Standing waves for nonlinear Schrödinger equations with a general nonlinearity

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Abstract

For elliptic equations $\varepsilon^2 \Delta u - V(x)u + f(u) = 0, x \in \mathbf{R}^N$, $N \geq 3$, we develop a new variational approach to construct localized positive solutions concentrating at an isolated component of positive local minimum points of V, as $\varepsilon \to 0$, under conditions on f we believe to be almost optimal.

1 Introduction

We are concerned in standing waves for the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \Delta \psi - V(x)\psi + f(\psi) = 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^N, \tag{1}$$

where $\hbar$ denotes the Plank constant, i the imaginary unit. For the physical background for this equation, we refer to the introduction in [6]. We assume that $f(\exp(i\theta)v) = \exp(i\theta)f(v)$ for $v \in \mathbf{R}$. A solution of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. Then, $\psi(x,t)$ is a solution of (1) if and only if the function v satisfies

$$\frac{\hbar^2}{2}\Delta v - (V(x) - E)v + f(v) = 0 \quad \text{in} \quad \mathbf{R}^N. \tag{2}$$

In this paper we are interested in positive solutions in $H^1(\mathbf{R}^N)$ for small $\hbar > 0$. For small $\hbar > 0$, these standing waves are referred as semi-classical states. For simplicity and without loss of generality, we write $V - E$ as V, i.e., we shift E to 0. Thus, we consider the following equation

$$\varepsilon^2\Delta v - V(x)v + f(v) = 0, \quad v > 0, \quad v \in H^1(\mathbf{R}^N) \tag{3}$$

when $\varepsilon > 0$ is sufficiently small. We assume that the potential function V satisfies the following condition

(V1) $V \in C(\mathbf{R}^N, \mathbf{R})$ and $\inf_{x\in\mathbf{R}^N} V(x) = V_0 > 0$.

For future reference we observe that defining $u(x) = v(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, equation (3) is equivalent to

$$\Delta u - V_\varepsilon u + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}^N). \tag{4}$$

An interesting class of solutions of (3) are families of solutions which concentrate and develop spike layers, peaks, around certain points in $\mathbf{R}^N$ while vanishing elsewhere as $\varepsilon \to 0$. The existence of single peak solutions was first studied by Floer and Weinstein [16]. For $N = 1$ and $f(u) = u^3$, they construct a single peak solution concentrating around any given non-degenerate critical point of the potential $V(x)$. Oh [25] extended this result in higher dimension and for $f(u) = |u|^{p-1}u$, $1 < p < \frac{N+2}{N-2}$. Furthermore, Oh [26] proved the existence of multi-peak solutions which are concentrating around any finite subsets of the non-degenerate critical points of V.

The arguments in [16, 25, 26] are based on a Lyapunov-Schmidt reduction and rely on the uniqueness and non-degeneracy of the ground state solutions, namely of the positive least energy solutions, for the autonomous problems : for fixed $x_0 \in \mathbf{R}^N$,

$$\Delta v - V(x_0)v + f(v) = 0 \quad \text{in} \quad \mathbf{R}^N \quad \text{and} \quad v \in H^1(\mathbf{R}^N). \tag{5}$$

Subsequently reduction methods were also found suitable to find solutions of (3) concentrating around possibly degenerate critical points of $V(x)$, when the ground state solutions of the limit problems (5) are unique and non-degenerate. More precisely, Ambrosetti, Badiale and Cingolani [1] consider concentration phenomena at isolated local minima and maxima with polynomial degeneracy and in [23] Y.Y. Li deals with C^1-stable critical points of V. See also [2, 10, 11, 22], for further related results.

However, the uniqueness and non-degeneracy of the ground state solutions of (5) are, in general, rather difficult to check. They are known so far only for a rather restricted class of nonlinearities f. To attack the existence of positive solutions of (3) without these assumptions, the variational approach, initiated by Rabinowitz [27], proved to be successful. In [27] he proves, by a mountain pass argument, the existence of positive solutions of (3) for small $\varepsilon > 0$ whenever

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbf{R}^N} V(x).$$

These solutions concentrate around the global minimum points of V when $\varepsilon \to 0$, as it was shown by X. Wang [28]. Later, del Pino and Felmer [12] by introducing a penalization approach prove a localized version of the result of Rabinowitz and Wang (see also [13, 14, 15, 19] for related results). In [12], assuming (V1) and the following condition,

(V2) there is a bounded domain O such that

$$m \equiv \inf_{x\in O} V(x) < \min_{x\in\partial O} V(x)$$

they show the existence of a single peak solution concentrating around the minimum points of V in O. They assume that the nonlinearity f satisfies the assumptions $(f1), (f2)$ below and the so called *global Ambrosetti-Rabinowitz condition:* for some $\mu > 2$, $0 < \mu \int_0^t f(s)ds < f(t)t,\ t > 0$. Also the monotonicity of the function $\xi \to f(\xi)/\xi$ is required (see [12]). Recently, it has been shown in [7] and [21] that the monotonicity condition is not necessary.

The motivation of this paper is to explore what are the essential features which guarantee the existence of localized bound state solutions. Specially, we are concerned with single peak solutions concentrating around local minimum points, as $\varepsilon \to 0$, since the corresponding standing waves of (1) are possible candidates to be orbitally stable. To state our main result we need the followings. Let

$$\mathcal{M} \equiv \{x \in O \mid V(x) = m\}$$

and assume that $f : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies

(f1) $\lim_{t\to 0^+} f(t)/t = 0$;

(f2) there exists some $p \in (1, (N+2)/(N-2)), N \geq 3$ such that $\limsup_{t\to\infty} f(t)/t^p < \infty$;

(f3) there exists $T > 0$ such that $\frac{1}{2}mT^2 < F(T)$, where $F(t) = \int_0^t f(s)ds$.

