in (0, \bar\delta_0)$ there exist $\lambda_\delta > 0$, $\rho_\delta > 0$ and $\ell_\delta > 0$ for which, for any $i \in \{1, \dots, m\}$, if $q \in \Gamma^i$ and $F(q) \le c(i) + \lambda_\delta$ then*

*(i) if $\min_{1\le j\le m} |q(x) - \sigma_j| \ge \delta$ for every $x \in (s, p)$ then $p - s \le \ell_\delta$,*

*(ii) $\inf_{x\in\mathbf{R}} |q(x) - \sigma_j| > \delta$ for every $j \in \{1, \dots, m\} \setminus \{i, j(i)\}$,*

*(iii) if $|q(x_-) - \sigma_i| \le \delta$ and $|q(x_+) - \sigma_{j(i)}| \le \delta$ for some $x_-$, $x_+ \in \mathbf{R}$, then $|q(x) - \sigma_i| < \rho_\delta$ for every $x \le x_-$ and $|q(x) - \sigma_{j(i)}| < \rho_\delta$ for every $x \ge x_+$.*

*Moreover $\lambda_\delta + \rho_\delta \to 0$ as $\delta \to 0$.*

*Proof.* Note that $(i)$ plainly follows by (2.3). To prove $(ii)$ and $(iii)$ we first fix some notations. Given $\delta > 0$, let

$$\lambda_\delta = \tfrac{1}{2}\delta^2 + \max_{x\in\mathbf{R}} a(x) \cdot \max_{|s-\sigma_i|\le\delta} W(s) \quad \text{and} \quad r_\delta = \inf\{r > 0 \,|\, \mu_r \ge \lambda_\delta\}.$$

Since $\lambda_\delta \to 0$ as $\delta \to 0$, there exists $\delta_1 \in (0, \rho_0)$ such that $\{r > 0 \,|\, \mu_r \ge \lambda_{\delta_1}\} \ne \emptyset$ and so $r_\delta$ is well defined for any $\delta \in (0, \delta_1)$. In fact, $r_\delta$ is non decreasing on $(0, \delta_1)$ and $r_\delta \to 0$ as $\delta \to 0$. Set $\rho_\delta = \max\{\delta, r_\delta\} + \frac{\lambda_\delta}{\sqrt{\mu_{r_\delta}}}$. Since by definition, $\mu_{r_\delta} \ge \lambda_\delta$, we have that $\rho_\delta \to 0$ as $\delta \to 0$. Let $\bar\delta_0 \in (0, \delta_1)$ be such that $\rho_\delta < \rho_0$ and $\lambda_\delta \le \frac{1}{8}\min\{c(j) \,|\, 1 \le j \le m\}$ for all $\delta \in (0, \bar\delta_0)$.
Now let $\delta \in (0, \bar\delta_0)$ and $q \in \Gamma^i$, $x_- \in \mathbf{R}$ be such that $|q(x_-) - \sigma_i| \le \delta$ and $F(q) \le c(i) + \lambda_\delta$. We define

$$q_-(x) = \begin{cases} \sigma_i & \text{if } x < x_- - 1, \\ (x_- - x)\sigma_i + (x - x_- + 1)q(x_-) & \text{if } x_- - 1 \le x \le x_- \\ q(x) & \text{if } x \ge x_- \end{cases}$$

and note that since $q_- \in \Gamma^i$, $F(q_-) \ge c(i)$. Moreover we have

$$\int_{x_- - 1}^{x_-} \tfrac{1}{2}|\dot q_-|^2 + a(x)W(q_-)\,dx \le \lambda_\delta$$

and

$$F(q_-) = F(q) - \int_{-\infty}^{x_-} \tfrac{1}{2}|\dot q|^2 + a(x)W(q)\,dx + \int_{x_- - 1}^{x_-} \tfrac{1}{2}|\dot q_-|^2 + a(x)W(q_-)\,dx,$$

from which we obtain

$$\int_{-\infty}^{x_-} \tfrac{1}{2}|\dot q|^2 + a(x)W(q)\,dx \le 2\lambda_\delta. \tag{2.4}$$

To prove $(iii)$ we assume by contradiction that there exists $x < x_-$ such that $|q(x) - \sigma_i| \geq \rho_\delta$. Then, by (2.3), we get

$$\int_{-\infty}^{x_-} \tfrac{1}{2}|\dot{q}|^2 + a(x)W(q)\,dx \geq \sqrt{2\mu_{r_\delta}}(\rho_\delta - r_\delta) \geq 2\sqrt{2}\lambda_\delta$$

which contradicts (2.4). This proves that $|q(x) - \sigma_i| < \rho_\delta$ for any $x \leq x_-$. Analogously, if $|q(x_+) - \sigma_{j(i)}| \leq \delta$ then $|q(x) - \sigma_{j(i)}| < \rho_\delta$ for any $x \geq x_+$.
To establish $(ii)$ we prove that $\inf_{x\in(x_-,x_+)} |q(x) - \sigma_j| > \delta$ for any $j \in \{1, \dots, m\} \setminus \{i, j(i)\}$ and $x_-, x_+$ such that $|q(x_-) - \sigma_i| \leq \delta$ and $|q(x_+) - \sigma_{j(i)}| \leq \delta$. By $(iii)$ this will imply $(ii)$. Assume by contradiction that there exists $\bar{x} \in (x_-, x_+)$ and $\iota \in \{1, \dots, m\} \setminus \{i, j(i)\}$ such that $|q(\bar{x}) - \sigma_\iota| = \delta$. We define the functions

$$q_-(x) = \begin{cases} q(x) & \text{if } x < \bar{x}, \\ (\bar{x} + 1 - x)q(\bar{x}) + (x - \bar{x})\sigma_\iota & \text{if } \bar{x} \leq x \leq \bar{x} + 1 \\ \sigma_\iota & \text{if } x \geq \bar{x} + 1, \end{cases}$$

$$q_+(x) = \begin{cases} \sigma_\iota & \text{if } x < \bar{x} - 1, \\ (\bar{x} - x)\sigma_\iota + (x - \bar{x} + 1)q(\bar{x}) & \text{if } \bar{x} - 1 \leq x \leq \bar{x} \\ q(x) & \text{if } x \geq \bar{x}. \end{cases}$$

Clearly $q_- \in \Gamma_{i,\iota}$, $q_+ \in \Gamma_{\iota,j(i)}$ and thus, on one hand, $F(q_-) + F(q_+) \geq c(i) + c(\iota)$. On the other hand, arguing as above, one checks that $F(q_-) + F(q_+) \leq F(q) + 4\lambda_\delta$. By the definition of $\lambda_\delta$, this gives the contradiction $c(i) + c(\iota) \leq F(q) + 4\lambda_\delta \leq c(i) + 5\lambda_\delta \leq c(i) + \frac{5}{8}\min\{c(j) \,|\, 1 \leq j \leq m\}$. □

According to Lemma 2.2, we fix $\bar{\delta} \in (0, \bar{\delta}_0)$ such that $\bar{\rho} = \rho_{\bar{\delta}} \leq \frac{\rho_0}{4}$ and we denote $\bar{\lambda} = \lambda_{\bar{\delta}}$ and $\bar{\ell} = \ell_{\bar{\delta}}$.

To exploit compactness properties of $F$ in $\Gamma^i$, by Lemma 2.2, it will be useful to introduce the function $X : E \to \mathbf{R} \cup \{+\infty\}$ given by

$$X(q) = \sup\{x \in \mathbf{R} \,|\, \min_{1\leq j\leq m} |q(x) - \sigma_j| \geq \rho_0\}.$$

Note that if $q_n \to q_0$ weakly in $E$ then, by the Sobolev Immersion Theorem, $q_n \to q_0$ in $L^\infty_{loc}(\mathbf{R})$. This implies that if $X(q_n) \to X_0 \in \mathbf{R}$, then $\min_{1\leq j\leq m} |q_0(X_0) - \sigma_j| = \rho_0$ and $X(q_0) \geq X_0$ follows by definition.

By Lemma 2.2, we can give a further characterization of the sublevels of $F$ on the classes $\Gamma^i$. To this aim we define, for $i \in \{1, \dots, m\}$, the function

$$Q_i(x) = \begin{cases} \sigma_i & \text{if } x < 0, \\ (1 - x)\sigma_i + x\sigma_{j(i)} & \text{if } 0 \leq x \leq 1 \\ \sigma_{j(i)} & \text{if } x > 1 \end{cases} \tag{2.5}$$

noting that $Q_i \in \Gamma^i$. Then we can show that the minimizing sequences for $F$ in $\Gamma^i$ are precompact in the following sense.

