Standing waves for nonlinear Schrödinger equations with a general nonlinearity: one and two dimensional cases

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April 11, 2008

Abstract

For $N = 1, 2$, we consider singularly perturbed elliptic equations $\varepsilon^2\Delta u - V(x)u + f(u) = 0$, $u(x) > 0$ on $\mathbf{R}^N$, $\lim_{|x|\to\infty} u(x) = 0$. For small $\varepsilon > 0$, we show the existence of a localized bound state solution concentrating at an isolated component of positive local minimum of V under conditions on f we believe to be almost optimal; when $N \geq 3$, it was shown in [6].

1 Introduction

In this paper, we deal with standing waves for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - V(x)\psi + f(\psi) = 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^N. \tag{1}$$

Here $\hbar$ denotes the Plank constant, i the imaginary unit. For the physical background of this equation, we refer to the introduction in [8]. We assume that $f(\exp(i\theta)s) = \exp(i\theta)f(s)$ for $s \in \mathbf{R}$. A solution of the form $\psi(x, t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. Then, $\psi(x, t)$ is a solution of (1) if and only if the function v satisfies

$$\frac{\hbar^2}{2}\Delta v - (V(x) - E)v + f(v) = 0 \quad \text{in} \quad \mathbf{R}^N. \tag{2}$$

We are interested in positive solutions in $H^1(\mathbf{R}^N)$ for small $\hbar > 0$. For small $\hbar > 0$, these standing waves are referred to as semi-classical states. For simplicity and without loss of generality, we write $V - E$ as V, i.e., we shift E to 0. Thus, we consider the following equation

$$\varepsilon^2\Delta v - V(x)v + f(v) = 0, \quad v > 0, \quad v \in H^1(\mathbf{R}^N) \tag{3}$$

where $\varepsilon > 0$ is sufficiently small. Throughout the paper, the potential V will be assumed to satisfy

(V1) $V \in C(\mathbf{R}^N, \mathbf{R})$, $V_0 \equiv \inf_{\mathbf{R}^N} V(x) \geq 0$ and $\liminf_{|x|\to\infty} V(x) > 0$.

An interesting class of solutions of (3) are families of solutions which concentrate and develop spike layers, peaks, around certain points in $\mathbf{R}^N$ while vanishing elsewhere as $\varepsilon \to 0$. In the case $V_0 > 0$, the existence of single peak solutions was first studied by Floer and Weinstein [19].

For $N = 1$ and $f(u) = u^3$, using a finite dimensional reduction method, they construct, for any given non-degenerate critical point $x_0 \in \mathbf{R}$ of $V(x)$, a positive solution u_ε having a single peak located at x_ε, and such that $x_\varepsilon \to x_0 \in \mathbf{R}$ as $\varepsilon \to 0$. Precisely they show that $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges to the unique positive solution of

$$\Delta u - V(x_0)u + u^3 = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}).$$

Motivated by the approach in [19], many authors have obtained refined results on (3) in higher dimension and for more general f (see [2, 13, 14, 25, 26, 28, 29]). Let $x_0 \in \mathbf{R}^N$ denote the point where concentration occurs. In the above papers it is necessary to assume that there exists a unique positive solution U of

$$\Delta u - V(x_0)u + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}). \tag{4}$$

Moreover if $\Delta\phi - V(x_0)\phi + f'(U)\phi = 0$, then ϕ must be of the form: $\phi = \sum_{i=1}^N a_i \frac{\partial U}{\partial x_i}$ for some $a_i \in \mathbf{R}$. These uniqueness and nondegeneracy conditions are known to hold only for a restricted class of f. On the contrary it is known from [4, 5] that under weak conditions on f a positive least energy solution of (4) exists. In addition these conditions are almost optimal.

In a different direction, a variational approach was initiated by Rabinowitz [32] and developed further by several authors (see [8, 9, 10, 11, 15, 16, 17, 18, 21, 24]). But in this approach rather strong conditions of f are still necessary.

It is noticed in [22] and [23] that the least energy solutions of (4) found in [4] and [5] are nothing but mountain pass solutions. Since a mountain pass solution is structurally stable, it is expected that the solution would continue to exist under some *perturbations*. In fact, it is proved in [6] that for $N \geq 3$, this expectation is true. In this paper we prove the same phenomenon for $N = 1, 2$. Thus this paper is complementary to [6]. More generally than in [6], we allow the potential V to be zero on a bounded set. More precisely in addition to (V1) we assume on V.

(V2) There is a bounded open set $O \subset \mathbf{R}^N$ such that

$$0 < m \equiv \inf_{x \in O} V(x) < \min_{x \in \partial O} V(x).$$

We define

$$\mathcal{M} \equiv \{x \in O \mid V(x) = m\}$$

and set $\mathcal{Z} \equiv \{x \in \mathbf{R}^N | V(x) = 0\}$.

We also assume that $f : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and satisfies the following conditions.

(f1) $\lim_{t\to 0^+} f(t)/t = 0$;

(f1′) $\limsup_{t\to 0^+} \dfrac{f(t)}{t^{1+\mu}} < \infty$ for some $\mu > 0$;

(f2) if $N = 2$, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|f(t)| \le C_\alpha \exp(\alpha t^2)$ for all $t \in \mathbf{R}^+$;

(f3) there exists $t_0 > 0$ such that if $N = 2$, $\frac{1}{2}mt_0^2 < F(t_0)$ and if $N = 1$, $\frac{1}{2}mt^2 > F(t)$ for $t \in (0, t_0)$, $\frac{1}{2}mt_0^2 = F(t_0)$ and $mt_0 < f(t_0)$, where $F(t) = \int_0^t f(s)ds$.

We consider the following limiting equation

$$\Delta u - mu + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}^N). \tag{5}$$

If (f1-3) hold, it is known from [4, 5] that (5) has a least energy solution.

Theorem 1 *Let $N = 1, 2$ and assume that (V1-2) and (f1-3) hold. If $\mathcal{Z} \neq \emptyset$ assume furthermore (f1′). Then for sufficiently small $\varepsilon > 0$, there exists a positive solution v_ε of (3) such that for a maximum point x_ε of v_ε (which is unique for $N = 1$),*

$$\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{M}) = 0,$$

and $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + x_\varepsilon)$ converges (up to a subsequence for $N = 2$) uniformly to a least energy solution of (5). In addition for some $c, C > 0$,

$$v_\varepsilon(x) \le C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon|).$$

In [4, 5], the authors proved that condition (f3) is necessary for the existence of a non-trivial solution of the associated problem (5). In the case $\mathcal{Z} \neq \emptyset$ we need an additional decay condition on f at 0, but when $\mathcal{Z} = \emptyset$, the conditions (f1-3) are the same as in [4]. Thus, basically, the concentration phenomena occurs as soon as the equation (5) has a non-trivial solution.

The proof of Theorem 1 follows the approach introduced in [6], but is more involved. Indeed our approach requires to prove that the set S_m of least energy solutions U of (5) satisfying $U(0) = \max_{x\in\mathbf{R}^N} U(x)$ is compact. For $N = 2$ it is more involved to show the compactness than for $N \geq 3$. Also at the heart of the proof in [6] is the construction of a *good sample path.* Such a path is easy to construct when $N \geq 3$ since it is given $\gamma(t) = U(\cdot/t)$ for some approximate solution U. However the path $\gamma(t) = U(\cdot/t)$ does not belong to the class of admissible paths when $N = 1$ or $N = 2$ and in the two cases a different technical construction is required (see Proposition 2). Finally when we allow $V = 0$ on a compact set, there is no constant $C > 0$, independent of $u \in C_0^\infty(\mathbf{R}^N)$ and of $\varepsilon > 0$ small, such that

$$\int_{\mathbf{R}^N} \varepsilon^2|\nabla u|^2 + V(x)u^2dx \geq C\int_{\mathbf{R}^N} \varepsilon^2|\nabla u|^2 + u^2dx.$$

This difficulty, arising for any $N \in \mathbf{N}$, requires additional technicalities with respect to [6].

Defining $u(x) = v(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, equation (3) is equivalent to

$$\Delta u - V_\varepsilon u + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}^N). \tag{6}$$

In our approach we take into account the shape and location of the solutions we expect to find. Thus on one hand we benefit from the advantage of the Lyapounov-Schmidt reduction approaches, which is to discover the solution around a small neighborhood of a well chosen first approximation. On the other hand we do not need the uniqueness nor non-degeneracy of the least energy solutions of (5). Our approach is indeed purely variational.