Theorem 1 *Let $N \geq 3$ and suppose that (V1-2) and (f1-3) hold. Then for sufficiently small $\varepsilon > 0$, there exists a positive solution v_ε of (3) satisfying*

(i) *there exists a maximum point x_ε of v_ε such that $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{M}) = 0$, and $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon(x - x_\varepsilon))$ converges (up to a subsequence) uniformly to a least energy solution of*

$$\Delta u - mu + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}^N). \tag{6}$$

(ii) $v_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon|)$ *for some* $c, C > 0$.

In [4] Berestycki and Lions proved that there exists a least energy solution of (6) if (f1),(f2) and (f3) are satisfied, and also, using the Pohozaev's identity, they showed that conditions (f2) and (f3) are necessary for existence of a non-trivial solution of the associated problem (6). Thus, basically, the concentration phenomena occurs as soon as (6) has a least energy solution and our result answers positively a conjecture of N. Dancer [9]. We should also mention [3], where it is proved that if (V1),(V2) and (f1),(f2) and (f3) are satisfied there exists a sequence $\{\varepsilon_n\}_n$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that the conclusion of Theorem 1 holds for $\varepsilon = \varepsilon_n$. Actually, it seems hopeless that the techniques of [3] could be used to get the result for any small $\varepsilon > 0$. Finally we point out that contrary to the works [3, 12, 21] we do not assume f in $C^{0,1}(\mathbf{R})$ but just continuous. Without this additional regularity we do not know if the positive solutions of (6) are radially symmetric (see [17]). Thus, it is more involved to prove the compactness, modulo translations, of the set of least energy solutions of (6) (see Proposition 1). In turn this compactness is necessary to show the exponential decay of Theorem 1 (ii).

The approaches of [3, 6, 7, 12, 21] have in common to look for solutions of (4), for $\varepsilon > 0$ small, independently of their suspected shape (the location itself is somehow prescribed by the penalization). Then, a posteriori, it is shown that they converge, up to a subsequence, to a ground state of the limiting problem (6). Here, we propose a completely different approach. We search directly solutions of (4) in a neighborhood of the set of least energy solution of (6) whose *mass* stays closed to $\mathcal{M}$. Namely in our approach we take into account the shape and location of the solutions we expect to find. This is reminiscent of the perturbation type approaches developed in [1, 16, 23, 25, 26] but we point out that no uniqueness nor non-degeneracy of the least energy solutions of (6) are required. Our approach is indeed purely variational.

2 Proof of Theorem 1.

The variational framework is the following. Let H_ε be the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon = \Big(\int_{\mathbf{R}^N} |\nabla u|^2 + V_\varepsilon u^2 dx\Big)^{1/2}.$$

We define a norm $\|\cdot\|$ on $H^1(\mathbf{R}^N)$ by

$$\|u\|^2 = \int_{\mathbf{R}^N} |\nabla u|^2 + V_0 u^2 dx.$$

Since $\inf_{\mathbf{R}^N} V(x) = V_0 > 0$, we clearly have $H_\varepsilon \subset H^1(\mathbf{R}^N)$. From now on, for any set $B \subset \mathbf{R}^N$ and $\varepsilon > 0$, we define $B_\varepsilon \equiv \{x \in \mathbf{R}^N \mid \varepsilon x \in B\}$. For $u \in H_\varepsilon$, let

$$P_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V_\varepsilon u^2 dx - \int_{\mathbf{R}^N} F(u) dx \tag{7}$$

(since we seek positive solutions, we assume without loss of generality that $f(t) = 0$ for all $t \leq 0$).

Fixing an arbitrary $\mu > 0$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in O_\varepsilon \\ \varepsilon^{-\mu} & \text{if } x \notin O_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(u) = \Big(\int_{\mathbf{R}^N} \chi_\varepsilon u^2 dx - 1 \Big)_+^{\frac{p+1}{2}}. \tag{8}$$

The functional Q_ε will act as a penalization to force the concentration phenomena to occur inside O. This type of penalization was first introduced in [7]. Finally let $\Gamma_\varepsilon : H_\varepsilon \to \mathbf{R}$ be given by

$$\Gamma_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u). \tag{9}$$

It is standard to see that $\Gamma_\varepsilon \in C^1(H_\varepsilon)$. Clearly a critical point of P_ε corresponds to a solution of (4). To find solutions of (4) which *concentrate* in O as $\varepsilon \to 0$, we shall search critical points of Γ_ε for which Q_ε is zero. As we shall see the functional Γ_ε enjoys a mountain pass geometry for any $\varepsilon > 0$ small. First we study some properties of the solutions of (6).

Without loss of generality, we may assume that $0 \in \mathcal{M}$. For any set $B \subset \mathbf{R}^N$ and $\delta > 0$, we define $B^\delta \equiv \{x \in \mathbf{R}^N | \mathrm{dist(x,B)} \le \delta\}$. As we already mention, the following equations for $a > 0$ are limiting equations of (4)

$$\Delta u - au + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}^N). \tag{10}$$

We define an energy functional for the limiting problems (10) by

$$L_a(u) = \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u|^2 + au^2 dx - \int_{\mathbf{R}^N} F(u) dx, \quad u \in H^1(\mathbf{R}^N). \tag{11}$$

In [4] Berestycki and Lions proved that, for any $a > 0$, there exists a least energy solution of (10) if (f1),(f2) and (f3) with $m = a$ are satisfied. Also they showed that each solution U of (10) satisfies the Pohozaev's identity

$$\frac{N-2}{2} \int_{\mathbf{R}^N} |\nabla U|^2 dx + N \int_{\mathbf{R}^N} a\frac{u^2}{2} - F(u) dx = 0. \tag{12}$$

Let S_a be the set of least energy solutions U of (10) satisfying $U(0) = \max_{x \in \mathbf{R}^N} U(x)$. Then, we obtain the following compactness of S_a.