**Lemma 2.3** *If $(q_n) \subset \Gamma^i$ is such that $F(q_n) \to c(i)$ and $X(q_n) \to X_0 \in \mathbf{R}$, then there exists $q_0 \in \Gamma^i$ such that, along a subsequence, $\|q_n - q_0\|_{H^1(\mathbf{R})} \to 0$, $F(q_0) = c(i)$ and $X(q_0) = X_0$.*

*Proof.* First note that since $F(q_n) \leq C$, we have $\int_{\mathbf{R}} |\dot{q}_n|^2\,dx \leq C$. Moreover, by Lemma 2.1, we obtain $\|q_n\|_{L^\infty} \leq C'$ for any $n \in \mathbf{N}$ and therefore that $(q_n)$ is bounded in $E$. So we can conclude that there exists $q_0 \in E$ such that along a subsequence (still denoted $q_n$) $q_n \to q_0$ weakly in $E$ and, by semicontinuity, $F(q_0) \leq c(i)$.
We have to prove that $q_0 \in \Gamma^i$ and $\|q_n - q_0\|_{H^1(\mathbf{R})} \to 0$. To this aim note that for any $\varepsilon > 0$ there exist $\lambda \in (0, \bar{\lambda})$, $\ell > \bar{\ell}$ such that if $q \in \Gamma^i \cap \{F \leq c(i) + \lambda\}$ then

$$\int_{|x-X(q)|\geq\ell} |\dot{q}|^2 + |q(x) - Q_i(x - X(q))|^2 dx \leq \varepsilon \text{ and } \int_{|x-X(q)|\geq\ell} W(q(x))dx \leq \varepsilon. \quad (2.6)$$

Indeed, setting $\bar{a} = \min_{\mathbf{R}} a(x)$, by Lemma 2.2 we can choose $\delta < \bar{\delta}$ such that $\lambda_\delta \leq \frac{\varepsilon}{3}\min\{\frac{1}{2}, \bar{a}w_0\}$. The same lemma says that if $q \in \Gamma^i \cap \{F \leq c(i) + \lambda_\delta\}$ then $|q(x) - \sigma_i| \leq \rho_\delta \leq \bar{\rho}$ for any $x \leq X(q) - \ell_\delta$ and $|q(x) - \sigma_i| \leq \rho_\delta \leq \bar{\rho}$ for any $x \geq X(q) + \ell_\delta$. In fact, by Lemma 2.2-$(i)$, there exist $x_- \in (X(q) - \ell_\delta, X(q))$ and $x_+ \in (X(q), X(q) + \ell_\delta)$ such that $|q(x_-) - \sigma_i|$, $|q(x_+) - \sigma_{j(i)}| \leq \delta$. We define the function

$$\bar{q}(x) = \begin{cases} \sigma_i & \text{if } x < x_- - 1, \\ (x_- - x)\sigma_i + (x - x_- + 1)q(x_-) & \text{if } x_- - 1 \leq x \leq x_- \\ q(x) & \text{if } x_- \leq x \leq x_+ \\ (x_+ + 1 - x)q(x_+) + (x - x_+)\sigma_{j(i)} & \text{if } x_+ \leq x \leq x_+ + 1 \\ \sigma_{j(i)} & \text{if } x > x_+ + 1, \end{cases}$$

and arguing as in the proof of Lemma 2.2, since $F(\bar{q}) \geq c(i)$, we obtain

$$\int_{|x-X(q)|\geq\ell_\delta+1} \tfrac{1}{2}|\dot{q}|^2 + a(x)W(q)\,dx \leq F(q) - F(\bar{q}) + 2\lambda_\delta \leq 3\lambda_\delta \leq \varepsilon\min\{\tfrac{1}{2}, \bar{a}w_0\}.$$

Then, observing that $W(q(x)) \geq w_0|q(x) - Q_i(x - X(q))|^2$ for any $|x - X(q)| \geq \ell_\delta + 1$, (2.6) follows setting $\ell = \ell_\delta + 1$ and $\lambda = \lambda_\delta$.
Now, since $X(q_n) \to X_0$ and $F(q_n) \to c(i)$, by (2.6), we derive that for any $\varepsilon > 0$ there exists $\ell > 0$ for which $\int_{|x-X_0|\geq\ell} |q_n(x) - Q_i(x - X_0)|^2\,dx \leq \varepsilon$ and $\int_{|x-X_0|\geq\ell} W(q_n(x))\,dx \leq \varepsilon$ for any $n \in \mathbf{N}$ and then

$$\int_{|x-X_0|\geq\ell} |q_0(x) - Q_i(x - X_0)|^2\,dx \leq \varepsilon \text{ and } \int_{|x-X_0|\geq\ell} W(q_0(x))\,dx \leq \varepsilon. \quad (2.7)$$

The first inequality in (2.7) implies by Lemma 2.1 that $q_0 \in \Gamma^i$ and then that $F(q_0) = c(i)$. Moreover, since $q_n \to q_0$ in $L^\infty_{loc}(\mathbf{R})$, the arbitrariness of $\varepsilon$ in (2.7) implies also that $q_n - q_0 \to 0$ in $L^2(\mathbf{R})$ and $\int_{\mathbf{R}} W(q_n(x))\,dx \to \int_{\mathbf{R}} W(q_0(x))\,dx$ as $n \to \infty$. Then, since $F(q_n) \to c(i) = F(q_0)$, we obtain $\int_{\mathbf{R}} |\dot{q}_n|^2\,dx \to \int_{\mathbf{R}} |\dot{q}_0|^2\,dx$ which together with the fact that $\dot{q}_n \to \dot{q}_0$ weakly in $L^2(\mathbf{R})$, implies $\|q_n - q_0\|_{H^1(\mathbf{R})} \to 0$.
To complete the proof we have to show that $X(q_0) = X_0$. Recalling that $X_0 \leq X(q_0)$, we assume by contradiction that $X_0 < X(q_0)$. By definition of $X(q_0)$, $|q_0(x) - \sigma_{j(i)}| < \rho_0$ for all $x > X(q_0)$ and therefore, by (2.2), since $q_0$ is a solution to problem (2.1), one can prove that $|\ddot{q}_0(x)| > 0$ for all $x > X(q_0)$. Then, since $\dot{q}_0(x) \to 0$ as $x \to +\infty$, we derive that $|\dot{q}_0(X(q_0))| > 0$ and hence the existence of a $x_0 \in (X_0, X(q_0))$ such that $|q_0(x_0) - \sigma_{j(i)}| > \rho_0$. Therefore, by uniform convergence, we obtain $|q_n(x_0) - \sigma_{j(i)}| > \rho_0$ for $n$ large enough, a contradiction since $X(q_n) \to X_0$. $\square$

As direct consequence of Lemma 2.3 we plainly obtain the following estimate which will be essential in the next section. For all $r > 0$ there exists $h_r > 0$ such that

$$\text{if } q \in \Gamma^i \text{ and } \inf_{z \in \mathcal{K}^i} \|q - z\|_{H^1(\mathbf{R})} \geq r \text{ then } F(q) \geq c(i) + h_r \tag{2.8}$$

Moreover, we remark that since the function $a$ is periodic, there always exists a minimizing sequence $(q_n) \subset \Gamma^i$ for $F$ such that $X(q_n) \in [0, T]$ for any $n \in \mathbf{N}$, where $T$ is the period of $a$. Then, by Lemma 2.3, the set $\mathcal{K}^i = \{q \in \Gamma^i \,|\, F(q) = c(i)\}$ is not empty for any $i \in \{1, \ldots, m\}$. Clearly any minimum of $F$ on $\Gamma^i$ is a solution of (2.1) and so a one dimensional solution of (1.2).

Finally, we note that, by the definition of $\Gamma^i$ and of the function $Q_i$ in (2.5), since $W$ behaves quadratically around each $\sigma_i$, we have

$$\int_{\mathbf{R}} |q(x) - Q_i(x)|^2 dx < +\infty, \quad \forall q \in \Gamma^i.$$

Therefore the following metric is well defined on $\Gamma^i$

$$\mathsf{d}(q_1, q_2) = \Big(\int_{\mathbf{R}} |q_1(x) - q_2(x)|^2 \, dx\Big)^{\frac{1}{2}}, \quad \forall q_1, q_2 \in \Gamma^i.$$

Note that the metric space $(\Gamma^i, \mathsf{d})$ is not complete and we will denote by $Y^i$ its completion. Moreover, given $A, B \subset \Gamma^i$, we set $diam(A) = \sup\{\mathsf{d}(q_1, q_2) \,|\, q_1, q_2 \in A\}$ and $\mathsf{dist}(A, B) = \inf\{\mathsf{d}(q_1, q_2) \,|\, q_1 \in A,\, q_2 \in B\}$.

# 3 Two dimensional solutions

In this section we study the existence of two dimensional solutions of (1.2). This study is done assuming, a priori, some discreteness on the set $\mathcal{K}^i$, $i = 1, \ldots, m$, which will be essential to recover sufficient compactness in the problem. Letting $T$ be the period of $a$, we assume that there exists $i \in \{1, \ldots, m\}$ such that

$(*)_i$ there exists $\mathcal{K}^i_0 \subset \mathcal{K}^i$ such that, setting $\mathcal{K}^i_\xi = \{q(\cdot - T\xi) \,|\, q \in \mathcal{K}_{i,0}\}$ for $\xi \in \mathbf{Z}$, there results $\mathcal{K}^i = \cup_{\xi \in \mathbf{Z}} \mathcal{K}^i_\xi$ and moreover

- $(i)$ if $(q_n) \subset \mathcal{K}^i_0$, there exists $q \in \mathcal{K}^i_0$ such that, along a subsequence, $\|q_n - q\|_{H^1(\mathbf{R})} \to 0$,
- $(ii)$ there exists $\alpha > 0$ such that if $\xi \neq \xi'$ then $\mathsf{dist}(\mathcal{K}^i_\xi, \mathcal{K}^i_{\xi'}) \geq \alpha$.

Postponing the discussion of assumption $(*)_i$ to the next section, we now make precise the variational setting.