2 Preliminaries

As we already mention, the following equations for $m > 0$ are limiting equations of (6)

$$\Delta u - mu + f(u) = 0, \quad u > 0, \quad u \in H^1(\mathbf{R}^N). \tag{7}$$

We define an energy functional for the limiting problems (7) by

$$L_m(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + mu^2 dx - \int_{\mathbf{R}^N} F(u)dx, \quad u \in H^1(\mathbf{R}^N). \tag{8}$$

In [4] and [5], the authors proved that, for any $m > 0$, there exists a least energy solution of (7) if (f1-3) are satisfied. Also they showed that each solution U of (7) satisfies the Pohozaev's identity

$$\frac{N-2}{2}\int_{\mathbf{R}^N} |\nabla U|^2 dx + N\int_{\mathbf{R}^N} m\frac{u^2}{2} - F(u)dx = 0. \tag{9}$$

Let S_m be the set of least energy solutions U of (7) satisfying $U(0) = \max_{x\in\mathbf{R}^N} U(x)$ and denote by E_m the least energy level:

$$E_m = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla U|^2 + mU^2\, dx - \int_{\mathbf{R}^N} F(U)\, dx, \quad U \in S_m.$$

Then, we obtain the following compactness of S_m.

Proposition 1 *Suppose that (f1-3) are satisfied. For each $m > 0$, S_m is compact in $H^1(\mathbf{R}^N)$ and there exist $C, c > 0$, independent of $U \in S_m$ such that*

$$U(x) \le C\exp(-c|x|) \quad \text{for all } x \in \mathbf{R}^N.$$

Moreover, if $N = 1$, S_m consists of only one element, that is, there exists a unique solution of (7) up to a translation.

Proposition 1 is proved in [6] for $N \ge 3$. For $N = 1$ we refer to [4] for the existence and [23] for the uniqueness. To prove Proposition 1 for $N = 2$, we use the following lemma. We use the notation $B(x, r) = \{y \in \mathbf{R}^N \mid |y-x| < r\}$ for $x \in \mathbf{R}^N$ and $r > 0$.

Lemma 1 *Assume $N = 2$ and that $G(t) \in C(\mathbf{R}, \mathbf{R})$ satisfies*

(a)

$$\lim_{t\to 0} \frac{G(t)}{t^2} = 0. \tag{10}$$

(b) For any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$|G(t)| \le C_\alpha e^{\alpha t^2} \quad \textit{for all } t \in \mathbf{R}^+. \tag{11}$$

Then, for any H^1-bounded sequence $\{u_n\} \subset H^1(\mathbf{R}^2)$ such that

$$\sup_{y \in \mathbf{R}^2} \int_{B(y,1)} |u_n|^2 \, dx \to 0, \tag{12}$$

it holds that

$$\int_{\mathbf{R}^2} G(u_n) \, dx \to 0 \quad \textit{as } n \to \infty. \tag{13}$$

Proof. Let $\alpha \in (0, 4\pi)$ and set $\Psi(t) = e^{\alpha t^2} - 1$. It is proved in [1] (see also [30]) that there exists $C_\alpha > 0$ such that

$$||\nabla u||_{L^2}^2 \int_{\mathbf{R}^2} \Psi(\frac{u}{||\nabla u||_{L^2}}) \, dx \le C_\alpha ||u||_{L^2}^2 \quad \text{for all } u \in H^1(\mathbf{R}^2) \setminus \{0\}. \tag{14}$$

For $u \in H^1(\mathbf{R}^2)$ satisfying $||\nabla u||_{L^2} \le M$, we have

$$M^2 \Psi(\frac{u}{M}) = \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{u^{2j}}{M^{2(j-1)}} \le \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{u^{2j}}{||\nabla u||_{L^2}^{2(j-1)}} = ||\nabla u||_{L^2}^2 \Psi(\frac{u}{||\nabla u||_{L^2}}).$$

Thus we have

$$\int_{\mathbf{R}^2} \Psi(\frac{u}{M}) \, dx \le C_\alpha M^{-2} ||u||_{L^2}^2 \quad \text{for } ||\nabla u||_{L^2} \le M. \tag{15}$$

Under the assumptions (10)–(11), for any δ, $M > 0$ there exists $C_{\delta,M} > 0$ such that for all $t \in \mathbf{R}$

$$|G(t)| \le \delta \Psi(\frac{t}{M}) + C_{\delta,M} t^4. \tag{16}$$

Let $\{u_n\} \subset H^1(\mathbf{R}^2)$ be a sequence such that $||u_n||_{H^1} \le M$ and (12) holds. By a result of Lions [27, Lemma I.1], (12) implies

$$\int_{\mathbf{R}^2} |u_n|^4 \, dx \to 0. \tag{17}$$

Thus by (15)–(17), we have

$$\limsup_{n \to \infty} \int_{\mathbf{R}^2} G(u_n) \, dx \le \delta C_\alpha.$$

Since $\delta > 0$ is arbitrary, we have $\int_{\mathbf{R}^2} G(u_n) \, dx \to 0$. □

Remark 1 *(i) A statement similar to Lemma 1 also holds for $N = 1$. More precisely, assume $G(t) \in C(\mathbf{R}, \mathbf{R})$ satisfies (10). Then for any H^1-bounded sequence $\{u_n\} \subset H^1(\mathbf{R})$ such that $\sup_{y \in \mathbf{R}} \int_{B(y,1)} |u_n|^2\,dx \to 0$, it holds that*

$$\int_{\mathbf{R}} G(u_n)\,dx \to 0 \quad \text{as } n \to \infty.$$

In fact, $\{u_n\}$ is bounded in $L^\infty(\mathbf{R})$ since $H^1(\mathbf{R}) \subset L^\infty(\mathbf{R})$. Thus for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|G(u_n)| \le \varepsilon |u_n|^2 + C_\varepsilon |u_n|^4 \quad \text{for all } n \in \mathbf{N} \text{ and } x \in \mathbf{R}$$

and we can prove that $\int_{\mathbf{R}} G(u_n)\,dx \to 0$ as in Lemma 1.
(ii) In the spirit of the above proofs, under the condition $\limsup_{t\to 0} \frac{|G(t)|}{t^2} < \infty$ and in addition (11) if $N = 2$, we can see that $\int_{\mathbf{R}^N} G(u)\,dx$ stays bounded if $||u||_{H^1}$ is bounded. Moreover under the condition (10) if $N = 1$ and (10)–(11) if $N = 2$, it holds that

$$\lim_{||u||_{H^1} \to 0} \frac{1}{||u||_{H^1}^2} \int_{\mathbf{R}^N} G(u)\,dx \to 0.$$

Proof of Proposition 1 for N=2. From (9), we see that for any $U \in S_m$,

$$\int_{\mathbf{R}^2} \frac{m}{2} U^2 - F(U)\,dx = 0 \quad \text{and} \quad \frac{1}{2} \int_{\mathbf{R}^N} |\nabla U|^2 dx = L_m(U) = E_m. \tag{18}$$

Thus, $\{\int_{\mathbf{R}^N} |\nabla U|^2 dx \mid U \in S_m\}$ is bounded. We claim that $\{\int_{\mathbf{R}^N} U^2 dx \mid U \in S_m\}$ is also bounded. Assume by contradiction that there exist $\{U_j\} \subset S_m$ satisfying $\lambda_j \equiv \|U_j\|_{L^2} \to \infty$ as $j \to \infty$. We define $\tilde{U}_j(x) \equiv U_j(\lambda_j x)$. Then, we see that

$$\|\nabla \tilde{U}_j\|_{L^2}^2 = \|\nabla U\|_{L^2}^2 = 2E_m \ \text{ and } \ \|\tilde{U}_j\|_{L^2} = 1. \tag{19}$$

Thus, $\{\tilde{U}_j\}$ is bounded in $H^1(\mathbf{R}^2)$ and satisfies

$$\frac{1}{\lambda_j^2} \Delta \tilde{U}_j - m\tilde{U}_j + f(\tilde{U}_j) = 0 \quad \text{in } \mathbf{R}^2. \tag{20}$$

For any sequence $\{x_j\} \subset \mathbf{R}^2$, we may assume that after taking a subsequence $\tilde{U}_j(x + x_j) \to \tilde{U}_0(x)$ weakly in $H^1(\mathbf{R}^2)$. It follows from (20) that $m\tilde{U}_0 = f(\tilde{U}_0)$

in $\mathbf{R}^2$ from which we see that $\tilde{U}_0 \equiv 0$. Indeed, since $\tilde{U}_0 \in H^1$ satisfies $m\tilde{U}_0 = f(\tilde{U}_0)$, the rearrangement of $\tilde{U}_0$ — say U^* — satisfies $U^* \in H^1_r(\mathbf{R}^2) \subset C(\mathbf{R}^2 \backslash 0)$ and $mU^* = f(U^*)$. Since $z = 0$ is an isolated solution of $mz = f(z)$, $U^* \in H^1_r(\mathbf{R}^2)$ must be identically 0 and it implies $\tilde{U}_0 \equiv 0$.

Since $\{x_j\} \subset \mathbf{R}^2$ is arbitrary, we have

$$\lim_{j\to\infty} \sup_{y\in\mathbf{R}^2} \int_{B(y,1)} |\tilde{U}_j(x)|^2\, dx = 0.$$

Thus by Lemma 1,

$$||f(\tilde{U}_j)||^2_{L^2} = \int_{\mathbf{R}^2} |f(\tilde{U}_j)|^2\, dx \to 0 \quad \text{as } j \to \infty.$$

By (20),

$$\begin{aligned} m||\tilde{U}_j||^2_{L^2} &\leq \int_{\mathbf{R}^2} \frac{1}{\lambda_j^2}|\nabla\tilde{U}_j|^2 + m|\tilde{U}_j|^2\, dx = \int_{\mathbf{R}^2} f(\tilde{U}_j)\tilde{U}_j\, dx \\ &\leq ||f(\tilde{U}_j)||_{L^2}||\tilde{U}_j||_{L^2} \to 0. \end{aligned}$$

This is a contradiction to (19). Therefore S_m is bounded in $H^1(\mathbf{R}^2)$.