Proposition 1 *For each $a > 0$ and $N \ge 3$, S_a is compact in $H^1(\mathbf{R}^N)$. Moreover, there exist $C, c > 0$, independent of $U \in S_a$ such that*

$$U(x) \le C \exp(-c|x|).$$

Proof. From (12), we see that for any $U \in S_a$,

$$\frac{1}{N}\int_{\mathbf{R}^N} |\nabla U|^2 dx = L_a(U). \tag{13}$$

Thus, $\{\int_{\mathbf{R}^N} |\nabla U|^2 dx \mid U \in S_a\}$ is bounded. Note that for any $U \in S_a$,

$$\int_{\mathbf{R}^N} |\nabla U|^2 + aU^2 dx = \int_{\mathbf{R}^N} f(U)U dx. \tag{14}$$

By (f1) and (f2), we see that there exists $C > 0$ satisfying

$$\int_{\mathbf{R}^N} f(U)U dx \le \frac{a}{2}\int_{\mathbf{R}^N} U^2 dx + C\int_{\mathbf{R}^N} U^{\frac{2N}{N-2}} dx. \tag{15}$$

Thus, it follows from (14) and (15) that

$$\frac{a}{2}\int_{\mathbf{R}^N} U^2 dx \le C\int_{\mathbf{R}^N} U^{\frac{2N}{N-2}} dx. \tag{16}$$

Then, by the Sobolev inequality, we see that $\{\int_{\mathbf{R}^N} U^2 dx \mid U \in S_a\}$ is bounded. Thus, S_a is bounded in $H^1(\mathbf{R}^N)$. Then, we see from elliptic estimates (see [18]) that S_a is bounded in $L^\infty(\mathbf{R}^N)$. Moreover, from the maximum principle, we see that S_a is bounded away from 0 in $L^\infty(\mathbf{R}^N)$. We claim that $\lim_{|x|\to\infty} U(x) = 0$ uniformly for $U \in S_a$. To the contrary, we assume that for some $\{U_k\}_{k=1}^\infty \subset S_a$ and $\{x_k\}_{k=1}^\infty \subset \mathbf{R}^N$ with $\lim_{k\to\infty} |x_k| = \infty$, it holds $\liminf_{k\to\infty} U_k(x_k) > 0$. Define $V_k(x) = U_k(x + x_k)$. We see from elliptic estimates that for some $\beta > 0$, $\{U_k, V_k\}_{k=1}^\infty$ is bounded in $C^{1,\beta}(\mathbf{R}^N)$. Then, taking a subsequence if it is necessary, we can assume that for some $U, V \in H^1(\mathbf{R}^N)$, U_k and V_k converge to U and V in $C^1_{loc}(\mathbf{R}^N)$ and weakly in $H^1(\mathbf{R}^N)$, respectively. This implies that U and V are solutions of (10) and we have

$$L_a(U), L_a(V) \ge L_a(W) \ \text{ for any } \ W \in S_a.$$

Note that

$$L_a(U_1) = L_a(U_2) = \cdots = \frac{1}{N}\int_{\mathbf{R}^N} |\nabla U_1|^2 dx = \frac{1}{N}\int_{\mathbf{R}^N} |\nabla U_2|^2 dx = \cdots.$$

Thus, for each $2R \le |x_k|$,

$$\begin{aligned} &L_a(U_k) \\ &= \tfrac{1}{N}\textstyle\int_{\mathbf{R}^N} |\nabla U_k|^2 dx \ge \tfrac{1}{N}\int_{B(0,R)} |\nabla U_k|^2 dx + \tfrac{1}{N}\int_{B(x_k,R)} |\nabla U_k|^2 dx \\ &\qquad\qquad\qquad\qquad = \tfrac{1}{N}\textstyle\int_{B(0,R)} |\nabla U_k|^2 + |\nabla V_k|^2 dx. \end{aligned}$$

Taking $R > 0$ large enough we reach a contradiction. Thus, $\lim_{|x|\to\infty} U(x) = 0$ uniformly for $U \in S_a$. Then, by the comparison principle and the elliptic estimates, we see that there exists $C, c > 0$ satisfying

$$U(x) + |\nabla U(x)| \le C \exp(-c|x|), \quad x \in \mathbf{R}^N, U \in S_a.$$

Thus, for any $\delta > 0$, there exists $R > 0$ such that

$$\int_{|x|\ge R} |\nabla U|^2 + aU^2 dx \le \delta \quad \text{for} \quad U \in S_a. \tag{17}$$

Let $\{U_k\}_{k=1}^\infty$ be a sequence in S_a. Taking a subsequence if it is necessary, we can assume that U_k converges weakly to some U in $H^1(\mathbf{R}^N)$. Note, then, that U is a solution of (10). From (f2), it is standard to see that as $k \to \infty$,

$$\int_{|x|\le R} f(U_k)U_k dx \to \int_{|x|\le R} f(U)U dx.$$

Since

$$\int_{\mathbf{R}^N} |\nabla U_k|^2 + a(U_k)^2 - f(U_k)U_k dx = \int_{\mathbf{R}^N} |\nabla U|^2 + a(U)^2 - f(U)U dx = 0,$$

it follows from (17) that

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} |\nabla U_k|^2 + a(U_k)^2 dx = \int_{\mathbf{R}^N} |\nabla U|^2 + a(U)^2 dx.$$

This implies that $U_k \to U \in S_a$ in $H^1(\mathbf{R}^N)$. This completes the proof that S_a is compact for $N \ge 3, a > 0$. $\square$

Let $E_m = L_m(U)$ for $U \in S_m$ and $10\delta = \text{dist}(\mathcal{M}, \mathbf{R}^N \setminus O)$. We fix a $\beta \in (0, \delta)$ and a cutoff $\varphi \in C_0^\infty(\mathbf{R}^N)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| \le \beta$ and $\varphi(x) = 0$ for $|x| \ge 2\beta$. Also, setting $\varphi_\varepsilon(y) = \varphi(\varepsilon y), y \in \mathbf{R}^N$, for each $x \in \mathcal{M}^\beta$ and $U \in S_m$, we define

$$U_\varepsilon^x(y) \equiv \varphi_\varepsilon(y - \frac{x}{\varepsilon})U(y - \frac{x}{\varepsilon}).$$