Let

$$X^i = \{u \in H^1_{loc}(\mathbf{R}^2) \,|\, u(\cdot, y) \in \Gamma^i \text{ for a.e. } y \in \mathbf{R}\}$$

and note that if $u \in X^i$, then the function $y \to \int_{\mathbf{R}} \frac{1}{2}|\nabla u(x,y)|^2 + a(x)W(u(x,y))\,dx$ is measurable and greater than or equal to $c(i)$ for a.e. $y \in \mathbf{R}$. Therefore the functional $\varphi : X^i \to \mathbf{R} \cup \{+\infty\}$ given by

$$\varphi(u) = \int_{\mathbf{R}} [\int_{\mathbf{R}} \tfrac{1}{2}|\nabla u(x,y)|^2 + a(x)W(u(x,y))\,dx - c(i)]\,dy\,, \qquad u \in X^i,$$

is well defined. It can be rewritten in the more enlightening form

$$\varphi(u) = \int_{\mathbf{R}} [\int_{\mathbf{R}} \tfrac{1}{2}|\partial_y u(x,y)|^2\,dx + F(u(\cdot,y)) - c(i)]\,dy, \qquad u \in X^i.$$

We will look for non trivial two phase solutions of (1.1) as minima of $\varphi$ on suitable subsets of $X^i$. To this end we first discuss some preliminary properties of $\varphi$.
First of all we note that $\varphi(u) \geq 0$ for all $u \in X^i$ and if $q \in \mathcal{K}^i$, then the function $u(x,y) = q(x)$ belongs to $X^i$ and $\varphi(u) = 0$, i.e., the one dimensional solutions of (1.1) are global minima of $\varphi$ on $X^i$. Moreover, $\varphi$ satisfies the following semicontinuity property.

**Lemma 3.1** *Let $(u_n) \subset X^i$ and $u \in X^i$ be such that $u_n \to u$ weakly in $H^1(\Omega)$ for every $\Omega \subset\subset \mathbf{R}^2$, then $\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n)$.*

*Proof.* Let $M, L > 0$ and $Q_{M,L} = (-M,M) \times (-L,L)$. Since $u_n \to u$ weakly in $H^1(Q_{M,L})$ we have $u_n \to u$ in $L^2(Q_{M,L})$ and $\nabla u_n \to \nabla u$ weakly in $L^2(Q_{M,L})$.
By Lusin and Egoroff Theorems, given any $\varepsilon > 0$, there exists a compact set $K \subset Q_{M,L}$ such that $\nabla u$ is continuous on $K$, $u_n \to u$ uniformly on $K$ and

$$\int_K \tfrac{1}{2}|\nabla u|^2 + aW(u)\,dx\,dy \geq \int_{Q_{M,L}} \tfrac{1}{2}|\nabla u|^2 + aW(u)\,dx\,dy - \varepsilon.$$

Now if $\liminf_{n\to\infty} \varphi(u_n) = +\infty$ the lemma is trivially true and thus we can assume that $\varphi(u_n) \leq C$ for any $n \in \mathbf{N}$. In this case, as $n \to \infty$, we obtain

$$\begin{aligned}
\varphi(u_n) &\geq \int_{Q_{M,L}} \tfrac{1}{2}|\nabla u_n|^2 + a\,W(u_n)\,dx\,dy - 2c(i)L \\
&\geq \int_K \tfrac{1}{2}|\nabla u_n|^2 + a\,W(u_n)\,dx\,dy - 2c(i)L \\
&\geq \int_K \tfrac{1}{2}|\nabla u|^2\,dx\,dy + \int_K \tfrac{1}{2}|\nabla u|(|\nabla u_n| - |\nabla u|)\,dx\,dy + \int_K a\,W(u_n)\,dx\,dy - 2c(i)L \\
&= \int_K \tfrac{1}{2}|\nabla u|^2\,dx\,dy + o(1) + \int_K a\,W(u) + a\,(W(u_n) - W(u))\,dx\,dy - 2c(i)L \\
&= \int_K \tfrac{1}{2}|\nabla u|^2 + a\,W(u)\,dx\,dy - 2c(i)L + o(1) \\
&\geq \int_{Q_{M,L}} \tfrac{1}{2}|\nabla u|^2 + a\,W(u)\,dx\,dy - 2c(i)L + o(1) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$\liminf_{n\to\infty} \varphi(q_n) \geq \int_{-L}^{L} [\int_{-M}^{M} \tfrac{1}{2}|\nabla u|^2 + a\,W(u)\,dx - c(i)]\,dy, \quad \forall M, L > 0$$

and since $u \in X^i$ the lemma follows. □

Concerning the coerciveness of $\varphi$, we have first to consider some estimates which will be useful to characterize the compactness properties of sublevels of $\varphi$.
First we note that if $u \in X^i$ then $F(u(\cdot,y)) \geq c(i)$ for a.e. $y \in \mathbf{R}$ and so

$$\|\partial_y u\|^2_{L^2(\mathbf{R}^2)} \leq 2\varphi(u) \qquad \forall u \in X^i. \tag{3.1}$$

Moreover, since $W(s) \geq 0$ for any $s \in \mathbf{R}$, setting

$$T_L = \{(x,y) \in \mathbf{R}^2 \,|\, |y| < L\}, \quad L > 0,$$

we have

$$\begin{aligned}\varphi(u) &\geq \int_{-L}^{L} F(u(\cdot,y)) - c(i)\, dy + \tfrac{1}{2}\|\partial_y u\|^2_{L^2(\mathbf{R}^2)} \\ &\geq \tfrac{1}{2}\|\partial_x u\|^2_{L^2(T_L)} + \tfrac{1}{2}\|\partial_y u\|^2_{L^2(\mathbf{R}^2)} - 2c(i)L\end{aligned}$$

from which we derive

$$\|\nabla u\|^2_{L^2(T_L)} \leq 2\varphi(u) + 4c(i)L \qquad \forall u \in X^i \text{ and } \forall L > 0. \tag{3.2}$$

Now note that, by Fubini Theorem, if $u \in X^i$ then $u(x,\cdot) \in H^1_{loc}(\mathbf{R})$ for a.e. $x \in \mathbf{R}$. Therefore, if $y_1 < y_2 \in \mathbf{R}$ then $u(x,y_2) - u(x,y_1) = \int_{y_1}^{y_2} \partial_y u(x,y)\, dy$ holds for any $u \in X^i$, for a.e. $x \in \mathbf{R}$. So, if $u \in X^i$, by (3.1) we obtain for $y_1, y_2 \in \mathbf{R}$ that

$$\begin{aligned}\int_{\mathbf{R}} |u(x,y_2) - u(x,y_1)|^2\, dx &= \int_{\mathbf{R}} |\int_{y_1}^{y_2} \partial_y u(x,y)\, dy\,|^2 dx \\ &\leq |y_2 - y_1| \int_{\mathbf{R}}\int_{\mathbf{R}} |\partial_y u(x,y)|^2\, dy\, dx \leq 2\varphi(u)|y_2 - y_1|.\end{aligned}$$

Given $u \in X^i$, by definition, the function $u(\cdot,y) \in \Gamma^i$ for a.e. $y \in \mathbf{R}$. If $\varphi(u) < +\infty$, by the previous estimates, the function $y \to u(\cdot,y)$ is Holder continuous from a dense subset of $\mathbf{R}$ with values in $\Gamma^i$ and so it can be extended to a continuous function on $\mathbf{R}$ considering as target space the complete metric space $Y^i$. According to that, any function $u \in X^i \cap \{\varphi < +\infty\}$ defines a continuous trajectory in $Y^i$ verifying

$$\mathsf{d}(u(\cdot,y_2), u(\cdot,y_1))^2 \leq 2\varphi(u)|y_2 - y_1|, \quad \forall\, y_1, y_2 \in \mathbf{R}. \tag{3.3}$$

In particular, by (3.3), we obtain that if $A \subset \Gamma^i$ and $u \in X^i \cap \{\varphi < +\infty\}$, then

$$|\mathsf{dist}(u(\cdot,y_1), A) - \mathsf{dist}(u(\cdot,y_2), A)| \leq \sqrt{2\varphi(u)}\, |y_2 - y_1|^{\frac{1}{2}} \qquad \forall y_1, y_2 \in \mathbf{R}. \tag{3.4}$$

Another important estimate is a kind of counterpart for the functional $\varphi$ of the estimate (2.3) on $F$ given in Section 2.
By (2.8), if $(y_1, y_2) \subset \mathbf{R}$ and $u \in X^i$ are such that $\inf_{z \in \mathcal{K}^i} \|u(\cdot,y) - z\|_{H^1(\mathbf{R})} \geq r > 0$ for a.e. $y \in (y_1, y_2)$, then

$$\varphi(u) \geq \int_{y_1}^{y_2} [\int_{\mathbf{R}} \tfrac{1}{2}|\partial_y u(x,y)|^2\, dx + F(u(\cdot,y)) - c(i)]\, dy$$

$$\begin{aligned}
&\geq \int_{y_1}^{y_2} \tfrac{1}{2} \int_{\mathbf{R}} |\partial_y u(x,y)|^2 \, dx \, dy + h_r(y_2 - y_1) \\
&\geq \tfrac{1}{2(y_2-y_1)} \int_{\mathbf{R}} (\int_{y_1}^{y_2} |\partial_y u(x,y)| \, dy)^2 \, dx + h_r(y_2 - y_1) \qquad (3.5)\\
&\geq \tfrac{1}{2(y_2-y_1)} \mathsf{d}(u(\cdot,y_1),u(\cdot,y_2))^2 + h_r(y_2-y_1) \geq \sqrt{2h_r}\mathsf{d}(u(\cdot,y_1),u(\cdot,y_2)).
\end{aligned}$$

In the next lemma, using (3.5) and assumption $(*)_i$, we will prove that for any $u \in X^i \cap \{\varphi \leq C\}$ the functions $y \to u(\cdot,y)$ are uniformly bounded with respect to $\mathsf{d}$.