To show the compactness of S_m, we first show that for any $\delta > 0$ there exists $R > 0$ such that

$$\sup_{|x|\geq R} |U(x)| \leq \delta \quad \text{for all } U \in S_m. \tag{21}$$

If not, there exists $\{U_j\} \subset S_m$, $\{y_j\} \subset \mathbf{R}^2$ such that $|y_j| \to \infty$ and $\liminf_{j\to\infty} U_j(y_j) > 0$. After extracting a subsequence, we may assume that $U_j(x) \to U(x)$, $U_j(x + x_j) \to V(x)$ weakly in $H^1(\mathbf{R}^2)$ with both $U(x)$ and $V(x)$ nontrivial critical points of L_m. In particular

$$L_m(U),\ L_m(V) \geq E_m.$$

Therefore

$$\begin{aligned} L_m(U_j) &= \frac{1}{2}||\nabla U_j||^2_{L^2} \geq \frac{1}{2}\int_{B(0,R)} |\nabla U_j|^2\, dx + \frac{1}{2}\int_{B(y_j,R)} |\nabla U_j|^2\, dx \\ &\geq \frac{1}{2}\int_{B(0,R)} |\nabla U|^2\, dx + \frac{1}{2}\int_{B(0,R)} |\nabla V|^2\, dx + o(1) \end{aligned}$$

as $j \to \infty$ by weak convergence. Now since $R > 0$ is arbitrary, we have

$$\liminf_{j\to\infty} L_m(U_j) \geq \frac{1}{2}||\nabla U||_{L^2}^2 + \frac{1}{2}||\nabla V||_{L^2}^2 = 2E_m.$$

This is a contradiction and thus (21) holds.

By a classical comparison argument, we can derive from (21) that

$$U(x) + |\nabla U(x)| \leq C\exp(-c|x|) \quad \text{for all } x \in \mathbf{R}^2 \text{ and } U \in S_m.$$

Thus for any $\delta > 0$ there exists $R > 0$ such that

$$\int_{|x|\geq R} |\nabla U|^2 + mU^2\, dx \leq \delta \quad \text{for all } U \in S_m.$$

¿From this fact we can easily derive the compactness of S_m in $H^1(\mathbf{R}^2)$. $\square$

Proposition 2 *Suppose that (f1-3) are satisfied. There exists some $T > 0$ such that for any $\delta > 0$, there exists a path $\gamma^\delta : [0,T] \to H^1(\mathbf{R}^N)$ satisfying*

(i) $\gamma^\delta(0) = 0$, $L_m(\gamma^\delta(T)) < -1$, $\max_{t\in[0,T]} L_m(\gamma^\delta(t)) = E_m$;

(ii) *there exists $T_0 \in (0,T)$ such that $\gamma^\delta(T_0) \in S_m$, $L_m(\gamma^\delta(T_0)) = E_m$ and $L_m(\gamma^\delta(t)) < E_m$ for $\|\gamma^\delta(t) - \gamma^\delta(T_0)\| \geq \delta$;*

(iii) *there exist $C, c > 0$ such that for any $t \in [0,T]$,*

$$|\gamma^\delta(t)(x)| + |\nabla_x \gamma^\delta(t)(x)| dy \leq C\exp(-c|x|).$$

Proof. For $N \geq 3$, it is easy to see from (9) that for $U \in S_m$, the path defined by $\gamma(t)(x) = U(\frac{x}{t})$ satisfies the properties (i)-(iii) for any $\delta > 0$. To establish the proposition we use some elements of [22, 23]. First we deal with the case $N = 1$. Then S_m consists of one element $U \in H^1(\mathbf{R})$ and in addition $U(0) = t_0$ where $t_0 > 0$ is given in (f3) (see [23]). Let $\varepsilon_0 > 0$ and define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \begin{cases} U(x) & : \ x \in [0,\infty), \\ x^4 + U(0) & : \ x \in [-\varepsilon_0, 0], \\ \varepsilon_0^4 + U(0) & : \ x \in (-\infty, -\varepsilon_0]. \end{cases}$$

Then, from (f3), and since $U(0) = t_0$, we can choose $\varepsilon_0 > 0$ so that for $x \in [-\varepsilon_0, 0)$,

$$\frac{1}{2}|h'(x)|^2 + \frac{m}{2}(h(x))^2 - F(h(x)) = 8x^6 + \frac{m}{2}(x^4 + U(0))^2 - F(x^4 + U(0)) < 0. \tag{22}$$

Now defining $\gamma : (0, T] \to H^1(\mathbf{R})$ by

$$\gamma(t)(x) = h(|x| - \ln t)$$

and $\gamma(0) = 0$, we see that $\gamma : [0, T] \to H^1(\mathbf{R})$ is continuous. It is easy to see, using (22), that for $t > 1$,

$$L_m(\gamma(t)) = E_m + 2\int_{-\ln t}^{0} \frac{1}{2}|h'(x)|^2 + \frac{m}{2}(h(x))^2 - F(h(x))dx < E_m.$$

Also, using (f3), we have for $t \in (0, 1)$,

$$\begin{aligned} L_m(\gamma(t)) &= E_m - \int_{\ln t}^{-\ln t} \frac{1}{2}|U'(x)|^2 + \frac{m}{2}(U(x))^2 - F(U(x))dx \\ &< E_m. \end{aligned}$$

Finally, from (22), it follows that

$$\begin{aligned} &L_m(\gamma(t)) \\ &\le E_m + 2\int_{-\ln t + \varepsilon_0}^{0} \frac{1}{2}|h'(x)|^2 + \frac{m}{2}(h(x))^2 - F(h(x))dx \\ &= E_m + 2(\ln t - \varepsilon_0)\Big(\frac{m}{2}(U(0) + \varepsilon_0)^2 - F(U(0) + \varepsilon_0)\Big) \to -\infty \end{aligned}$$

as $t \to \infty$. Thus, for any large $T > 0$, the path $\gamma : [0, T] \to H^1(\mathbf{R})$ satisfies (i)-(iii) with $T_0 = 1$ and for any $\delta > 0$.

We now deal with the case $N = 2$. Here we use an idea developed in [22]. However for the property (ii) to hold we need to construct a path which is slightly different from the one defined in [22]. We use the notation: $h(t) = -mt + f(t)$, $H(t) = -\frac{m}{2}t^2 + F(t)$. For a fixed $U \in S_m$, we define $g(\theta, s) : (0, \infty) \times (0, \infty) \to \mathbf{R}$ by

$$g(\theta, s) = L_m(\theta U(\cdot/s)) = \frac{\theta^2}{2}||\nabla U||_{L^2}^2 - s^2 \int_{\mathbf{R}^2} H(\theta u)\, dx.$$

We have

$$\begin{aligned}
g_\theta(\theta,s) &= \theta||\nabla U||_{L^2}^2 - s^2\int_{\mathbf{R}^2} h(\theta U)U\,dx,\\
g_s(\theta,s) &= -2s\int_{\mathbf{R}^2} H(\theta U)\,dx,\\
\frac{\partial}{\partial\theta}\int_{\mathbf{R}^2} H(\theta U)\,dx &= \int_{\mathbf{R}^2} h(\theta U)U\,dx.
\end{aligned}$$

By (7) and (18), we have $\int_{\mathbf{R}^2} H(U)\,dx = 0$, $\int_{\mathbf{R}^2} h(U)U\,dx = ||\nabla U||_{L^2}^2 > 0$. Thus there exist constants $0<\theta_1<1<\theta_2$, such that

$$\frac{\partial}{\partial\theta}\int_{\mathbf{R}^2} H(\theta U)\,dx > 0 \quad \text{for } \theta\in(\theta_1,\theta_2). \tag{23}$$

Thus we have

$$\int_{\mathbf{R}^2} H(\theta U)\,dx \begin{cases} < 0 & \text{for } [\theta_1,1),\\ > 0 & \text{for } (1,\theta_2] \end{cases}$$

and

$$g_s(\theta,s)\begin{cases} > 0 & \text{for } \theta\in[\theta_1,1),\ s\in(0,\infty),\\ = 0 & \text{for } \theta=1,\ s\in(0,\infty),\\ < 0 & \text{for } \theta\in(1,\theta_2],\ s\in(0,\infty). \end{cases} \tag{24}$$