We will find a solution near the set

$$X_\varepsilon = \{U_\varepsilon^x(y) \mid x \in \mathcal{M}^\beta, U \in S_m\}$$

for sufficiently small $\varepsilon > 0$. We note that $0 \in \mathcal{M}$, and define

$$W_\varepsilon(y) = \varphi_\varepsilon(y)U(y)$$

where $U \in S_m$ is arbitrary but fixed. Setting $W_{\varepsilon,t}(y) = U(\frac{y}{t})\varphi_\varepsilon(y)$, we see that $\Gamma_\varepsilon(W_{\varepsilon,t}) = P_\varepsilon(W_{\varepsilon,t})$ for $t \geq 0$. Also, from (12), we see that for $U_t(x) = U(\frac{x}{t})$ we have

$$\begin{aligned} L_m(U_t) &= \int_{\mathbf{R}^N} \frac{t^{N-2}}{2}|\nabla U|^2 + m\frac{t^N}{2}U^2 - t^N F(U)dx \\ &= \left(\frac{t^{N-2}}{2} - \frac{(N-2)t^N}{2N}\right)\int_{\mathbf{R}^N}|\nabla U|^2 dx. \end{aligned}$$

Thus, there exists $t_0 > 0$ such that $L_m(U_t) < -2$ for $t \geq t_0$.

Finally we define

$$C_\varepsilon = \inf_{\gamma\in\Phi_\varepsilon}\max_{s\in[0,1]}\Gamma_\varepsilon(\gamma(s))$$

where $\Phi_\varepsilon = \{\gamma \in C([0,1], H_\varepsilon) \mid \gamma(0) = 0, \gamma(1) = W_{\varepsilon,t_0}\}$. We easily check that $\Gamma_\varepsilon(\gamma(1)) < -2$ for any $\varepsilon > 0$ sufficiently small.

Proposition 2

$$\limsup_{\varepsilon\to 0} C_\varepsilon \leq E_m.$$

Proof. Defining $W_{\varepsilon,0} = \lim_{t\to 0} W_{\varepsilon,t}$, we see that $W_{\varepsilon,0} = 0$. Thus setting $\gamma(s) = W_{\varepsilon,st_0}$ we have $\gamma \in \Phi_\varepsilon$. Now,

$$C_\varepsilon \leq \max_{s\in[0,1]}\Gamma_\varepsilon(\gamma(s)) = \max_{t\in[0,t_0]}\Gamma_\varepsilon(W_{\varepsilon,t}) = \max_{t\in[0,t_0]}P_\varepsilon(W_{\varepsilon,t})$$

and it is standard to show that

$$\lim_{\varepsilon\to 0}\max_{t\in[0,t_0]}P_\varepsilon(W_{\varepsilon,t}) \leq E_m$$

(see for example Proposition 6.1 of [21]). $\square$

Proposition 3

$$\liminf_{\varepsilon\to 0} C_\varepsilon \geq E_m.$$

Proof. To the contrary, we assume that $\liminf_{\varepsilon\to 0} C_\varepsilon < E_m$. Then, there exists $\alpha > 0$, $\varepsilon_n \to 0$ and $\gamma_n \in \Phi_{\varepsilon_n}$ satisfying $\Gamma_{\varepsilon_n}(\gamma_n(s)) < E_m - \alpha$ for $s \in [0,1]$. We fix an ε_n such that

$$\frac{m}{2}\varepsilon_n^\mu(1+(1+E_m)^{\frac{2}{p+1}}) < \min\{\alpha, 1\}$$

and $P_{\varepsilon_n}(\gamma_n(1)) < -2$ and denote ε_n by ε and γ_n by γ.

Since $P_\varepsilon(\gamma(0)) = 0$ we can find $s_0 \in (0,1)$ such that $P_\varepsilon(\gamma(s)) \geq -1$ for $s \in [0, s_0]$ and $P_\varepsilon(\gamma(s_0)) = -1$. Then, for any $s \in [0, s_0]$,

$$Q_\varepsilon(\gamma(s)) \leq \Gamma_\varepsilon(\gamma(s)) + 1 \leq E_m - \alpha + 1.$$

This implies that

$$\int_{\mathbf{R}^N\setminus O_\varepsilon} (\gamma(s))^2 dx \leq \varepsilon^\mu(1+(1+E_m)^{\frac{2}{p+1}}) \ \text{ for } \quad s \in [0, s_0].$$

Then, for $s \in [0, s_0]$,

$$\begin{aligned} P_\varepsilon(\gamma(s)) &= \tfrac{1}{2}\textstyle\int_{\mathbf{R}^N} |\nabla\gamma(s)|^2 + m(\gamma(s))^2 dx - \int_{\mathbf{R}^N} F(\gamma(s))dx \\ &\qquad + \tfrac{1}{2}\textstyle\int_{\mathbf{R}^N}(V_\varepsilon(x) - m)(\gamma(s))^2 dx \\ &\geq \tfrac{1}{2}\textstyle\int_{\mathbf{R}^N} |\nabla\gamma(s)|^2 + m(\gamma(s))^2 dx - \int_{\mathbf{R}^N} F(\gamma(s))dx \\ &\qquad + \tfrac{1}{2}\textstyle\int_{\mathbf{R}^N\setminus O_\varepsilon}(V_\varepsilon(x) - m)(\gamma(s))^2 dx \\ &\geq \tfrac{1}{2}\textstyle\int_{\mathbf{R}^N} |\nabla\gamma(s)|^2 + m(\gamma(s))^2 dx - \int_{\mathbf{R}^N} F(\gamma(s))dx \\ &\qquad - \tfrac{m}{2}\textstyle\int_{\mathbf{R}^N\setminus O_\varepsilon}(\gamma(s))^2 dx \\ &\geq L_m(\gamma(s)) - \tfrac{m}{2}\varepsilon^\mu(1+(1+E_m)^{\frac{2}{p+1}}). \end{aligned}$$