By (2.8), corresponding to $r_0 = \frac{\alpha}{3}$, let us fix $h_0 > 0$ such that

$$\text{if } q \in \Gamma^i \text{ and } \inf_{z \in \mathcal{K}^i} \|q - z\|_{H^1(\mathbf{R})} \geq \tfrac{r_0}{2} \text{ then } F(q) \geq c(i) + h_0. \qquad (3.6)$$

Then, we have

**Lemma 3.2** *For any $C > 0$ there exists $C' > 0$ such that if $u \in X^i \cap \{\varphi \leq C\}$, then $\mathsf{d}(u(\cdot,y_1),u(\cdot,y_2)) \leq C'$ for any $y_1$, $y_2 \in \mathbf{R}$.*

*Proof.* Denoting $\gamma(y) = u(\cdot,y)$, $y \in \mathbf{R}$, we can consider $\gamma$ as a path in $Y^i$. For any $y_1 < y_2 \in \mathbf{R}$, by compactness, $\gamma([y_1,y_2])$ intersects only a finite number of sets $B_{r_0}(\mathcal{K}^i_\xi)$, $\xi \in \mathbf{Z}$. Let $\{B_i \,|\, i = 1,\dots,k\}$ be the family in $\{B_{r_0}(\mathcal{K}^i_\xi) \,|\, B_{r_0}(\mathcal{K}^i_\xi) \cap \gamma([y_1,y_2]) \neq \emptyset, \, \xi \in \mathbf{Z}\}$ such that if $\gamma(y) \notin \cup_{i=1}^k B_i$, $y \in [y_1,y_2]$, then $\mathsf{dist}(\gamma(y),\mathcal{K}^i) \geq r_0$ and $B_i \neq B_{i+1}$ for all $i \in \{1,\dots,k-1\}$.

Noting that $\text{diam}(B_i) = b_0 > 0$ for any $i \in \{1,\dots,k\}$ and $\mathsf{dist}(B_{i+1},B_i) \geq r_0$ for any $i \in \{1,\dots,k-1\}$, from (3.5) and (3.6) one obtains that

$$\varphi(u) \geq \sqrt{2h_0} \max\{\mathsf{d}(\gamma(y_1),\gamma(y_2)) - kb_0, (k-1)r_0\}.$$

Therefore, if $\varphi(u) \leq C$ then $k \leq \frac{1}{\sqrt{2h_0}r_0}(C + r_0\sqrt{2h_0})$ and hence $\mathsf{d}(\gamma(y_1),\gamma(y_2)) \leq \frac{1}{\sqrt{2h_0}}(C + \frac{b_0}{r_0}(C + r_0\sqrt{2h_0})) = C'$. $\square$

Another consequence of (3.5) and of the assumption $(*)_i$ is that they provide information on the asymptotic behavior of the functions in the sublevels of $\varphi$ as $y \to \pm\infty$. More precisely we have

**Lemma 3.3** *If $u \in X^i \cap \{\varphi < +\infty\}$, there exist $\xi^\pm \in \mathbf{Z}$ such that*

$$\mathsf{dist}(u(\cdot,y),\mathcal{K}^i_{\xi^\pm}) \to 0 \quad \text{as} \quad y \to \pm\infty.$$

*Proof.* First of all note that since $\varphi(u) < +\infty$ then, by (3.5),

$$\liminf_{y \to \pm\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i) = 0.$$

Since, by Lemma 3.2, the path $y \to u(\cdot,y)$ is bounded in $Y^i$, there exist $\xi_\pm \in \mathbf{Z}$ such that

$$\liminf_{y \to \pm\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_{\xi_\pm}) = 0.$$

Considering the case $y \to +\infty$ we assume that $\limsup_{y\to\pm\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_{\xi_+}) > 0$. Then, arguing as in Lemma 2.1 the path $y \to u(\cdot,y)$ crosses an annulus of positive thickness around the set $\mathcal{K}^i_{\xi_+}$ and contained in the set $B_{r_0}(\mathcal{K}^i_{\xi_+})$ infinitely many times. This allows us to use (3.5) to conclude that $\varphi(u) = +\infty$, a contradiction which proves that $\lim_{y\to+\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_{\xi_+}) = 0$. Similarly one can prove that $\lim_{y\to-\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_{\xi_-}) = 0$. $\square$

By Lemma 3.3 we can restrict ourselves to consider the elements in $X^i$ which have prescribed limits as $y \to \pm\infty$. By periodicity it is sufficient to consider, for $\xi \in \mathbf{Z}$, the classes

$$\mathcal{M}^i_\xi = \{u \in X^i \mid \lim_{y\to-\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_0) = \lim_{y\to+\infty} \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_\xi) = 0\}.$$

By Lemma 3.2 we deduce that for any $C > 0$ there exists $C' > 0$ such that

$$\text{if } u \in \mathcal{M}^i_\xi \text{ and } \varphi(u) \le C \text{ then } \mathsf{dist}(u(\cdot,y),\mathcal{K}^i_0) \le C' \text{ for any } \xi \in \mathbf{Z},\ y \in \mathbf{R}. \quad (3.7)$$

The following lemma describes a first compactness property of the functional $\varphi$.

**Lemma 3.4** *Let $\xi \in \mathbf{Z}$ and $(u_n) \subset \mathcal{M}^i_\xi \cap \{\varphi \le C\}$. Then there exists $u \in X^i$ such that, up to a subsequence, $u_n \to u$ weakly in $H^1(\Omega)$ for any $\Omega \subset\subset \mathbf{R}^2$.*

*Proof.* Pick any function $z_0 \in \mathcal{K}^i_0$. Letting $\mathrm{diam}(\mathcal{K}^i_0) = d_0$, by (3.7), there exists $C' > 0$ such that $\mathsf{d}(u_n(\cdot,y),z_0) \le C' + d_0$ for any $y \in \mathbf{R}$. By (3.2), for any $L > 0$, we obtain

$$\int_{T_L} |u_n(x,y) - z_0(x)|^2 dx\, dy + \|\nabla u_n\|^2_{L^2(T_L)} \le (2(C'+d_0)^2 + 4c(i))L + 2C$$

from which we deduce that $(u_n - z_0)$ is bounded in $H^1(T_L)$. This implies that there exists a subsequence $(u_{n_k})$ of $(u_n)$ and a function $u$ such that $u - z_0 \in \cap_{L>0} H^1(T_L)$ and $u_{n_k} - z_0 \to u - z_0$ weakly in $H^1(\Omega)$ for every $\Omega \subset\subset \mathbf{R}^2$. Since $u - z_0 \in \cap_{L>0} H^1(T_L)$ we have that $u(\cdot,y) - z_0 \in H^1(\mathbf{R})$ for a. e. $y \in \mathbf{R}$ and $u \in X^i$ follows. $\square$

Moreover, we have

**Lemma 3.5** *Let $(u_n) \subset X^i$ and $u \in X^i$ be such that $u_n \to u$ weakly in $H^1(\Omega)$ for every $\Omega \subset\subset \mathbf{R}^2$. Then, for all $\xi \in \mathbf{Z}$, $\mathsf{dist}(u(\cdot,y),\mathcal{K}^i_\xi) \le \liminf_{n\to\infty} \mathsf{dist}(u_n(\cdot,y),\mathcal{K}^i_\xi)$ for a.e. $y \in \mathbf{R}$.*

*Proof.* Fix any $z_0 \in \mathcal{K}^i_\xi$. Since $u_n \to u$ in $L^2_{loc}(\mathbf{R}^2)$, there exists a function $w \in L^2_{loc}(\mathbf{R}^2)$ such that $w(x,y) \ge |u_n(x,y) - z_0(x)|$ for a.e. $(x,y) \in \mathbf{R}^2$. By the Fubini Theorem there exists $A \subset \mathbf{R}$ with $\mathrm{meas}(A) = 0$ such that $w(\cdot,y) \in L^2_{loc}(\mathbf{R})$ for any $y \in \mathbf{R} \setminus A$. Let us fix $y \in \mathbf{R} \setminus A$ and a sequence $(z_n) \subset \mathcal{K}^i_\xi$ such that $\mathsf{dist}(u_n(\cdot,y),\mathcal{K}^i_\xi) = \mathsf{d}(u_n(\cdot,y),z_n) + o(1)$ as $n \to \infty$. By $(*)_i$ there exists

$z \in \mathcal{K}^i_\xi$ and a subsequence $(z_{n_k}) \subset (z_n)$ such that $\|z_{n_k} - z\|_{H^1(\mathbf{R})} \to 0$ and moreover we can assume that $\lim \mathsf{d}(u_{n_k}(\cdot, y), z_{n_k}) = \liminf \mathsf{dist}(u_n(\cdot, y), \mathcal{K}^i_\xi)$. We have $|u_{n_k}(x, y) - z_{n_k}(x)| \le w(x, y) + |z_{n_k}(x) - z_0(x)| \le w(x, y) + C$ for some $C > 0$, for any $y \in \mathbf{R}\backslash A$ and for a.e. $x \in \mathbf{R}$. Then for any $M > 0$, since $w(\cdot, y) \in L^2((-M, M))$, we can use the dominated convergence Theorem to get

$$\int_{-M}^{M} |u(x,y)-z(x)|^2 dx = \lim_{k\to\infty} \int_{-M}^{M} |u_{n_k}(x,y)-z_{n_k}(x)|^2 dx \le \liminf_{k\to\infty} \mathsf{dist}(u_n(\cdot,y), \mathcal{K}^i_\xi)$$

and so to conclude that $\mathsf{dist}(u(\cdot, y), \mathcal{K}^i_\xi) \le \liminf_{n\to\infty} \mathsf{dist}(u_n(\cdot, y), \mathcal{K}^i_\xi)$ for any $y \in \mathbf{R} \setminus A$. $\square$

By Lemma 3.4 every sequence in $\mathcal{M}^i_\xi \cap \{\varphi \le C\}$ admits a subsequence which converge in the specified sense to some $u \in X^i$ and along which, by Lemma 3.1, the functional is lower semicontinuous.
The next step in our proof is to show, using again assumption $(*)_i$, that there exist particular classes $\mathcal{M}^i_\xi$ on which suitable sublevels of $\varphi$ satisfy additional compactness properties.