Since $g_\theta(1,s) = ||\nabla U||_{L^2}^2 - s^2\int_{\mathbf{R}^2} h(U)U\,dx = (1-s^2)||\nabla U||_{L^2}^2$, for any $s\neq 1$ there exists $\theta_s>0$ such that

$$g_\theta(\theta,s)\begin{cases} > 0 & \text{for } s\in(0,1),\ \theta\in[1-\theta_s,1+\theta_s],\\ < 0 & \text{for } s\in(1,\infty),\ \theta\in[1-\theta_s,1+\theta_s]. \end{cases} \tag{25}$$

We can also find a small $s_0\in(0,1)$ such that

$$g_\theta(\theta,s) = \theta\left(||\nabla U||_{L^2}^2 - s^2\int_{\mathbf{R}^2}\frac{h(\theta U)}{\theta U}U^2\,dx\right) > 0 \quad \text{for } s\in[0,s_0],\ \theta\in[0,1]. \tag{26}$$

For a fixed small $\varepsilon>0$ to be precise later let $\zeta(t) = (\theta(t),s(t)) : [0,\infty)\to\mathbf{R}^2_{(\theta,s)}$ be a piece-wise linear curve joining

$$\begin{aligned}
&(0,s_0)\to(1-\theta_0,s_0)\to(1-\theta_0,1-\varepsilon)\\
\to\ &(1,1-\varepsilon)\to(1,1)\to(1,1+\varepsilon)\\
\to\ &(1+\theta_0,1+\varepsilon)\to(1+\theta_0,\infty).
\end{aligned}$$

Here θ_0 is chosen such that $1-\theta_0 \in [\theta_1, 1)$ and $1+\theta_0 \in (1, \theta_2]$. We remark that each segment is horizontal or vertical. Let $0 \equiv t_0 < t_1 < \cdots < t_6 < t_7 \equiv \infty$ be such that for each $i = 0, \cdots, 7$, $\zeta(t_i)$ is an end point of a linear segment of the piece-wise linear curve ζ. We set

$$\hat{\gamma}_\epsilon(t)(x) = \theta(t)U(x/s(t)).$$

Then we see that the function $t \mapsto L_m(\hat{\gamma}_\epsilon(t)) = g(\zeta(t))$ is strictly increasing on $(t_0, t_1), (t_1, t_2), (t_2, t_3)$ by (26), (24), (25), respectively. We also see that the function is constant on $(t_3, t_4), (t_4, t_5)$ by (24), and strictly decreasing on $(t_5, t_6), (t_6, t_7)$ by (25), (24), respectively. Lastly, we note that $L_m(\hat{\gamma}_\varepsilon(t)) = \frac{1+\theta_\varepsilon}{2}||\nabla U||_{L^2}^2 - s(t)^2 \int_{\mathbf{R}^2} H((1+\theta_\varepsilon)U)\, dx \to -\infty$ as $t \to \infty$.

Thus for a given $\delta > 0$ choosing $\varepsilon = \varepsilon(\delta) > 0$ so that

$$||U(x/s) - U(x)|| < \delta \quad \text{for } |s| \leq \varepsilon,$$

we see $\gamma^\delta(t) = \hat{\gamma}_{\varepsilon(\delta)}(t) : [0, T] \to H^1(\mathbf{R}^2)$ satisfies the properties (i)–(iii). This ends the proof of Proposition 2. □

3 Proof of Theorem 1.

The variational framework follows the one of [6]. Let $\widetilde{m} > 0$ be a number such that

$$\widetilde{m} < \min\{m, \liminf_{|x|\to\infty} V(x)\} \tag{27}$$

and we define $\tilde{V}_\varepsilon(x) \equiv \max\{\widetilde{m}, V_\varepsilon(x)\}$. Let H_ε be the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon = \Big(\int_{\mathbf{R}^N} |\nabla u|^2 + \tilde{V}_\varepsilon u^2 dx\Big)^{1/2}.$$

We also denote by $||\cdot||_\varepsilon^*$ the corresponding dual norm on H_ε^*, that is,

$$||f||_\varepsilon^* = \sup_{||\varphi||_\varepsilon \leq 1, \varphi \in H_\varepsilon} \Big|\langle f, \varphi\rangle\Big| \quad \text{for } f \in H_\varepsilon^*.$$

We define a norm $\|\cdot\|$ on $H^1(\mathbf{R}^N)$ by

$$\|u\| = \Big(\int_{\mathbf{R}^N} |\nabla u|^2 + \widetilde{m} u^2 dx\Big)^{1/2}.$$

We clearly have $H_\varepsilon \subset H^1(\mathbf{R}^N)$. From now on, for any set $B \subset \mathbf{R}^N$ and $\varepsilon > 0$, we define $B_\varepsilon \equiv \{x \in \mathbf{R}^N \mid \varepsilon x \in B\}$. For $u \in H_\varepsilon$, let

$$P_\varepsilon(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + V_\varepsilon u^2 dx - \int_{\mathbf{R}^N} F(u)dx. \tag{28}$$

Since we are concerned with positive solutions, we may assume without loss of generality that $f(t) = 0$ for all $t \le 0$. For $\nu > 0$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in O_\varepsilon \\ \varepsilon^{-\nu} & \text{if } x \notin O_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(u) = \Big(\int_{\mathbf{R}^N} \chi_\varepsilon u^2 dx - 1\Big)_+^2. \tag{29}$$

We take $\nu = 6/\mu$ if $\mathcal{Z} \neq \emptyset$, and any $\nu > 0$ if $\mathcal{Z} = \emptyset$. The functional Q_ε will act as a penalization to force the concentration phenomena to occur inside O. This type of penalization was first introduced in [9]. Finally let $\Gamma_\varepsilon : H_\varepsilon \to \mathbf{R}$ be given by

$$\Gamma_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u). \tag{30}$$

It is standard to see that $\Gamma_\varepsilon \in C^1(H_\varepsilon)$. Clearly a critical point of P_ε corresponds to a solution of (6). To find solutions of (6) which *concentrate* in O as $\varepsilon \to 0$, we shall search critical points of Γ_ε for which Q_ε is zero. As we shall see the functional Γ_ε enjoys a mountain pass geometry for any $\varepsilon > 0$ small.

For any set $B \subset \mathbf{R}^N$ and $\delta > 0$, we define $B^\delta \equiv \{x \in \mathbf{R}^N | \text{dist}(x, B) \le \delta\}$. Let $10\beta = \text{dist}(\mathcal{M}, \mathbf{R}^N \setminus O)$ and fix a cutoff function $\varphi \in C_0^\infty(\mathbf{R}^N)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| \le \beta$ and $\varphi(x) = 0$ for $|x| \ge 2\beta$. Also, define $\varphi_\varepsilon(y) = \varphi(\varepsilon y), y \in \mathbf{R}^N$.

Without loss of generality we may assume that $0 \in \mathcal{M}$. We shall find a solution near the set

$$X_\varepsilon = \{\varphi_\varepsilon(y - \frac{x}{\varepsilon})U(y - \frac{x}{\varepsilon}) \mid x \in \mathcal{M}^\beta, U \in S_m\}$$

for sufficiently small $\varepsilon > 0$. For the curve γ^δ constructed in Proposition 2, we define

$$\gamma_\varepsilon^\delta(t)(x) = \varphi_\varepsilon(x)\gamma^\delta(t)(x). \tag{31}$$

We see that $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) = P_\varepsilon(\gamma_\varepsilon^\delta(t))$ for $t \in [0,T]$ and $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(0)) = 0$. Finally we define

$$C_\varepsilon = \inf_{\gamma\in\Phi_\varepsilon} \max_{s\in[0,1]} \Gamma_\varepsilon(\gamma(s)) \tag{32}$$

where $\Phi_\varepsilon = \{\gamma \in C([0,1], H_\varepsilon) \,|\gamma(0) = 0, \gamma(1) = \gamma_\varepsilon^\delta(T)\}$. We easily check that $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(T)) < 0$ for any sufficiently small $\varepsilon > 0$.