Thus, $L_m(\gamma(s_0)) < 0$ and recalling that for equation (6) the mountain pass level corresponds to the least energy level (see [20]) we have that

$$\max_{s\in[0,1]} L_m(\gamma(s)) \geq E_m.$$

Then we deduce that

$$\begin{aligned} E_m - \alpha &\geq \max\nolimits_{s\in[0,1]} \Gamma_\varepsilon(\gamma(s)) \\ &\geq \max\nolimits_{s\in[0,1]} P_\varepsilon(\gamma(s)) \\ &\geq \max\nolimits_{s\in[0,s_0]} P_\varepsilon(\gamma(s)) \\ &\geq \max\nolimits_{s\in[0,1]} L_m(\gamma(s)) - \tfrac{m}{2}\varepsilon^\mu(1+(1+E_m)^{\frac{2}{p+1}}) \\ &\geq E_m - \tfrac{m}{2}\varepsilon^\mu(1+(1+E_m)^{\frac{2}{p+1}}) \end{aligned}$$

and this contradiction completes the proof. $\Box$

Propositions 2 and 3 imply that $\lim_{\varepsilon\to 0}(\max_{s\in[0,1]}\Gamma_\varepsilon(\gamma_\varepsilon(s))-C_\varepsilon)=0$, where $\gamma_\varepsilon(s)=W_{\varepsilon,st_0}$ for $s\in(0,1]$ and $\gamma_\varepsilon(0)=0$. For future reference we denote

$$D_\varepsilon \equiv \max_{s\in[0,1]}\Gamma_\varepsilon(\gamma_\varepsilon(s)).$$

Then, we see that

$$C_\varepsilon \le D_\varepsilon \quad \text{and} \quad \lim_{\varepsilon\to 0} C_\varepsilon = \lim_{\varepsilon\to 0} D_\varepsilon = E_m.$$

Now define

$$\Gamma_\varepsilon^\alpha = \{u\in H_\varepsilon \mid \Gamma_\varepsilon(u)\le\alpha\}$$

and for a set $A\subset H_\varepsilon$ and $\alpha>0$, let $A^\alpha\equiv\{u\in H_\varepsilon \mid \inf_{v\in A}\|u-v\|_\varepsilon\le\alpha\}$.

Proposition 4 *Let $\{\varepsilon_i\}_{i=1}^\infty$ be such that $\lim_{i\to\infty}\varepsilon_i=0$ and $\{u_{\varepsilon_i}\}\in X_{\varepsilon_i}^d$ such that*

$$\lim_{i\to\infty}\Gamma_{\varepsilon_i}(u_{\varepsilon_i})\le E_m \text{ and } \lim_{i\to\infty}\Gamma'_{\varepsilon_i}(u_{\varepsilon_i})=0.$$

Then, for sufficiently small $d>0$, there exists, up to a subsequence, $\{y_i\}_{i=1}^\infty\subset\mathbf{R}^N$, $x\in\mathcal{M}$, $U\in S_m$ such that

$$\lim_{i\to\infty}|\varepsilon_i y_i - x|=0 \text{ and } \lim_{i\to\infty}\|u_{\varepsilon_i}-\varphi_{\varepsilon_i}(\cdot-y_i)U(\cdot-y_i)\|_{\varepsilon_i}=0.$$

Proof. For convenience' sake, we write ε for ε_i. By compactness of S_m and $\mathcal{M}^\beta$, there exist $Z\in S_m$ and $x\in\mathcal{M}^\beta$ such that

$$\|u_\varepsilon-\varphi_\varepsilon(\cdot-x/\varepsilon)Z(\cdot-x/\varepsilon)\|_\varepsilon\le 2d \tag{18}$$

for small $\varepsilon>0$. We denote $u_\varepsilon^1=\varphi_\varepsilon(\cdot-x/\varepsilon)u_\varepsilon$ and $u_\varepsilon^2=u_\varepsilon-u_\varepsilon^1$. As a first step in the proof of the Proposition we shall prove that

$$\Gamma_\varepsilon(u_\varepsilon)\ge\Gamma_\varepsilon(u_\varepsilon^1)+\Gamma_\varepsilon(u_\varepsilon^2)+O(\varepsilon). \tag{19}$$

Suppose there exist $x_\varepsilon\in B(x/\varepsilon,2\beta/\varepsilon)\backslash B(x/\varepsilon,\beta/\varepsilon)$ and $R>0$ satisfying $\liminf_{\varepsilon\to 0}\int_{B(x_\varepsilon,R)}(u_\varepsilon)^2dy>0$. Taking a subsequence, we can assume that $\varepsilon x_\varepsilon\to x_0$ with x_0 in the closure of $B(x,2\beta)\backslash B(x,\beta)$ and that $u_\varepsilon(\cdot+x_\varepsilon)\to\tilde W$ weakly in $H^1(\mathbf{R}^N)$ for some $\tilde W\in H^1(\mathbf{R}^N)$. Moreover $\tilde W$ satisfies

$$\Delta\tilde W(y)-V(x_0)\tilde W(y)+f(\tilde W(y))=0 \text{ for } y\in\mathbf{R}^N.$$

By definition, $L_{V(x_0)}(\tilde{W}) \geq E_{V(x_0)}$. Also, for large $R > 0$

$$\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^2 dy \geq \frac{1}{2} \int_{\mathbf{R}^N} |\nabla \tilde{W}|^2 dy. \tag{20}$$

Now, recalling from [20] that $E_a > E_b$ if $a > b$, we see that $E_{V(x_0)} \geq E_m$, since $V(x_0) \geq m$. Thus, from (13) and (20) we get that

$$\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^2 dy \geq \frac{N}{2} L_{V(x_0)}(\tilde{W}) \geq \frac{N}{2} E_m > 0.$$

Then, taking $d > 0$ sufficiently small, we get a contradiction with (18). Since there does not exist such a sequence $\{x_\varepsilon\}_\varepsilon$ we deduce from a result of P.L. Lions (see [24, Lemma I.1]) that

$$\liminf_{\varepsilon \to 0} \int_{B(x/\varepsilon, 2\beta/\varepsilon) \setminus B(x/\varepsilon, \beta/\varepsilon)} |u_\varepsilon|^{p+1} dy = 0. \tag{21}$$