We define

$$m_{i,\xi} = \inf_{u\in\mathcal{M}^i_\xi} \varphi(u), \quad \xi \in \mathbf{Z},$$

noting that one can plainly prove, using suitable test functions, that $m_{i,\xi} < +\infty$ for any $\xi \in \mathbf{Z}$. Moreover

**Lemma 3.6** *For $\xi \neq 0$, $m_{i,\xi} \ge \sqrt{2h_0}r_0$ and $m_{i,\xi} \to +\infty$ as $|\xi| \to \infty$.*

*Proof.* We observe that $\mathsf{dist}(\mathcal{K}^i_0, \mathcal{K}^i_\xi) \to +\infty$ as $|\xi| \to \infty$ and thus, from the definition of $\mathcal{M}^i_\xi$ and Lemma 3.2, it readily follows that $m_{i,\xi} \to +\infty$ as $|\xi| \to \infty$. To prove the first estimate let $\xi \neq 0$ and $u \in \mathcal{M}^i_\xi$. Then $\mathsf{dist}(u(\cdot, y), \mathcal{K}^i_0) \to 0$ as $y \to -\infty$ while $\liminf_{y\to+\infty} \mathsf{dist}(u(\cdot, y), \mathcal{K}^i_0) \ge \mathsf{dist}(\mathcal{K}^i_0, \mathcal{K}^i_\xi) \ge 3r_0$. By (3.4) there exists $(y_1, y_2) \subset \mathbf{R}$ such that $\mathsf{d}(u(\cdot, y_1), u(\cdot, y_2)) \ge r_0$ and, for any $y \in (y_1, y_2)$ $r_0 \le \mathsf{dist}(u(\cdot, y), \mathcal{K}^i_0) \le 2r_0$. So in particular, by $(*)_i$, $r_0 \le \mathsf{dist}(u(\cdot, y), \mathcal{K}^i)$, thus using (3.6) and (3.5) we derive that $\varphi(u) \ge \sqrt{2h_0}r_0$ and the lemma follows. $\square$

By Lemma 3.6 there exists $\xi_1 \in \mathbf{Z}$ such that

$$m_{i,\xi_1} = \min_{\xi\neq0} m_{i,\xi}.$$

Letting $[\xi_1] = \{j\xi_1 \,:\, j \in \mathbf{N} \cup \{0\}\}$, still by Lemma 3.6, there exists $\xi_2 \in \mathbf{Z} \setminus [\xi_1]$ such that

$$m_{i,\xi_2} = \min_{\xi\in\mathbf{Z}\setminus[\xi_1]} m_{i,\xi}.$$

Assuming that $\xi_1, \xi_2, \ldots, \xi_{n-1}$ have been already defined and that $[\xi_1, \xi_2, \ldots, \xi_{n-1}] = \{\sum_{\iota=1}^{n-1} j_\iota \xi_\iota \,|\, j_\iota \in \mathbf{N} \cup \{0\}\} \neq \mathbf{Z}$, by Lemma 3.6, there exists $\xi_n \in \mathbf{Z} \setminus [\xi_1, \xi_2, \ldots, \xi_{n-1}]$ such that

$$m_{i,\xi_n} = \min_{\xi\in\mathbf{Z}\setminus[\xi_1,\xi_2,\ldots,\xi_{n-1}]} m_{i,\xi}.$$

Since we must have $m_{i,\xi_n} \leq \max\{m_{i,1}, m_{i,-1}\}$ the process must ends after a finite number of steps generating a finite set of integers $\{\xi_1, \xi_2, \dots, \xi_l\}$ such that $[\xi_1, \dots, \xi_l] = \mathbf{Z}$. By definition we have that if $k_1, k_2 \in \mathbf{Z}$ are such that $k_1 + k_2 = \xi_\iota$ for some $\iota \in \{1, \dots, l\}$ it is impossible that both $k_1, k_2$ belong to $[\xi_1, \dots, \xi_{\iota-1}]$. Therefore we conclude that

$$\text{if} \quad k_1, k_2 \in \mathbf{Z},\ k_1 + k_2 = \xi_\iota, \quad \text{then} \quad \max\{m_{i,k_1}, m_{i,k_2}\} \geq m_{i,\xi_\iota}. \tag{3.8}$$

As we will see in the next lemma, the property (3.8) allows to further characterize the functions in $\mathcal{M}^i_{\xi_\iota}$ whose action is close to $m_{i,\xi_\iota}$.

**Lemma 3.7** *There exist $\delta_0 \in (0, \frac{r_0}{2})$ and $\lambda_0 > 0$ such that if $\iota \in \{1, \dots, l\}$, $u \in \mathcal{M}^i_{\xi_\iota}$ and $\varphi(u) \leq m_{i,\xi_\iota} + \lambda_0$ then*

*(i) if $\inf_{z \in \mathcal{K}^i_0} \|u(\cdot, y_0) - z\|_{H^1(\mathbf{R})} \leq \delta_0$ then $\mathsf{dist}(u(\cdot, y), \mathcal{K}^i_0) \leq r_0$ for all $y \leq y_0$,*

*(ii) if $\inf_{z \in \mathcal{K}^i_{\xi_\iota}} \|u(\cdot, y_0) - z\|_{H^1(\mathbf{R})} \leq \delta_0$ then $\mathsf{dist}(u(\cdot, y), \mathcal{K}^i_{\xi_\iota}) \leq r_0$ for all $y \geq y_0$.*

*(iii) if $\xi \in \mathbf{Z} \setminus \{0, \xi_\iota\}$ then $\inf_{z \in \mathcal{K}^i_\xi} \|u(\cdot, y) - z\|_{H^1(\mathbf{R})} > \delta_0$ for all $y \in \mathbf{R}$.*

*Proof.* Set $\lambda_0 = \frac{1}{8} \min\{\sqrt{\frac{h_0}{2}} r_0, m_{i,\xi_1}\}$ and let $\delta_0 \in (0, \min\{\frac{r_0}{2}, \sqrt{\lambda_0}\})$ be such that

$$\sup\{F(q) \,|\, \inf_{z \in \mathcal{K}^i_0} \|q - z\|_{H^1(\mathbf{R})} \leq \delta_0\} \leq c(i) + \lambda_0.$$

Let $u \in \mathcal{M}^i_{\xi_\iota}$ be such that $\varphi(u) \leq m_{i,\xi_\iota} + \lambda_0$ and assume that $y_0 \in \mathbf{R}$ is such that $\inf_{z \in \mathcal{K}^i_0} \|u(\cdot, y_0) - z\|_{H^1(\mathbf{R})} \leq \delta_0$. By $(*)_i$ there exists $z_0 \in \mathcal{K}^i_0$ such that $\|u(\cdot, y_0) - z_0\|_{H^1(\mathbf{R})} \leq \delta_0$. We define

$$\tilde{u}(x, y) = \begin{cases} z_0(x) & \text{if } y \leq y_0 - 1, \\ u(x, y_0)(y - y_0 + 1) + z_0(x)(y_0 - y) & \text{if } y_0 - 1 \leq y \leq y_0, \\ u(x, y) & \text{if } y \geq y_0. \end{cases}$$

We have $\tilde{u} \in \mathcal{M}^i_{\xi_\iota}$ and so $\varphi(\tilde{u}) \geq m_{i,\xi_\iota}$. Then, setting

$$\varphi^{y_0}_{-\infty}(u) = \int_{-\infty}^{y_0} [\int_{\mathbf{R}} \tfrac{1}{2} |\partial_y u|^2 \, dx + F(u(\cdot, y) - c(i)] \, dy,$$

we obtain

$$\begin{aligned} m_{i,\xi_\iota} \quad \leq \quad & \varphi(\tilde{u}) = \varphi(u) - \varphi^{y_0}_{-\infty}(u) + \int_{y_0-1}^{y_0} \tfrac{1}{2} \int_{\mathbf{R}} |u(x, y_0) - z_0(x)|^2 dx \, dy + \\ & + \int_{y_0-1}^{y_0} F(u(x, y_0)(y - y_0 + 1) + z_0(x)(y_0 - y)) - c(i) \, dy \\ & \leq \varphi(u) - \varphi^{y_0}_{-\infty}(u) + \tfrac{1}{2} \delta_0^2 + \lambda_0 \end{aligned}$$

from which, since $\varphi(u) \leq m_{i,\xi_\iota} + \lambda_0$ we conclude that $\varphi^{y_0}_{-\infty}(u) \leq 3\lambda_0$. Assume by contradiction that there is $y_1 \leq y_0$ such that $\mathsf{dist}(u(\cdot, y_1), \mathcal{K}^i_0) \geq r_0$. Then by (3.4)

there exists $(y_1', y_0') \subset (y_1, y_0)$ such that $\mathsf{d}(u(\cdot, y_1'), u(\cdot, y_0')) \ge \frac{r_0}{2}$ and, for a.e $y \in (y_1', y_0')$, $\mathsf{dist}(u(\cdot, y), \mathcal{K}_0^i) \in (\frac{r_0}{2}, r_0)$. In particular $\inf_{z \in \mathcal{K}^i} \|u(\cdot, y) - z\|_{H^1(\mathbf{R})} \ge \frac{r_0}{2}$ for a.e. $y \in (y_1', y_0')$ and using (3.6) and (3.5) we get the contradiction $3\lambda_0 \ge \varphi_{-\infty}^{y_0}(u) \ge \sqrt{\frac{h_0}{2}} r_0 \ge 8\lambda_0$. Similarly one can show that if $\inf_{z \in \mathcal{K}_{\xi_\iota}^i} \|u(\cdot, y_0) - z\|_{H^1(\mathbf{R})} \le \delta_0$ then $\mathsf{dist}(u(\cdot, y), \mathcal{K}_{\xi_\iota}^i) \le r_0$ for all $y \ge y_0$.
To prove the last part of the lemma we argue again by contradiction assuming that there exist $y_0 \in \mathbf{R}$, $\xi' \in \mathbf{Z} \setminus \{0, \xi_\iota\}$ and $z_0 \in \mathcal{K}_{\xi'}^i$ such that $\|u(\cdot, y_0) - z_0\|_{H^1(\mathbf{R})} \le \delta_0$. We define

$$u_1(x, y) = \begin{cases} u(x, y) & \text{if } y \le y_0, \\ u(x, y_0)(y_0 + 1 - y) + z_0(x)(y - y_0) & \text{if } y_0 \le y \le y_0 + 1, \\ z_0(x) & \text{if } y \ge y_0 + 1, \end{cases}$$

and

$$u_2(x, y) = \begin{cases} z_0(x) & \text{if } y \le y_0 - 1, \\ u(x, y_0)(y - y_0 + 1) + z_0(x)(y_0 - y) & \text{if } y_0 - 1 \le y \le y_0, \\ u(x, y) & \text{if } y \ge y_0. \end{cases}$$