Proposition 3

$$\limsup_{\varepsilon\to 0} C_\varepsilon \le E_m.$$

Proof. Obviously, we see that

$$C_\varepsilon \le \max_{s\in[0,T]} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)).$$

Note that V_ε converges uniformly to m on each bounded set. Thus, from the properties (ii), (iii) of γ^δ in Proposition 2, we see that

$$\lim_{\varepsilon\to 0} \max_{s\in[0,T]} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)) \le E_m.$$

This completes the proof. □

Proposition 4

$$\liminf_{\varepsilon\to 0} C_\varepsilon \ge E_m.$$

Proof. This was proved in [6, Proposition 3]. In fact, the proof in [6] does not depend on the space dimension and holds also for the case $V_0 \equiv \inf_{x\in\mathbf{R}^N} V(x) = 0$. □

We denote

$$D_\varepsilon^\delta = \max_{s\in[0,T]} \Gamma_\varepsilon(\gamma_\varepsilon^\delta(s)). \tag{33}$$

By the argument in Propositions 3 and 4, we have

$$C_\varepsilon \le D_\varepsilon^\delta \quad \text{and} \quad \lim_{\varepsilon\to 0} C_\varepsilon = \lim_{\varepsilon\to 0} D_\varepsilon^\delta = E_m. \tag{34}$$

Now we define

$$\Gamma_\varepsilon^\alpha = \{u \in H_\varepsilon \mid \Gamma_\varepsilon(u) \le \alpha\} \quad \text{for } \alpha \in \mathbf{R}$$

and

$$X_\varepsilon^d = \{u \in H_\varepsilon | \inf_{v\in X_\varepsilon} ||u - v||_\varepsilon \le d\} \quad \text{for } d > 0.$$

Proposition 5 *There exists a small* $d_0 > 0$ *such that for any* $\{\varepsilon_i\}$ *and* $\{u_{\varepsilon_i}\}$ *satisfying*

$$\begin{aligned} u_{\varepsilon_i} &\in X_{\varepsilon_i}^{d_0}, \\ \lim_{i\to\infty} \varepsilon_i &= 0, \\ \lim_{i\to\infty} \Gamma_{\varepsilon_i}(u_{\varepsilon_i}) &\leq E_m, \\ \lim_{i\to\infty} ||\Gamma'_{\varepsilon_i}(u_{\varepsilon_i})||_{\varepsilon_i}^* &= 0, \end{aligned}$$

there exists, up to a subsequence, $\{y_i\} \subset \mathbf{R}^N$, $x \in \mathcal{M}$, $U \in S_m$ *such that*

$$\lim_{i\to\infty} |\varepsilon_i y_i - x| = 0 \ \textit{and} \ \lim_{i\to\infty} \|u_{\varepsilon_i} - \varphi_{\varepsilon_i}(\cdot - y_i)U(\cdot - y_i)\|_{\varepsilon_i} = 0.$$

Proof. For notational convenience, we write ε for ε_i. Assume that $\{\varepsilon_i\}$ and $\{u_{\varepsilon_i}\} \subset X_{\varepsilon_i}^{d_0}$ satisfy the conditions in the statement of Proposition 5 (we will choose $d_0 > 0$ later sufficiently small).

By compactness of S_m and $\mathcal{M}^\beta$, there exist $Z \in S_m$ and $x \in \mathcal{M}^\beta$ such that

$$\|u_\varepsilon - \varphi_\varepsilon(\cdot - x/\varepsilon)Z(\cdot - x/\varepsilon)\|_\varepsilon \leq 2d_0 \tag{35}$$

for small $\varepsilon > 0$. The proof of Proposition 5 consists of several steps.

Step 1: For any $R > 0$ *we have*

$$\lim_{\varepsilon\to 0} \sup_{z\in A(\frac{x}{\varepsilon};\frac{\beta}{2\varepsilon},\frac{3\beta}{\varepsilon})} \int_{B(z,R)} |u_\varepsilon|^2 \, dy = 0.$$

Here we use the notation:

$$A(x; r_1, r_2) = \{y \in \mathbf{R}^N \,|\, r_1 \leq |y - x| \leq r_2\} \quad \textit{for } x \in \mathbf{R}^N \textit{ and } 0 < r_1 < r_2.$$

Indeed, suppose by contradiction that there exist $R > 0$ and a sequence $\{x_\varepsilon\} \subset A(\frac{x}{\varepsilon}; \frac{\beta}{2\varepsilon}, \frac{3\beta}{\varepsilon})$ satisfying $\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} |u_\varepsilon|^2 dy > 0$. Taking a subsequence, we can assume that $\varepsilon x_\varepsilon \to x_0$ with $x_0 \in A(x; \frac{\beta}{2}, 3\beta)$ and that $u_\varepsilon(\cdot + x_\varepsilon) \to \tilde{W}$ weakly in $H^1(\mathbf{R}^N)$ for some $\tilde{W} \in H^1(\mathbf{R}^N) \setminus \{0\}$. Moreover $\tilde{W}$ satisfies

$$\Delta\tilde{W} - V(x_0)\tilde{W} + f(\tilde{W}) = 0 \ \text{ for } \ y \in \mathbf{R}^N.$$

By definition, $L_{V(x_0)}(\tilde{W}) \geq E_{V(x_0)}$. Also, for large $R > 0$

$$\liminf_{\varepsilon\to 0} \int_{B(x_\varepsilon,R)} |\nabla u_\varepsilon|^2 dy \geq \frac{1}{2} \int_{\mathbf{R}^N} |\nabla\tilde{W}|^2 dy. \tag{36}$$

Now, recalling from [22] that $E_a > E_b$ if $a > b$, we see that $E_{V(x_0)} \geq E_m$, since $V(x_0) \geq m$. Also, from (8), (9) we see that $\int_{\mathbf{R}^N} |\nabla \tilde{W}|^2 dy = N L_{V(x_0)}(\tilde{W})$. Thus we get that

$$\liminf_{\varepsilon \to 0} \int_{B(x_\varepsilon, R)} |\nabla u_\varepsilon|^2 dy \geq \frac{N}{2} L_{V(x_0)}(\tilde{W}) \geq \frac{N}{2} E_m > 0.$$

Then, taking $d_0 > 0$ sufficiently small, we get a contradiction with (35).

Step 2: Let $u_\varepsilon^1 = \varphi_\varepsilon(\cdot - x/\varepsilon) u_\varepsilon$ *and* $u_\varepsilon^2 = u_\varepsilon - u_\varepsilon^1$. *Then*

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1). \tag{37}$$

We have

$$\begin{aligned} &\Gamma_\varepsilon(u_\varepsilon) \\ &= \quad \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + \int_{\mathbf{R}^N} \varphi_\varepsilon(1-\varphi_\varepsilon)|\nabla u_\varepsilon|^2 + V_\varepsilon \varphi_\varepsilon(1-\varphi_\varepsilon)|u_\varepsilon|^2 \, dy \\ &\qquad - \int_{\mathbf{R}^N} F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2) \, dy + o(1) \\ &\geq \quad \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) - \int_{\mathbf{R}^N} F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2) \, dy + o(1). \end{aligned}$$

Now

$$\left| \int_{\mathbf{R}^N} F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2) \, dy \right| \leq \int_{A(\frac{x}{\varepsilon}; \frac{\beta}{\varepsilon}, \frac{2\beta}{\varepsilon})} |F(u_\varepsilon)| + |F(u_\varepsilon^1)| + |F(u_\varepsilon^2)| \, dy.$$

We choose a cutoff function $\psi(x) \in C^\infty(\mathbf{R}^N, \mathbf{R})$ such that $\psi(x) = 1$ for $\beta \leq |x| \leq 2\beta$ and $\psi(x) = 0$ for $|x| \geq 3\beta$, $|x| \leq \beta/2$. Setting $w_\varepsilon(y) = \psi(\varepsilon y - x) u_\varepsilon(y)$ and applying to w_ε Lemma 1 when $N = 2$ and Remark 1 (i) when $N = 1$, it follows from Step 1 that

$$\int_{A(\frac{x}{\varepsilon}; \frac{\beta}{\varepsilon}, \frac{2\beta}{\varepsilon})} |F(u_\varepsilon)| \, dy \leq \int_{\mathbf{R}^N} |F(w_\varepsilon)| \, dy \to 0.$$

In a similar way, it follows that

$$\int_{A(\frac{x}{\varepsilon}; \frac{\beta}{\varepsilon}, \frac{2\beta}{\varepsilon})} |F(u_\varepsilon^1)| \, dy \to 0 \quad \text{and} \quad \int_{A(\frac{x}{\varepsilon}; \frac{\beta}{\varepsilon}, \frac{2\beta}{\varepsilon})} |F(u_\varepsilon^2)| \, dy \to 0.$$

Thus (37) is established.

Step 3: For small $d_0 > 0$,

$$\Gamma_\varepsilon(u_\varepsilon^2) \geq \frac{1}{4}||u_\varepsilon^2||_\varepsilon^2 + o(1).$$

We have

$$\begin{aligned}
\Gamma_\varepsilon(u_\varepsilon^2) &\geq P_\varepsilon(u_\varepsilon^2) \\
&= \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_\varepsilon^2|^2 + \tilde{V}_\varepsilon |u_\varepsilon^2|^2\, dy - \frac{1}{2}\int_{\mathbf{R}^N} (\tilde{V}_\varepsilon - V_\varepsilon)|u_\varepsilon^2|^2\, dy \\
&\quad - \int_{\mathbf{R}^N} F(u_\varepsilon^2)\, dy \\
&\geq \frac{1}{2}||u_\varepsilon^2||_\varepsilon^2 - \frac{m}{2}\int_{\mathbf{R}^N\setminus O_\varepsilon} |u_\varepsilon^2|^2\, dy - \int_{\mathbf{R}^N} F(u_\varepsilon^2)\, dy. \qquad (38)
\end{aligned}$$

Here we use the fact that $\tilde{V}_\varepsilon - V_\varepsilon = 0$ on O_ε and $|\tilde{V}_\varepsilon - V_\varepsilon| \leq \widetilde{m}$ on $\mathbf{R}^N \setminus O_\varepsilon$. Note that P_ε is uniformly bounded in $X_\varepsilon^{d_0}$. Thus so is Q_ε, which implies that

$$\int_{\mathbf{R}^N\setminus O_\varepsilon} |u_\varepsilon^2|^2\, dy \leq C\varepsilon^\nu. \qquad (39)$$