As a consequence, we can derive using (f1),(f2) and boundedness of $\{\|u_\varepsilon\|_{L^2}\}_\varepsilon$ that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^N} F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2) dy = 0.$$

At this point, writing

$$\begin{aligned}
\Gamma_\varepsilon(u_\varepsilon) &= \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) \\
&+ \textstyle\int_{\mathbf{R}^N} \varphi_\varepsilon(1-\varphi_\varepsilon)|\nabla u_\varepsilon|^2 + V_\varepsilon \varphi_\varepsilon(1-\varphi_\varepsilon) u_\varepsilon^2 dy \\
&- \textstyle\int_{\mathbf{R}^N} F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2) dy + O(\varepsilon),
\end{aligned}$$

the inequality (19) follows.

We now estimate $\Gamma_\varepsilon(u_\varepsilon^2)$. Since $\{u_\varepsilon\}_\varepsilon$ is bounded, we see from (18) that $\|u_\varepsilon^2\|_\varepsilon \leq 4d$ for small $\varepsilon > 0$. Then, it follows from Sobolev's inequality, that for some $C, c > 0$,

$$\begin{aligned}
\Gamma_\varepsilon(u_\varepsilon^2) \geq P_\varepsilon(u_\varepsilon^2) &\geq \frac{1}{2}\|u_\varepsilon^2\|_\varepsilon^2 - \frac{V_0}{4} \int_{\mathbf{R}^N} (u_\varepsilon^2)^2 dy - C \int_{\mathbf{R}^N} (u_\varepsilon^2)^{2N/(N-2)} dy \\
&\geq \frac{1}{4}\|u_\varepsilon^2\|_\varepsilon^2 - Cc\|u_\varepsilon^2\|_\varepsilon^{2N/(N-2)} \\
&\geq \|u_\varepsilon^2\|_\varepsilon^2 \Big(\frac{1}{4} - Cc(4d)^{4/(N-2)}\Big).
\end{aligned} \tag{22}$$

In particular, taking $d > 0$ small enough, we can assume that $\Gamma_\varepsilon(u_\varepsilon^2) \geq 0$.

Now let $W_\varepsilon(y) = u^1_\varepsilon(y + x/\varepsilon)$. Taking a subsequence we can assume that, $W_\varepsilon \to W$ weakly in $H^1(\mathbf{R}^N)$ for some $W \in H^1(\mathbf{R}^N)$. Moreover W satisfies

$$\Delta W(y) - V(x)W(y) + f(W(y)) = 0 \quad \text{for} \quad y \in \mathbf{R}^N.$$

From the maximum principle we see that W is positive. Let us prove that $W_\varepsilon \to W$ strongly in $H^1(\mathbf{R}^N)$. Suppose there exist $R > 0$ and a sequence $\{z_\varepsilon\}_\varepsilon$ with $z_\varepsilon \in B(x/\varepsilon, 2\beta/\varepsilon)$ satisfying

$$\liminf_{\varepsilon\to 0} |z_\varepsilon - x/\varepsilon| = \infty \quad \text{and} \quad \liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,R)} (u^1_\varepsilon)^2 dy > 0.$$

We may assume that $\varepsilon z_\varepsilon \to z_0 \in O$ as $\varepsilon \to 0$. Then, $\tilde{W}_\varepsilon(y) = u^1_\varepsilon(y + z_\varepsilon)$ converges weakly to $\tilde{W}$ in $H^1(\mathbf{R}^N)$ satisfying

$$\Delta\tilde{W} - V(z_0)\tilde{W} + f(\tilde{W}) = 0, \quad \text{for} \quad y \in \mathbf{R}^N.$$

At this point as before we get a contradiction and then using (f1),(f2) and [24, Lemma I.1] that

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^N} F(W_\varepsilon)dx \to \int_{\mathbf{R}^N} F(W)dx. \tag{23}$$

Then it follows from the weak convergence of W_ε to W in $H^1(\mathbf{R}^N)$ that

$$\begin{aligned}
& \lim_{\varepsilon\to 0} \Gamma_\varepsilon(u^1_\varepsilon) \\
\geq\ & \liminf_{\varepsilon\to 0} P_\varepsilon(u^1_\varepsilon) \\
=\ & \liminf_{\varepsilon\to 0} \frac{1}{2}\int_{\mathbf{R}^N} |\nabla W_\varepsilon(y)|^2 + V(\varepsilon y + x)W^2_\varepsilon(y)dy - \int_{\mathbf{R}^N} F(W_\varepsilon(y))dy \\
\geq\ & \frac{1}{2}\int_{\mathbf{R}^N} |\nabla W|^2 + V(x)W^2 dy - \int_{\mathbf{R}^N} F(W)dy \\
\geq\ & E_m.
\end{aligned} \tag{24}$$

Since $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon) \leq E_m$, $\Gamma_\varepsilon(u^2_\varepsilon) \geq 0$ and because of (19), we see that

$$\limsup_{\varepsilon\to 0} \Gamma_\varepsilon(u^1_\varepsilon) \leq E_m. \tag{25}$$

Then (24) implies that $L_{V(x)}(W) = E_m$. Also, from [20], we see that $x \in \mathcal{M}$. At this point it is clear that $W(y) = U(y - z)$ with $U \in S_m$ and $z \in \mathbf{R}^N$.