We note that $u_1 \in \mathcal{M}_{\xi'}^i$ and $u_2(\cdot - \xi' T, \cdot) \in \mathcal{M}_{\xi_\iota - \xi'}^i$ and thus, by (3.8) that $\varphi(u_1) + \varphi(u_2) \ge m_{i,\xi_\iota} + m_{i,\xi_1}$. Now, arguing as above, we obtain

$$\varphi(u) \ge \varphi(u_1) + \varphi(u_2) - \delta_0^2 - 2\lambda_0$$

which leads to the contradiction $4\lambda_0 \ge m_{i,\xi_1}$. □

Lemma 3.7 can be combined with (3.5) to derive the following concentration result which, together with Lemma 3.4, is the key compactness property of our problem.

**Lemma 3.8** *There exists $\ell_0 > 0$ such that if $\iota \in \{1, \dots, l\}$, $u \in \mathcal{M}_{\xi_\iota}^i$, $\varphi(u) \le m_{i,\xi_\iota} + \lambda_0$ and $\mathsf{dist}(u(\cdot, 0), \mathcal{K}_0^i) = \frac{3r_0}{2}$ then*

$$\mathsf{dist}(u(\cdot, y), \mathcal{K}_0^i) \le r_0 \ \forall\, y \le -\ell_0 \ \text{and} \ \mathsf{dist}(u(\cdot, y), \mathcal{K}_{\xi_\iota}^i) \le r_0 \ \forall\, y \ge \ell_0.$$

*Proof.* By (3.5) we can fix $\ell_0 > 0$ such that if $I$ is any real interval with length equal to $\ell_0$ and $u \in X^i$ is such that $\inf_{z \in \mathcal{K}^i} \|u(\cdot, y) - z\|_{H^1(\mathbf{R})} > \delta_0$ for a.e. $y \in I$ then $\varphi(u) \ge m_{i,\xi_l} + 2\lambda_0$. By Lemmas 3.3 and 3.7, since $\mathsf{dist}(u(\cdot, 0), \mathcal{K}_0^i) = \frac{3r_0}{2}$, there exist $y_- \in (-\ell_0, 0)$ and $y_+ \in (0, \ell_0)$ such that $\inf_{z \in \mathcal{K}_0^i} \|u(\cdot, y_-) - z\|_{H^1(\mathbf{R})} \le \delta_0$ and $\inf_{z \in \mathcal{K}_{\xi_\iota}^i} \|u(\cdot, y_-) - z\|_{H^1(\mathbf{R})} \le \delta_0$. Then the lemma follows applying again Lemma 3.7. □

Lemma 3.8 together with Lemmas 3.1 and 3.4 allows us to use the direct method of the Calculus of Variation to show that the functional $\varphi$ admits a minimum in each classes $\mathcal{M}_{\xi_\iota}^i$, $\iota = 1, \dots, l$. Indeed we have

**Lemma 3.9** *For any $\iota \in \{1, \dots, l\}$ there exists $u_\iota \in \mathcal{M}_{\xi_\iota}^i$ such that $\varphi(u_\iota) = m_{i,\xi_\iota}$ and $\|u_\iota\|_{L^\infty(\mathbf{R}^2)} \le R_0$.*

*Proof.* Let $(u_n) \subset \mathcal{M}^i_{\xi_\iota}$ be such that $\varphi(u_n) \to m_{i,\xi_\iota}$. By $(H_2)$ we can assume that $\|u_n\|_{L^\infty(\mathbf{R}^2)} \le R_0$, indeed otherwise we can consider the minimizing sequence $\tilde{u}_n = \max\{\min\{u_n, R_0\}, -R_0\}$. Since $\lim \mathsf{dist}(u_n(\cdot,y),\mathcal{K}^i_0) \to 0$ as $y \to -\infty$ and $\liminf_{y\to+\infty} \mathsf{dist}(u_n(\cdot,y),\mathcal{K}^i_0) \ge 3r_0$ for any $n \in \mathbf{N}$, by (3.4), we obtain that for any $n \in \mathbf{N}$ there exists $y_n \in \mathbf{R}$ such that $\mathsf{dist}(u_n(\cdot,y_n),\mathcal{K}^i_0) = \frac{3r_0}{2}$. Then, setting $v_n = u_n(\cdot,\cdot+y_n)$, we have $v_n \in \mathcal{M}^i_{\xi_\iota}$, $\varphi(v_n) = \varphi(u_n)$ and $\mathsf{dist}(v_n(\cdot,0),\mathcal{K}^i_0) = \frac{3r_0}{2}$. We can assume that $\varphi(v_n) \le m_{i,\xi_\iota} + \lambda_0$ for any $n \in \mathbf{N}$ and, by Lemma 3.8, we obtain that for any $n \in \mathbf{N}$

$$\mathsf{dist}(v_n(\cdot,y),\mathcal{K}^i_0) \le r_0,\ \forall y \le -\ell_0 \text{ and } \mathsf{dist}(v_n(\cdot,y),\mathcal{K}^i_{\xi_\iota}) \le r_0,\ \forall y \ge \ell_0. \tag{3.9}$$

By Lemma 3.4 there exists $u_\iota \in X^i$ such that along a subsequence $v_n \to u_\iota$ weakly in $H^1(\Omega)$ for any $\Omega \subset\subset \mathbf{R}^2$ and by Lemma 3.1 we have $\varphi(u_\iota) \le m_{i,\xi_\iota}$. Moreover, clearly $\|u_\iota\|_{L^\infty(\mathbf{R}^2)} \le R_0$. Finally, by Lemma 3.5 we have

$$\mathsf{dist}(u_\iota(\cdot,y),\mathcal{K}^i_0) \le r_0\ \forall y \le -\ell_0 \text{ and } \mathsf{dist}(u_\iota(\cdot,y),\mathcal{K}^i_{\xi_\iota}) \le r_0\ \forall y \ge \ell_0 \tag{3.10}$$

and so, by (3.10) and Lemma 3.3 we conclude that $\mathsf{dist}(u_\iota(\cdot,y),\mathcal{K}^i_0) \to 0$ as $y \to -\infty$ and $\mathsf{dist}(u_\iota(\cdot,y),\mathcal{K}^i_{\xi_\iota}) \to 0$ as $y \to +\infty$, i.e. $u_\iota \in \mathcal{M}^i_{\xi_\iota}$, and the lemma is proved. □

By Lemma 3.9, we can now conclude the proof of Theorem 1.2.

**Proof of Theorem 1.2** Let $\iota \in \{1,\dots,l\}$ and let $u_\iota$ be given by Lemma 3.9. By Lemma 3.9, we have

$$\int_{\mathbf{R}^2} |\nabla(u_\iota+h)|^2 + a(x)W(u_\iota+h) - |\nabla u_\iota|^2 + a(x)W(u_\iota)dxdy = \varphi(u_\iota+h) - \varphi(u_\iota) \ge 0$$

for all $h \in \mathcal{C}^\infty_0(\mathbf{R}^2)$. Since $\|u_\iota\|_{L^\infty(\mathbf{R}^2)} \le R_0$, we obtain that $u_\iota$ is a weak solution to $-\Delta u + a(x)W'(u) = 0$ on $\mathbf{R}^2$ and then, by standard elliptic arguments, that $u_\iota$ is a classical solution with $\|u_\iota\|_{\mathcal{C}^2(\mathbf{R}^2)} \le C$. The uniform $C^2$ estimates easily implies that $u_\iota$ satisfies the right boundary conditions. Indeed, assume by contradiction that $u_\iota(x,y)$ does not verify $u_\iota(x,y) \to \sigma_i$ as $x \to -\infty$ uniformly with respect to $y \in \mathbf{R}$. Then, there exist $\delta > 0$ and a sequence $(x_n,y_n) \in \mathbf{R}^2$ with $x_n \to -\infty$ such that $|u_\iota(x_n,y_n) - \sigma_i| \ge 2\delta$ for all $n \in \mathbf{N}$. The $C^2$-estimate above implies that there exists $\rho > 0$ such that

$$|u_\iota(x,y) - \sigma_i| \ge \delta \quad \forall (x,y) \in B_\rho(x_n,y_n),\ n \in \mathbf{N}. \tag{3.11}$$

If along a subsequence, $y_n \to y_0$ we easily obtain a contradiction. In fact in this case we have $|u_\iota(x,y) - \sigma_i| \ge \delta$ for all $(x,y) \in B_{\frac{\varrho}{2}}(x_n,y_0)$, $n \ge n_0$ which is not possible since $u_\iota(\cdot,y) \in \Gamma^i$ for a.e. $y \in \mathbf{R}$. Then, we must have $|y_n| \to \infty$. Assume for example that along a subsequence $y_n \to +\infty$. Since, by compactness, there exists $L > 0$ such that $\|z - \sigma_i\|_{L^\infty(-\infty,-L)} \le \frac{\delta}{2}$ for any $z \in \mathcal{K}^i_0$, by 3.11, we obtain

$$\limsup_{y\to+\infty} \mathsf{dist}(u_\iota(\cdot,y),\mathcal{K}^i_0) > 0$$

which is a contradiction since $u_\iota \in \mathcal{M}^i_0$. One argues analogously in the other cases. □

# 4 About the assumption $(*)_i$.

In this last section we discuss the assumption $(*)_i$. First of all we want to prove that $(*)_i$ is satisfied if and only if the set $\mathcal{K}^i$ is not a continuum, precisely

$$(*)_i \text{ does not hold } \iff \mathcal{K}^i \text{ is homeomorphic to } \mathbf{R} \tag{4.1}$$

To this aim we prove some properties of the function $X : \mathcal{K}^i \to \mathbf{R}$ introduced in Section 2.