Now, by Remark 1 (ii), we know that there exists $C_\rho > 0$ satisfying $C_\rho \to 0$ as $\rho \to 0$ such that

$$\int_{\mathbf{R}^N} F(u)\, dy \leq C_\rho ||u||^2 \leq C_\rho ||u||_\varepsilon^2 \quad \text{for all } ||u|| \leq \rho.$$

Also it follows from $u_\varepsilon \in X_\varepsilon^{d_0}$ that

$$||u_\varepsilon^2||_\varepsilon \leq 2d_0 \quad \text{for } \varepsilon > 0 \text{ small}.$$

Thus, choosing $d_0 > 0$ small, we have

$$\int_{\mathbf{R}^N} F(u_\varepsilon^2)\, dy \leq \frac{1}{4}||u_\varepsilon^2||_\varepsilon^2 \quad \text{for small } \varepsilon > 0. \qquad (40)$$

The conclusion of Step 3 follows from (38)–(40).
Step 4: $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^1) \geq E_m$.
Let $W_\varepsilon(y) = u_\varepsilon^1(y + x/\varepsilon)$. After extracting a subsequence, we may assume that $W_\varepsilon \to W$ weakly in $H^1(\mathbf{R}^N)$ for some $W \in H^1(\mathbf{R}^N) \setminus \{0\}$. Moreover

$$-\Delta W + V(x)W = f(W) \quad \text{in } \mathbf{R}^N.$$

Here we need to consider two cases

Case 1:

$$\lim_{\varepsilon\to 0} \sup_{z\in\mathbf{R}^N} \int_{B(z,1)} |W_\varepsilon(y) - W(y)|^2 \, dy = 0. \tag{41}$$

Case 2:

$$\lim_{\varepsilon\to 0} \sup_{z\in\mathbf{R}^N} \int_{B(z,1)} |W_\varepsilon(y) - W(y)|^2 \, dy > 0. \tag{42}$$

If Case 1 occurs, we have

$$\int_{\mathbf{R}^N} F(W_\varepsilon)\, dy \to \int_{\mathbf{R}^N} F(W)\, dy. \tag{43}$$

Indeed, we remark that

$$F(t) - F(w) = \int_w^t f(s)\, ds = (t-w) f(\theta t + (1-\theta)w).$$

Setting $g(t) = \max_{s\in[0,t]} |f(s)|$ and $g_\delta(t) = (g(t) - \delta t)_+$ for any given $\delta > 0$, we have

$$\begin{aligned} |F(t) - F(w)| &\le |t-w|(g(t) + g(w)) \\ &\le |t-w|(\delta(|t| + |w|) + g_\delta(t) + g_\delta(w)). \end{aligned}$$

Thus

$$\begin{aligned} & \int_{\mathbf{R}^N} |F(W_\varepsilon) - F(W)|\, dy \\ \le\ & \int_{\mathbf{R}^N} |W_\varepsilon - W|(\delta(|W_\varepsilon| + |W|) + g_\delta(u_\varepsilon^1) + g_\delta(W))\, dy \\ \le\ & \delta ||W_\varepsilon - W||_{L^2}(||W_\varepsilon||_{L^2} + ||W||_{L^2}) \\ & + ||W_\varepsilon - W||_{L^4} \left(\left(\int_{\mathbf{R}^N} g_\delta(W_\varepsilon)^{4/3}\, dy \right)^{3/4} + \left(\int_{\mathbf{R}^N} g_\delta(W)^{4/3}\, dy \right)^{3/4} \right). \end{aligned}$$

Now (41) implies $||W_\varepsilon - W||_{L^4} \to 0$ and $g_\delta(t)^{4/3}$ satisfies (10) when $N = 1$ and (10)–(11) when $N = 2$. Thus,

$$\limsup_{\varepsilon\to 0} \int_{\mathbf{R}^N} |F(W_\varepsilon) - F(W)|\, dy \le \delta \limsup_{\varepsilon\to 0} ||W_\varepsilon - W||_{L^2}(||W_\varepsilon||_{L^2} + ||W||_{L^2})$$

and since $\delta > 0$ is arbitrary, (43) hold. Now by (43), we have for any $R > 0$

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^1) \\
&\geq \liminf_{\varepsilon\to 0} P_\varepsilon(u_\varepsilon^1) \\
&\geq \liminf_{\varepsilon\to 0} \Big[\frac{1}{2}\int_{|y|\leq R} |\nabla W_\varepsilon|^2 + V(\varepsilon y + x)W_\varepsilon^2\, dy - \int_{\mathbf{R}^N} F(W_\varepsilon)\, dy\Big] \\
&\geq \frac{1}{2}\int_{|y|\leq R} |\nabla W|^2 + V(x)W^2\, dy - \int_{\mathbf{R}^N} F(W)\, dy.
\end{aligned}
$$

Thus since $R > 0$ is arbitrary, we have

$$
\liminf_{\varepsilon\to 0} \Gamma_\varepsilon(u_\varepsilon^1) \geq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla W|^2 + V(x)W^2\, dy - \int_{\mathbf{R}^N} F(W)\, dy \geq E_m. \tag{44}
$$

Next we show that Case 2 does not take place. Arguing indirectly, we assume that there exists $\{z_\varepsilon\} \subset \mathbf{R}^N$ such that

$$
\lim_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} |W_\varepsilon(y) - W(y)|^2\, dy > 0.
$$

Since $W_\varepsilon(y) \to W(y)$ weakly in $H^1(\mathbf{R}^N)$, we have

$$
|z_\varepsilon| \to \infty. \tag{45}
$$

Thus we have $\int_{B(z_\varepsilon,1)} |W(y)|^2\, dy \to 0$ and

$$
\lim_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} |u_\varepsilon^1(y + \frac{x}{\varepsilon})|^2\, dy > 0.
$$

Since $u_\varepsilon^1(y + x/\varepsilon) = \varphi_\varepsilon(y)u_\varepsilon(y + x/\varepsilon)$, it is also clear that $|z_\varepsilon| \leq \frac{3\beta}{\varepsilon}$. Thus, by Step 1, we have $|z_\varepsilon| \leq \frac{2}{3}\frac{\beta}{\varepsilon}$ for sufficiently small $\varepsilon > 0$. Extracting a subsequence, we may assume that

$$
\begin{aligned}
\varepsilon z_\varepsilon &\to z_0 \in \overline{B(x, 2\beta/3)} \subset O, \\
u_\varepsilon^1(y + z_\varepsilon + x/\varepsilon) &\to \tilde{W}(y) \not\equiv 0 \quad \text{weakly in } H^1(\mathbf{R}^N).
\end{aligned} \tag{46}
$$

For any $R > 0$ it follows from (46) that $u_\varepsilon^1(y + z_\varepsilon + \frac{x}{\varepsilon}) = u_\varepsilon(y + z_\varepsilon + \frac{x}{\varepsilon})$ in $B(0, R)$ for sufficiently small $\varepsilon > 0$. Thus it follows from $||\Gamma_\varepsilon'(u_\varepsilon)||_\varepsilon^* \to 0$ that

$$
-\Delta\tilde{W} + V(z_0 + x)\tilde{W} = f(\tilde{W}) \quad \text{in } \mathbf{R}^N.
$$

Now there exists $C > 0$ independent of z_0 and $\tilde{W}$ such that $||\tilde{W}|| \geq C$. Thus, because of (45), we get a contradiction with (35) if $d_0 > 0$ is sufficiently small. Thus Case 1 takes place and the conclusion of Step 4 holds.

Step 5: Conclusion

By the assumption $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(u_\varepsilon) \leq E_m$, Steps 2–4 show that $||u_\varepsilon^2||_\varepsilon \to 0$. By the argument in Step 4, in particular (43)–(44), we can see that $u_\varepsilon^1(y + \frac{x}{\varepsilon}) \to W(y)$ strongly in $H^1(\mathbf{R}^N)$. We can also see that $x \in \mathcal{M}$ and $W(y) = U(y - z)$ for some $U \in S_m$ and $z \in \mathbf{R}^N$. Setting $y_\varepsilon = \frac{x}{\varepsilon} + z$, we have $||u_\varepsilon^1 - \varphi(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon)||_\varepsilon \to 0$ and the proof is completed. □

As a corollary to Proposition 5 we have

Proposition 6 *Let $d_0 > 0$ be the number given in Proposition 5. Then for any $d \in (0, d_0)$ there exist $\varepsilon_d > 0$, $\rho_d > 0$ and $\omega_d > 0$ such that*

$$||\Gamma'_\varepsilon(u)||^*_\varepsilon \geq \omega_d$$

for all $\varepsilon \in (0, \varepsilon_d)$ and $u \in \Gamma_\varepsilon^{E_m+\rho_d} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^d)$.