Finally, using (23), (25) and the fact that $V \geq V(x)$ on O, we get from (24) that

$$\begin{aligned} \textstyle\int_{\mathbf{R}^N} |\nabla W|^2 + V(x)W^2 dy &\geq \limsup_{\varepsilon\to 0} \textstyle\int_{\mathbf{R}^N} |\nabla u_\varepsilon^1(y)|^2 + V(\varepsilon y)(u_\varepsilon^1)^2(y)dy \\ &\geq \limsup_{\varepsilon\to 0} \textstyle\int_{\mathbf{R}^N} |\nabla u_\varepsilon^1(y)|^2 + V(x)(u_\varepsilon^1)^2(y)dy \\ &\geq \limsup_{\varepsilon\to 0} \textstyle\int_{\mathbf{R}^N} |\nabla W_\varepsilon(y)|^2 + V(x)W_\varepsilon^2(y)dy. \end{aligned}$$

This proves the strong convergence of u_ε^1 to W in $H^1(\mathbf{R}^N)$. In particular setting $y_\varepsilon = x/\varepsilon + z$ we have $u_\varepsilon^1 \to \varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)$ strongly in $H^1(\mathbf{R}^N)$. This means that $u_\varepsilon^1 \to \varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)$ strongly in H_ε. To conclude the proof of the Proposition, it suffices to show that $u_\varepsilon^2 \to 0$ in H_ε. Now, using (19), (22) and $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon^1) = E_m$, we deduce that for some $C > 0$,

$$E_m \geq \lim_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon) \geq E_m + C\|u_\varepsilon^2\|_\varepsilon^2(1 - d^{4/N-2}) + O(\varepsilon).$$

This proves that $u_\varepsilon^2 \to 0$ in H_ε, and completes the proof. □

Proposition 5 *For sufficiently small* $d_1 > d_2 > 0$, *there exist constants* $\omega > 0$ *and* $\varepsilon_0 > 0$ *such that* $|\Gamma_\varepsilon'(u)| \geq \omega$ *for* $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^{d_1} \setminus X_\varepsilon^{d_2})$ *and* $\varepsilon \in (0, \varepsilon_0)$.

Proof. To the contrary, we suppose that for small $d_1 > d_2 > 0$, there exist $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\varepsilon_i = 0$ and $u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_1} \setminus X_{\varepsilon_i}^{d_2}$ satisfying $\lim_{i\to\infty}\Gamma_{\varepsilon_i}(u_{\varepsilon_i}) \leq E_m$ and $\lim_{i\to\infty}|\Gamma_{\varepsilon_i}'(u_{\varepsilon_i})| = 0$. For convenience' sake, we write ε for ε_i. By Proposition 4, there exists $\{y_\varepsilon\}_\varepsilon \subset \mathbf{R}^N$ such that for some $U \in S_m$ and $x \in \mathcal{M}$,

$$\lim_{\varepsilon\to 0}|\varepsilon y_\varepsilon - x| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\|u_\varepsilon - \varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)\|_\varepsilon = 0.$$

By the definition of X_ε, we see that $\lim_{\varepsilon\to 0}\operatorname{dist}(u_\varepsilon, X_\varepsilon) = 0$. This contradicts that $u_\varepsilon \notin X_\varepsilon^{d_2}$, and completes the proof. □

Following Proposition 5, we fix a $d > 0$ and corresponding $\omega > 0$ and $\varepsilon_0 > 0$ such that $|\Gamma_\varepsilon'(u)| \geq \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$ and $\varepsilon \in (0, \varepsilon_0)$. Then, we obtain the following proposition.

Proposition 6 *There exist* $\alpha > 0$ *such that for sufficiently small* $\varepsilon > 0$,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq C_\varepsilon - \alpha \textit{ implies that } \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}$$

where $\gamma_\varepsilon(s) = W_{\varepsilon,st_0}(s)$.

Proof. Since $\mathrm{supp}(\gamma_\varepsilon(s)) \subset \mathcal{M}_\varepsilon^{2\beta}$ for each $s \in [0,1]$, it follows that $\Gamma_\varepsilon(\gamma_\varepsilon(s)) = P_\varepsilon(\gamma_\varepsilon(s))$. Moreover, we see from the decay property of U and a change of variables that

$$\begin{aligned} P_\varepsilon(\gamma_\varepsilon(s)) &= \tfrac{1}{2}\int_{\mathbf{R}^N} |\nabla\gamma_\varepsilon(s)|^2 + V_\varepsilon(x)(\gamma_\varepsilon(s))^2 dx - \int_{\mathbf{R}^N} F(\gamma_\varepsilon(s))dx \\ &= \tfrac{1}{2}\int_{\mathbf{R}^N} |\nabla\gamma_\varepsilon(s)|^2 + m(\gamma_\varepsilon(s))^2 dx - \int_{\mathbf{R}^N} F(\gamma_\varepsilon(s))dx \\ &\quad + \tfrac{1}{2}\int_{\mathbf{R}^N} (V_\varepsilon(x) - m)(\gamma_\varepsilon(s))^2 dx \\ &= \tfrac{(st_0)^{N-2}}{2}\int_{\mathbf{R}^N} |\nabla U|^2 dx + \tfrac{(st_0)^N}{2}\int_{\mathbf{R}^N} mU^2 dx \\ &\quad -(st_0)^N \int_{\mathbf{R}^N} F(U)dx + O(\varepsilon). \end{aligned}$$

Then, from the Pohozaev identity (12), we see that

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) = P_\varepsilon(\gamma_\varepsilon(s)) = \Big(\frac{(st_0)^{N-2}}{2} - \frac{N-2}{2N}(st_0)^N\Big)\int_{\mathbf{R}^N} |\nabla U|^2 dx + O(\varepsilon).$$

Note that

$$\max_{t\in(0,\infty)} \Big(\frac{t^{N-2}}{2} - \frac{N-2}{2N}t^N\Big)\int_{\mathbf{R}^N} |\nabla U|^2 dx = E_m$$

and $\lim_{\varepsilon\to 0} C_\varepsilon = E_m$. Then, since, denoting $g(t) = \frac{t^{N-2}}{2} - \frac{N-2}{2N}t^N$,

$$g'(t) \begin{cases} > 0 & \text{for } \ t \in (0,1), \\ = 0 & \text{for } \ t = 1, \\ < 0 & \text{for } \ t > 1 \end{cases}$$

and $g''(1) = 2 - N < 0$, the conclusion follows. □

Proposition 7 *For sufficiently fixed small $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$ such that $\Gamma'_\varepsilon(u_n) \to 0$ as $n \to \infty$.*