**Remark 4.1** For any fixed $x_0 \in \mathbf{R}$ and $s \in \mathbf{R}$ such that $|s - \sigma_{j(i)}| \leq \rho_0$, consider the functional

$$F_{x_0}(q) = \int_{x_0}^{\infty} \tfrac{1}{2}|\dot{q}(x)|^2 + a(x)W(q(x))\,dx$$

on the class

$$\Gamma^i_{s,x_0} = \{q \in \sigma_{j(i)} + H^1([x_0,+\infty)) \mid q(x_0) = s,\ \|q - \sigma_{j(i)}\|_{L^\infty([x_0,+\infty))} \leq 2\rho_0\}$$

It is easy to prove that $F_{x_0}$ admits a minimum on $\Gamma^i_{s,x_0}$ and since, by (2.2), $F_{x_0}$ is strictly convex on $\Gamma^i_{s,x_0}$ such minimum is unique.

We have

**Lemma 4.2** *The function $X : \mathcal{K}^i \to \mathbf{R}$ is continuous and invertible on $X(\mathcal{K}^i)$ with continuous inverse.*

*Proof.* We consider the case $\sigma_i < \sigma_{j(i)}$ (we can argue in the same way if $\sigma_{j(i)} < \sigma_i$). First, note that since each $q \in \mathcal{K}^i$ is a minimum of $F$, using the uniqueness of the solution of the Cauchy problem related to $-\ddot{q}(x) + a(x)W'(q(x)) = 0$, we obtain

$$\sigma_i < q(x) < \sigma_{j(i)}, \quad \forall x \in \mathbf{R} \tag{4.2}$$

for every $q \in \mathcal{K}^i$.
Now, to prove that $X$ is one-to-one, let $q$, $\bar{q} \in \mathcal{K}^i$ be such that $X(q) = X(\bar{q}) \equiv x_0$. Then, by definition of $X$ and (4.2), we have $q(x_0) = \bar{q}(x_0) \equiv s$, $\sigma_{j(i)} - s = \rho_0$ and $\|q - \sigma_{j(i)}\|_{L^\infty([x_0,+\infty))} \leq \rho_0$, $\|\bar{q} - \sigma_{j(i)}\|_{L^\infty([x_0,+\infty))} \leq \rho_0$. Therefore $q_{|[x_0,+\infty)}, \bar{q}_{|[x_0,+\infty)} \in \Gamma^i_{s,x_0}$ and since $q$ and $\bar{q}$ are minima of $F$ on $\Gamma^i$, by Remark 4.1 it follows that $q(x) = \bar{q}(x)$ for all $x \in [x_0,+\infty)$. Then, by uniqueness of the Cauchy problem, we conclude $q \equiv \bar{q}$.
To prove that $X : \mathcal{K}^i \to \mathbf{R}$ is continuous, let $(q_n) \subset \mathcal{K}^i$ and $q_0 \in \mathcal{K}^i$ be such that $\|q_n - q_0\|_{H^1(\mathbf{R})} \to 0$. Then, by uniform convergence, $(X(q_n))$ is bounded in $\mathbf{R}$ and by Lemma 2.3, we conclude $X(q_n) \to X_0 = X(q_0)$.
Finally, to prove that $X^{-1} : X(\mathcal{K}^i) \to \mathcal{K}^i$ is continuous, let $(q_n) \subset \mathcal{K}^i$ and $q_0 \in \mathcal{K}^i$ be such that $X(q_n) \to X(q_0)$. Then, by Lemma 2.3, every subsequence of $(q_n)$ admits a subsequence which converges to some $\tilde{q}_0 \in \mathcal{K}^i$ and $X(\tilde{q}_0) = X(q_0)$. Since $X$ is one-to-one, $\tilde{q}_0 \equiv q_0$ and we can conclude $\|q_n - q_0\|_{H^1(\mathbf{R})} \to 0$. $\square$

The next result gives the essential link between the condition $(*)_i$ and the function $X$. It will be used to give some examples in which $(*)_i$ is verified.

**Lemma 4.3** *If $X(\mathcal{K}^i) \neq \mathbf{R}$ then $(*)_i$ holds.*

*Proof.* Let $x_0 \in \mathbf{R} \setminus X(\mathcal{K}^i)$. Then Lemma 2.3 readily implies that there exist $h_0 > 0$ and $\eta_0 \in (0, \frac{T}{2})$ such that

$$\text{if } q \in \Gamma^i \text{ and } X(q) \in [x_0 - \eta_0, x_0 + \eta_0] \text{ then } F(q) \geq c(i) + h_0. \tag{4.3}$$

Therefore, letting $J_0 = [x_0 - \eta_0, x_0 + \eta_0]$, there is no $q \in \mathcal{K}^i$ with $X(q) \in J_0$. By periodicity of the potential, we also obtain that if $q \in \Gamma^i$ and $X(q) \in J_\xi = \xi T + J_0$ for some $\xi \in \mathbf{Z}$ then $F(q) \geq c(i) + h_0$ and so $q \notin \mathcal{K}^i$. Denoting $I_\xi$ the connected component of $\mathbf{R} \setminus (\cup_{\xi \in \mathbf{Z}} J_\xi)$ which has on the right the interval $J_\xi$ and setting

$$\mathcal{K}^i_\xi = \{q \in \mathcal{K}^i \mid X(q) \in I_\xi\}, \quad \xi \in \mathbf{Z}$$

we obtain that $\mathcal{K}^i_\xi \cap \mathcal{K}^i_{\xi'} = \emptyset$ whenever $\xi \neq \xi'$ and that $\mathcal{K}^i = \cup_{\xi \in \mathbf{Z}} \mathcal{K}^i_\xi$. Hence, $(*)_i$ follows if for every $(q_n) \subset \mathcal{K}^i_0$ there exists $q \in \mathcal{K}^i_0$ such that, along a subsequence, $\|q_n - q\|_{H^1(\mathbf{R})} \to 0$.
To prove this last point, observe that since $I_0$ is bounded and $X(q_n) \in I_0$, up to a subsequence $X(q_n) \to X_0 \in \overline{I_0}$. Then, by Lemma 2.3, there exists $q \in \mathcal{K}^i$ such that along a subsequence, $\|q_n - q\|_{H^1(\mathbf{R})} \to 0$ and $X(q) = X_0$. Now, since $F(q) = c(i)$, by (4.3) it follows $X(q) \in I_0$ and thus $q \in \mathcal{K}^i_0$. $\square$

Lemmas 4.2 and 4.3 show that if $(*)_i$ does not hold then $X$ is an homeomorphism from $\mathcal{K}^i$ onto $\mathbf{R}$. Finally, since $(*)_i$ implies that $\mathcal{K}^i$ is not connected, if $(*)_i$ holds, there is no homeomorphism between $\mathcal{K}^i$ and $\mathbf{R}$. Thus (4.1) is verified.

We shall now give some examples in which $(*)_i$ is satisfied. First we mention that if $a$ is a small continuous non-constant periodic perturbation of a positive constant, the assumption $(*)_i$ can be checked by mean of Poincaré Melnikov methods. We refer to [6] for this kind of arguments. Here, following [3] and [4], we show that it is always possible, given a non negative periodic function $a$, to perturb it with a function, having $L^\infty$ norm as small as we want, in order that for the perturbed equation the condition $(*)_i$ is satisfied for any $i \in \{1, \ldots, m\}$.

More precisely, let $a$ be any continuous non negative $T$-periodic function and $b$ be a continuous positive non constant $T$-periodic function. Without loss of generality, we can assume $T = 1$. Let $a_n(x) = a(x) + b(\frac{x}{n})$, $n \in \mathbf{N}$, and note that $a_n$ is $n$-periodic. We consider the perturbed equation

$$-\Delta u(x, y) + a_n(x) W'(u(x, y)) = 0, \qquad (x, y) \in \mathbf{R}^2, \tag{4.4}$$

and we study the structure of the corresponding set of one dimensional minimal solutions.

First of all note that the functional corresponding to the one dimensional problem

$$F_n(q) = \int_{\mathbf{R}} \tfrac{1}{2}|\dot q(x)|^2 + a_n(x)W(q(x))\,dx, \quad q\in E$$

satisfies all the properties stated in Section 2 for all $n\in\mathbf{N}$. In particular, for any $i\in\{1,\dots,m\}$ there exists $j_n(i)\in\{1,\dots,m\}\setminus\{i\}$ such that the functional $F_n$ admits a minimum on the set

$$\Gamma_n^i = \{q\in\{F_n<+\infty\}\,|\,\lim_{t\to-\infty} q(t)=\sigma_i,\ \lim_{t\to+\infty} q(t)=\sigma_{j_n(i)}\}$$

at level $c_n(i)$. We will prove that the corresponding minimal set $\mathcal{K}^{i,n}$ satisfies assumption $(*)_i$ for all $i\in\{1,\dots,m\}$ whenever $n$ is sufficiently large.