Proof. We argue indirectly and suppose that for some $d \in (0, d_0)$ there exist sequences $\{\varepsilon_n\} \subset (0, 1/n)$ and $\{u_n\} \subset \Gamma_{\varepsilon_n}^{E_m+1/n} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^d)$ such that

$$||\Gamma'_{\varepsilon_n}(u_n)||^*_{\varepsilon_n} < \frac{1}{n}.$$

By Proposition 5, there exists $\{y_n\} \subset \mathbf{R}^N$, $U \in S_m$ and $x \in \mathcal{M}$ such that

$$\varepsilon_n y_n \to x \quad \text{and} \quad ||u_{\varepsilon_n} - \varphi_{\varepsilon_n}(\cdot - y_n)U(\cdot - y_n)||_{\varepsilon_n} \to 0.$$

Thus by the definition of X_{ε_n}, we have $u_{\varepsilon_n} \in X^d_{\varepsilon_n}$ for sufficiently large n, which is a contradition to $u_n \in X_\varepsilon^{d_0} \setminus X_\varepsilon^d$. □

We recall the definition (31) of $\gamma_\varepsilon^\delta(t)$. The following proposition follows from Proposition 2.

Proposition 7 *There exists $M_0 > 0$ independent of $\delta > 0$ with the following property: for any $\delta > 0$ there exist $\alpha_\delta > 0$ and $\bar{\varepsilon}_\delta \in (0, 1]$ such that for $\varepsilon \in (0, \bar{\varepsilon}_\delta]$*

$$\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) \geq E_m - \alpha_\delta \quad \textit{implies} \quad \gamma_\varepsilon^\delta(t) \in X_\varepsilon^{M_0\delta}.$$

Proof. First we remark that there exists $M_0 > 0$ independent of $\varepsilon \in (0,1]$ such that

$$||\varphi_\varepsilon v||_\varepsilon \leq M_0 ||v|| \quad \text{for all } \varepsilon \in (0,1] \text{ and } v \in H_\varepsilon. \tag{47}$$

By Proposition 2, there exists $\alpha_\delta > 0$ such that

$$L_m(\gamma^\delta(t)) \geq E_m - 2\alpha_\delta \quad \text{implies} \quad ||\gamma^\delta(t) - \gamma^\delta(T_0)|| \leq \delta. \tag{48}$$

We also remark that $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) = P_\varepsilon(\gamma_\varepsilon^\delta(t))$ and

$$\sup_{t\in[0,T]} |\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) - L_m(\gamma^\delta(t))| = \sup_{t\in[0,T]} |P_\varepsilon(\gamma_\varepsilon^\delta(t)) - L_m(\gamma^\delta(t))| \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus there exists $\bar{\varepsilon}_\delta > 0$ such that

$$\sup_{t\in[0,T]} |\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) - L_m(\gamma^\delta(t))| \leq \alpha_\delta \quad \text{for } \varepsilon \in (0, \bar{\varepsilon}_\delta]. \tag{49}$$

For $\varepsilon \in (0, \bar{\varepsilon}_\delta]$, by (49), $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) \geq E_m - \alpha_\delta$ implies

$$L_m(\gamma^\delta(t)) \geq \Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) - |\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) - L_m(\gamma^\delta(t))| \geq E_m - 2\alpha_\delta$$

and thus, by (48), we have $||\gamma^\delta(t) - \gamma^\delta(T_0)|| \leq \delta$. Therefore by (47),

$$\begin{aligned} ||\gamma_\varepsilon^\delta(t) - \varphi_\varepsilon \gamma^\delta(T_0)||_\varepsilon &= ||\varphi_\varepsilon(\gamma^\delta(t) - \gamma^\delta(T_0))||_\varepsilon \leq M_0 ||\gamma^\delta(t) - \gamma^\delta(T_0)|| \\ &\leq M_0 \delta. \end{aligned}$$

Recording that, $\gamma^\delta(T_0) \in S_m$, we have $\gamma_\varepsilon^\delta(t) \in X_\varepsilon^{M_0\delta}$. Thus $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) \geq E_m - \alpha_\delta$ implies $\gamma_\varepsilon^\delta(t) \in X_\varepsilon^{M_0\delta}$ and this completes the proof. □

Now we take $d_1 \in (0, \frac{1}{3}d_0)$ such that

$$L_{\widetilde{m}}(u) \geq 0 \quad \text{for all } ||u|| \leq 3d_1. \tag{50}$$

By Proposition 6, there exist numbers ε_1, ρ_1, $\omega_1 > 0$ such that

$$\inf_{u \in \Gamma_\varepsilon^{E_m+\rho_1} \cap (X_\varepsilon^{d_0} \setminus X_\varepsilon^{d_1})} ||\Gamma_\varepsilon'(u)||_\varepsilon^* \geq \omega_1 \quad \text{for } \varepsilon \in (0, \varepsilon_1).$$

Set $\delta_1 = d_1/M_0$ and let $D_\varepsilon^{\delta_1}$ be the number defined in (33). We have the following

Proposition 8 *For sufficiently small $\varepsilon > 0$,*

$$\inf_{u \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}} ||\Gamma'_\varepsilon(u)||_\varepsilon^* = 0.$$

Proof. By Proposition 7, there exists $\alpha_{d_1} > 0$ such that for small $\varepsilon > 0$

$$\Gamma_\varepsilon(\gamma_\varepsilon^{\delta_1}(t)) \geq E_m - \alpha_{d_1} \quad \text{implies} \quad \gamma_\varepsilon^{\delta_1}(t) \in X_\varepsilon^{M_0\delta_1} \subset X_\varepsilon^{d_1}. \tag{51}$$

By (34) for small $\varepsilon > 0$

$$D_\varepsilon^{\delta_1} \leq E_m + \min\{\rho_1, \frac{1}{12}\omega_1 d_0\}, \tag{52}$$

$$C_\varepsilon \geq E_m - \frac{1}{2}\min\{\alpha_{\delta_1}, \frac{1}{12}\omega_1 d_0\}. \tag{53}$$

Here C_ε is the minimax value given in (32).

Arguing indirectly, we assume that

$$a(\varepsilon) \equiv \inf_{u \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}} ||\Gamma'_\varepsilon(u)||_\varepsilon > 0.$$

Then, we can construct a deformation flow $\eta : [0, \infty) \times \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \to \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}}$ such that

(i) $\eta(s, u) = u$ if $s = 0$ or $u \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \setminus X_\varepsilon^{d_0}$.

(ii) $||\frac{d}{ds}\eta(s, u)||_\varepsilon \leq 1$ for all (s, u).

(iii) $\frac{d}{ds}(\Gamma_\varepsilon(\eta(s, u))) \leq 0$ for all (s, u).

(iv) $\frac{d}{ds}(\Gamma_\varepsilon(\eta(s, u))) \leq -\frac{1}{2}\omega_1$ if $\eta(s, u) \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap (X_\varepsilon^{\frac{2}{3}d_0} \setminus X_\varepsilon^{d_1})$.

(v) $\frac{d}{ds}(\Gamma_\varepsilon(\eta(s, u))) \leq -\frac{1}{2}a(\varepsilon)$ if $\eta(s, u) \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}$.

We can observe from (i)–(v) that if $u \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}$ satisfies $\eta(s_0, u) \notin X_\varepsilon^{\frac{2}{3}d_0}$ for some $s_0 > 0$, then there exists an interval $[s_1, s_2] \subset [0, s_0]$ such that

$$\eta(s, u) \in X_\varepsilon^{\frac{2}{3}d_0} \setminus X_\varepsilon^{d_1} \quad \text{for } s \in [s_1, s_2],$$

$$|s_2 - s_1| \geq \frac{2}{3}d_0 - d_1 \geq \frac{1}{3}d_0.$$

Thus

$$\Gamma_\varepsilon(\eta(s_0,u)) \le \Gamma_\varepsilon(u) - \frac{1}{2}\omega_1(s_2 - s_1) \le \Gamma_\varepsilon(u) - \frac{1}{6}\omega_1 d_0. \tag{54}$$

We define $\tilde{\gamma}(t) = \eta(s, \gamma_\varepsilon^{\delta_1}(t))$ for a large $s > 0$. We can see that

$$\max_{t\in[0,T]} \Gamma_\varepsilon(\tilde{\gamma}(t)) \le E_m - \min\{\alpha_\delta, \frac{1}{12}\omega_1 d_0\}. \tag{55}$$

In fact, if $\Gamma_\varepsilon(\gamma_\varepsilon^{\delta_1}(t)) \le E_m - \alpha_{\delta_1}$, (55) follows from (iii). If $\Gamma_\varepsilon(\gamma_\varepsilon^{\delta_1}(t)) > E_m - \alpha_{\delta_1}$, then by (51), we have $\gamma_\varepsilon^{\delta_1}(t) \in X_\varepsilon^{d_1}$. Here we distinguish two cases:

(a) $\eta(s, \gamma_\varepsilon^{\delta_1}(t)) \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{\frac{2}{3}d_0}$ for all $s \in [0, \infty)$.

(b) $\eta(s_0, \gamma_\varepsilon^{\delta_1}(t)) \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \setminus X_\varepsilon^{\frac{2}{3}d_0}$ for some $s_0 > 0$.