Proof. By Proposition 6, there exists $\alpha > 0$ such that for sufficiently small $\varepsilon > 0$,

$$\Gamma_\varepsilon(\gamma_\varepsilon(s)) \geq C_\varepsilon - \alpha \quad \text{implies that} \quad \gamma_\varepsilon(s) \in X_\varepsilon^{d/2}.$$

If Proposition 7 does not hold for sufficiently small $\varepsilon > 0$, there exists $a(\varepsilon) > 0$ such that $|\Gamma'_\varepsilon(u)| \geq a(\varepsilon)$ on $X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$. Also we know from Proposition 5 that there exists $\omega > 0$, independent of $\varepsilon > 0$, such that $|\Gamma'_\varepsilon(u)| \geq \omega$ for $u \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$. Thus, recalling that $\lim_{\varepsilon\to 0}(C_\varepsilon - D_\varepsilon) = 0$, by a deformation argument, for sufficiently small $\varepsilon > 0$, it is possible to construct a path $\gamma \in \Phi_\varepsilon$ satisfying $\Gamma_\varepsilon(\gamma(s)) < C_\varepsilon, s \in [0,1]$. This contradiction proves the Proposition. □

Proposition 8 *For sufficiently small fixed $\varepsilon > 0$, Γ_ε has a critical point $u \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$.*

Proof. Let $\{u_n\}_{n=1}^\infty$ be a Palais-Smale sequence as given by Proposition 7 corresponding to a fixed small $\varepsilon > 0$. Since $\{u_n\}_{n=1}^\infty$ is bounded in H_ε, $u_n \to u$ weakly in H_ε, for some $u \in H_\varepsilon$. Then, it follows in a standard way that u is a critical point of Γ_ε. Now we write $u_n = v_n + w_n$ with $v_n \in X_\varepsilon$ and $\|w_n\|_\varepsilon \le d$. Since X_ε is compact, there exists $v \in X_\varepsilon$ such $v_n \to v$ in X_ε, up to a subsequence, as $n \to \infty$. Moreover, for some $w \in H_\varepsilon$, $w_n \to w$ weakly, up to a subsequence, in H_ε, as $n \to \infty$. Thus, $u = v + w$ and

$$\|u - v\|_\varepsilon = \|w\|_\varepsilon \le \liminf_{n\to\infty} \|w_n\|_\varepsilon \le d.$$

This proves that $u \in X_\varepsilon^d$.

To show that $\Gamma_\varepsilon(u) \le D_\varepsilon$, it suffices to show that $\limsup_{n\to\infty} \Gamma_\varepsilon(u_n) \ge \Gamma_\varepsilon(u)$. In fact, writing $u_n = u + o_n$, we deduce that

$$\begin{aligned}\|o_n\|_\varepsilon = \|u_n - v - w\|_\varepsilon &\le \|v_n - v\|_\varepsilon + \|w_n - w\|_\varepsilon \\ &\le \|v_n - v\|_\varepsilon + \|w_n\|_\varepsilon + \|w\|_\varepsilon \\ &\le 2d + o(1)\end{aligned}$$

and

$$\|u_n\|_\varepsilon^2 = \|u\|_\varepsilon^2 + \|o_n\|_\varepsilon^2.$$

It is standard(see the proof of Proposition 2.31 in [8] for example) to show that

$$\int_{\mathbf{R}^N} F(u_n)dx = \int_{\mathbf{R}^N} F(u)dx + \int_{\mathbf{R}^N} F(o_n)dx + o(1).$$

Thus we see that

$$P_\varepsilon(u_n) = \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbf{R}^N} F(u)dx + \frac{1}{2}\|o_n\|_\varepsilon^2 - \int_{\mathbf{R}^N} F(o_n)dx + o(1).$$

For sufficiently large $n > 0$ and small $d > 0$, we deduce, as in the proof of Proposition 4, that

$$\frac{1}{2}\|o_n\|_\varepsilon^2 - \int_{\mathbf{R}^N} F(o_n)dx \ge 0.$$

It follows that $\limsup_{n\to\infty} \Gamma_\varepsilon(u_n) \ge \Gamma_\varepsilon(u)$. This completes the proof. □

Completion of the Proof for Theorem 1. We see from Proposition 8 that there exist $d > 0$ and $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, Γ_ε has a critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$. Since u_ε satisfies

$$\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + f(u_\varepsilon) = (p+1)\Big(\int \chi_\varepsilon u_\varepsilon^2 dx - 1\Big)_+^{\frac{p-1}{2}} \chi_\varepsilon u_\varepsilon \ \text{ in } \ \mathbf{R}^N \tag{26}$$

and $f(t) = 0$ for $t \leq 0$, we deduce that $u_\varepsilon > 0$ in $\mathbf{R}^N$. Moreover, by elliptic estimates through Moser iteration scheme, we deduce that $\{\|u_\varepsilon\|_{L^\infty}\}_\varepsilon$ is bounded (see, for example, [5, Proposition 3.5] for such techniques). Now by Proposition 4, we see that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^2 + V_\varepsilon (u_\varepsilon)^2 dx = 0,$$

and thus, by elliptic estimates (see [18]), we see that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\mathbf{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta})} = 0.$$

Using a comparison principle, it follows that, for some $C, c > 0$,

$$u_\varepsilon(x) \leq C \exp(-c\mathrm{dist}(x, \mathcal{M}_\varepsilon^{2\delta})).$$

This implies that $Q_\varepsilon(u_\varepsilon) = 0$ and thus u_ε satisfies (4). Finally let x_ε be a maximum point of u_ε. By Propositions 1 and 4, we readily deduce that $\varepsilon x_\varepsilon \to x$ for some $x \in \mathcal{M}$ as $\varepsilon \to 0$, and that for some $C, c > 0$,

$$u_\varepsilon(x) \leq C \exp(-c|x - x_\varepsilon|).$$

This completes the proof. $\square$

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