To this aim we note that, setting $\overline{F}(q)=\int_{\mathbf{R}}\frac{1}{2}|\dot q(x)|^2+\overline{a}W(q(x))\,dx$ and $\underline{F}(q)=\int_{\mathbf{R}}\frac{1}{2}|\dot q(x)|^2+\underline{a}W(q(x))\,dx$, where $\overline{a}=\max_{\mathbf{R}} a+\max_{\mathbf{R}} b$ and $\underline{a}=\min_{\mathbf{R}} a+\min_{\mathbf{R}} b$, we have $\underline{F}(q)\le F_n(q)\le\overline{F}(q)$, for all $q\in E$, $n\in\mathbf{N}$. Then, using a simple comparison argument, one can see that the result stated in Lemma 2.2 holds true uniformly with respect to $n$. In particular, there exist $\bar\delta$, $\bar\lambda$ and $\bar\ell$ such that if $q\in\Gamma_n^i\cap\{F_n\le c_n(i)+\bar\lambda\}$ and $\min_{1\le j\le m}|q(x)-\sigma_j|\ge\bar\delta$ for all $x\in(s,p)\subset\mathbf{R}$ then $p-s\le\bar\ell$.

Now note that, since $b$ is not constant, there exists $x_+$ and $x_-$ in $[0,1)$ such that $\overline{b}=b(x_+)=\max_{x\in[0,1)}b(x)>\min_{x\in[0,1)}b(x)=b(x_-)=\underline{b}$. Then, setting $\beta=\overline{b}-\underline{b}$ there exists $\eta>0$ such that

$$\min_{|x-x_+|\le 2\eta} b(x)\ge\overline{b}-\tfrac{\beta}{4} \text{ and } \max_{|x-x_-|\le 2\eta} b(x)\le\underline{b}+\tfrac{\beta}{4}.$$

When we consider the function $b(\frac{x}{n})$ we obtain analogously

$$\min_{|x-nx_+|\le 2n\eta} b(\frac{x}{n})\ge\overline{b}-\tfrac{\beta}{4} \text{ and } \max_{|x-nx_-|\le 2n\eta} b(\frac{x}{n})\le\underline{b}+\tfrac{\beta}{4}. \tag{4.5}$$

Therefore we obtain

**Lemma 4.4** *There exist $n_0\in\mathbf{N}$, $n_0\ge\frac{\bar\ell}{\eta}$, and $h_0>0$ such that if $n\ge n_0$, $q\in\Gamma_n^i$ and $X(q)\in J_0^n=[n(x_+-\eta),n(x_++\eta)]$ then $F_n(q)\ge c_n(i)+h_0$.*

*Proof.* By Lemma 2.2, if $q\in\Gamma_n^i$ and $F_n(q)\le c_n(i)+\bar\lambda$, there exists $(s,p)\subset(X(q)-\bar\ell,X(q)+\bar\ell)$ such that the path $q((s,p))$ is outside the set $\cup_{1\le i\le m}B_{\bar\delta}(\sigma_i)$ and moreover $|q(s)-\sigma_i|=|q(p)-\sigma_{j(i)}|=\bar\delta$. Then

$$\tfrac{\rho_0}{2}\le|q(s)-q(p)|\le\int_s^p|\dot q(x)|\,dx\le\sqrt{(p-s)}(\int_s^p|\dot q(x)|^2\,dx)^{\frac12}\le\sqrt{2(p-s)(c_n(i)+\bar\lambda)}$$

Then, setting $\bar\mu=\inf\{W(x)\,|\,x\notin\cup_{1\le i\le m}B_{\bar\delta}(\sigma_i)\}$, if $n\ge\frac{\bar\ell}{\eta}$ we obtain

$$\int_{|x-X(q)|\le n\eta} W(q)\,dx\ge\int_s^p W(q)\,dx\ge\bar\mu(p-s)\ge\tfrac{\bar\mu\rho_0^2}{8(c_n(i)+\bar\lambda)}\ge\varepsilon_0,$$

for some $\varepsilon_0 > 0$.
Arguing as in the proof of Lemma 2.3, we can prove that there exists $\lambda_0 \in (0, \bar{\lambda})$ and $\ell_0 > \bar{\ell}$ such that if $q \in \Gamma_n^i$ and $F_n(q) \leq c_n(i) + \lambda_0$ then

$$\int_{|x-X(q)|\geq \ell_0} W(q)\,dx \leq \tfrac{\beta\varepsilon_0}{4\bar{b}}.$$

Now, let $q \in \Gamma_n^i$ with $X(q) \in [n(x_+ - \eta), n(x_+ + \eta)]$ and let $j \in \mathbf{Z}$ be such that $[n(x_+ - \eta) + j, n(x_+ + \eta) + j] \subset [n(x_- - \eta) - 1, n(x_- + \eta) + 1]$. Then, if $F_n(q) \leq c_n(i) + \lambda_0$, by the previous estimate, for $n_0 \geq \frac{\ell_0}{\eta}$, using (4.5), we obtain

$$\begin{aligned}
F_n(q) - F_n(q(\cdot + j)) \quad &= \int_{\mathbf{R}} (b(\frac{x}{n}) - b(\frac{x-j}{n}))W(q)\,dx \\
&\geq \tfrac{\beta}{2} \int_{|x-nx_+|\leq 2n\eta} W(q)\,dx - \bar{b} \int_{|x-nx_+|\geq 2n\eta} W(q)\,dx \\
&\geq \tfrac{\beta}{2} \int_{|x-X(q)|\leq n\eta} W(q)\,dx - \bar{b} \int_{|x-X(q)|\geq n\eta} W(q)\,dx \geq \frac{\beta\varepsilon_0}{4}.
\end{aligned}$$

Then the lemma follows choosing $n_0 \geq \frac{\ell_0}{\eta}$ and $h_0 = \min\{\lambda_0, \frac{\beta\varepsilon_0}{4}\}$. □

Lemma 4.4 says, in particular, that if $n \geq n_0$ there is no $q \in \mathcal{K}^{i,n}$ with $X(q) \in J_0^n$. Then, by Lemma 4.3, we obtain that $(*)_i$ holds for any $i \in 1, \ldots, m$.

Finally we note that in Lemma 4.4 no restriction is made on the $L^\infty$ norm of $b$. This shows that $(*)_i$ holds for a dense subset of the set of periodic, positive and continuous functions. Moreover, we remark that the above argument is based on the result stated in Lemma 4.4 which, as one can easily see, is still true if we perturb $a_n$ with a periodic function having an $L^\infty$ norm small enough. This shows that $(*)_i$ holds for an open and dense subset of the set of periodic, positive and continuous functions.

# References


[1] Alama, S. , Bronsard, L. , Gui, C. : *Stationary layered solutions in* $\mathbf{R}^2$ *for an Allen-Cahn system with multiple well potential*, Calc. of Var. and PDEs **5**, 359–390 (1997).

[2] Alessio, F. , Bertotti, M.L. , Montecchiari, P. : *Multibump solutions to possibly degenerate equilibria for almost periodic Lagrangian systems*, Z. Angew. Math. Phys., to appear.

[3] Alessio, F. , Caldiroli, P. , Montecchiari, P. : *Genericity of the multibump dynamics for almost periodic Duffing-like systems*, Proc. Roy. Soc. Edin., to appear.

[4] Alessio, F. , Montecchiari, P. : *Multibump solutions for a class of Lagrangian systems slowly oscillating at infinity*, Ann. IHP Anal. Nonlinéaire **16**, 107–155 (1999).

[5] Allen, S. M. , Cahn, J. W. : *A macroscopic theory for antiphase boundary motions and its application to antiphase domain coarsening*, Acta Metal. **27**, 1085–1095 (1979).

[6] Ambrosetti, A. , Badiale, M. : *Homoclinics: Poincaré-Melnikov type results via a variational approach*, C.R. Acad. Sci. Paris, t.**323**, Série I, 753–758 (1996); and Ann. Inst. H. Poincaré, Anal. non-lin. **15**, 233–252 (1998).

[7] Berestycki, H. , Hamel, F. , Monneau, R. : *One-dimensional symmetry for some bounded entire solutions of some elliptic equations*, preprint (1999).

[8] Bolotin, S.V. , Kozlov, V.V. : *Libration in systems with many degrees of freedom*, Prikl. Mat. Mekh. **42**, 245–250 (1978); English trans. in J. Appl. Math. Mech. **42**, 256–261 (1978).

[9] De Giorgi, E. : *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis, Rome, (1978), E. De Giorgi et al. (eds).

[10] Farina, A. : *Symmetry solutions of semilinear elliptic equation in* $\mathbf{R}^N$ *and related conjecture*, Ricerche di Matematica in memory of Ennio De Giorgi.

[11] Ghoussoub, N. , Gui, C. : *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311**, 481-491 (1998).

[12] Rabinowitz, P.H. : *Homoclinic and heteroclinic orbits for a class of Hamiltonian systems*, Calc. of Var. and PDEs **1**, 1–36 (1993).

[13] Rabinowitz, P.H. : *Heteroclinic for reversible Hamiltonian system*, Ergod. Th. and Dyn. Sys. **14**, 817–829 (1994).

[14] Rabinowitz, P.H. : *Solutions of heteroclinic type for some classes of semilinear elliptic partial differential equations*, J. Math. Sci. Univ. Tokio, **1**, 525–550 (1994).