If (a) occurs, we see that $\Gamma(\tilde{\gamma}(t)) \le \Gamma_\varepsilon(\gamma_\varepsilon^{\delta_1}(t)) - \frac{1}{2}\min\{a(\varepsilon), \omega_1\}s$ and we have (55) for large $s > 0$. If (b) occurs, by (54) and (52) we have

$$\Gamma_\varepsilon(\tilde{\gamma}(t)) \le \Gamma_\varepsilon(\gamma_\varepsilon^{\delta_1}(t)) - \frac{1}{6}\omega_1 d_0 \le E_m - \frac{1}{12}\omega_1 d_0,$$

that is, (55) holds. Since $\tilde{\gamma} \in \Phi_\varepsilon$, we have

$$C_\varepsilon \le \max \Gamma_\varepsilon(\tilde{\gamma}(t)) \le E_m - \min\{\alpha_{\delta_1}, \frac{1}{12}\omega_1 d_0\},$$

which is in contradition to (53). This completes the proof. □

Finally we have the following proposition.

Proposition 9 *For sufficiently small* $\varepsilon > 0$, $\Gamma_\varepsilon(u)$ *has a critical point in* $\Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}$.

Proof. By Proposition 8, for small $\varepsilon > 0$ there exists a sequence $\{u_n\} \subset \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}$ such that $||\Gamma'_\varepsilon(u_n)||_\varepsilon^* \to 0$.

Since $X_\varepsilon^{d_1}$ is bounded in H_ε, we can extract a subsequence — still denoted u_n — such that $u_n \to u_0$ weakly in H_ε. In a standard way, we see that u_0 is a critical point of Γ_ε. Now we write $u_n = v_n + w_n$ with $v_n \in X_\varepsilon$ and $\|w_n\|_\varepsilon \le d_1$. Since X_ε is compact, after extracting a subsequence if necessary, there exist

$v_0 \in X_\varepsilon$ and $w_0 \in H_\varepsilon$ such that $v_n \to v_0$ strongly in H_ε and $w_n \to w_0$ weakly in H_ε as $n \to \infty$. Thus, $u_0 = v_0 + w_0$ and

$$\|u_0 - v_0\|_\varepsilon = \|w_0\|_\varepsilon \le \liminf_{n\to\infty} \|w_n\|_\varepsilon \le d_1.$$

This proves that $u_0 \in X_\varepsilon^{d_1}$.

Next we show that $\Gamma_\varepsilon(u_0) \le D_\varepsilon^{\delta_1}$. Writing $u_n = u_0 + \sigma_n$, we have

$$||\sigma_n||_\varepsilon = ||u_n - u_0||_\varepsilon \le ||v_n - v_0||_\varepsilon + ||w_n||_\varepsilon + ||w_0||_\varepsilon \le 2d_1 + o(1) \le 3d_1 \quad (56)$$

for large $n \in \mathbf{N}$. Also $\sigma_n \to 0$ weakly in H_ε and then

$$\int_{\mathbf{R}^N} |\nabla u_n|^2 + V_\varepsilon u_n^2 \, dy - \int_{\mathbf{R}^N} |\nabla u_0|^2 + V_\varepsilon u_0^2 \, dy$$

$$= 2\int_{\mathbf{R}^N} \nabla u_0 \nabla \sigma_n + V_\varepsilon u_0 \sigma_n \, dy \to 0,$$

$$\int_{\mathbf{R}^N} F(u_n)\, dy - \int_{\mathbf{R}^N} F(u_0)\, dy - \int_{\mathbf{R}^N} F(\sigma_n)\, dy \to 0$$

(c.f. the proof of Proposition 2.31 in [12] for example). Thus $P_\varepsilon(u_n) - P_\varepsilon(u_0) - P_\varepsilon(\sigma_n) \to 0$ and since from the weak lower semi-continuity of $v \mapsto ||v||^2_{L^2(\mathbf{R}^N \setminus O_\varepsilon)}; H_\varepsilon \to \mathbf{R}$

$$Q_\varepsilon(u_0) \le \liminf_{n\to\infty} Q_\varepsilon(u_n)$$

we have that

$$\begin{aligned} D_\varepsilon^{\delta_1} &\ge \liminf_{n\to\infty} \Gamma_\varepsilon(u_n) = \liminf_{n\to\infty} (P_\varepsilon(u_n) + Q_\varepsilon(u_n)) \\ &\ge P_\varepsilon(u_0) + \liminf_{n\to\infty} P_\varepsilon(\sigma_n) + Q_\varepsilon(u_0) \\ &= \Gamma_\varepsilon(u_0) + \liminf_{n\to\infty} P_\varepsilon(\sigma_n). \end{aligned} \quad (57)$$

Next we estimate $P_\varepsilon(\sigma_n)$. We have

$$\begin{aligned} P_\varepsilon(\sigma_n) &= \frac{1}{2}||\sigma_n||_\varepsilon^2 - \int_{\mathbf{R}^N} F(\sigma_n)\, dy - \frac{1}{2}\int_{\mathbf{R}^N} (\tilde{V}_\varepsilon - V_\varepsilon)\sigma_n^2 \, dy \\ &\ge L_{\widetilde{m}}(\sigma_n) - \frac{1}{2}\int_{\mathbf{R}^N} (\tilde{V}_\varepsilon - V_\varepsilon)\sigma_n^2 \, dy. \end{aligned}$$

By (50) and (56), we have $\liminf_{n\to\infty} L_{\widetilde{m}}(\sigma_n) \geq 0$. We also observe from (27) that $\tilde{V}_\varepsilon - V_\varepsilon$ has a compact support. Thus from the weak convergence of σ_n in H_ε it follows that $\int_{\mathbf{R}^N}(\tilde{V}_\varepsilon - V_\varepsilon)\sigma_n^2\, dy \to 0$. Therefore we have $\liminf_{n\to\infty} P_\varepsilon(\sigma_n) = 0$, which implies, from (57), that

$$\Gamma_\varepsilon(u_0) \leq D_\varepsilon^{\delta_1}.$$

This completes the proof. □

Completion of the Proof for Theorem 1. We see from Proposition 9 that $\Gamma_\varepsilon(u)$ has a critical point $u_\varepsilon \in \Gamma_\varepsilon^{D_\varepsilon^{\delta_1}} \cap X_\varepsilon^{d_1}$. Since u_ε satisfies

$$\Delta u_\varepsilon - V_\varepsilon u_\varepsilon + f(u_\varepsilon) = 4\Big(\int \chi_\varepsilon u_\varepsilon^2 dx - 1\Big)_+ \chi_\varepsilon u_\varepsilon \ \text{ in } \ \mathbf{R}^N \tag{58}$$

and $f(t) = 0$ for $t \leq 0$, we deduce that $u_\varepsilon > 0$ in $\mathbf{R}^N$. Since $\{\|u_\varepsilon\|_\varepsilon\}$ and $\{Q_\varepsilon(u_\varepsilon)\}$ are bounded, it follows that $\{\|u_\varepsilon\|\}$ is bounded. Then, for $N = 1$, it follows easily that $\{\|u_\varepsilon\|_{L^\infty(\mathbf{R})}\}$ is bounded. For the case $N = 2$, taking a function $\phi \in C_0^\infty(\mathbf{R}^2, [0,1])$ satisfying $\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^\infty} + ||\Delta\phi||_{L^\infty} \leq 1$, we see that

$$\Delta(u_\varepsilon\phi) - V_\varepsilon(u_\varepsilon\phi) \geq -f(u_\varepsilon)\phi + 2\nabla u_\varepsilon \nabla\phi + u_\varepsilon\Delta\phi \equiv g_\varepsilon. \tag{59}$$

From the boundedness of $\{\|u_\varepsilon\|\}$, (f2) and Remark 1 (ii), we deduce that $\{\|f(u_\varepsilon)\|_{L^2}\}$ is bounded. This means that a set $\{\|g_\varepsilon\|_{L^2}\}$ is bounded uniformly for $\phi \in C_0^\infty(\mathbf{R}^2, [0,1])$ satisfying $\|\phi\|_{L^\infty} + \|\nabla\phi\|_{L^\infty} + ||\Delta\phi||_{L^\infty} \leq 1$. Then, since $V_\varepsilon \geq 0$, we deduce from [20, Theorem 8.15-16] that $\{\|u_\varepsilon\|_{L^\infty}\}$ is bounded. Now by Proposition 5, we see that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^N\setminus\mathcal{M}_\varepsilon^{2\beta}} |\nabla u_\varepsilon|^2 + \tilde{V}_\varepsilon(u_\varepsilon)^2 dx = 0,$$

and thus, by elliptic estimates (see [20]), we obtain that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\mathbf{R}^N\setminus\mathcal{M}_\varepsilon^{2\beta})} = 0. \tag{60}$$

This gives the following decay estimate for u_ε on $\mathbf{R}^N \setminus \mathcal{M}_\varepsilon^{2\beta} \cup \mathcal{Z}_\varepsilon^\beta$

$$u_\varepsilon(x) \leq C\exp(-c\,\mathrm{dist}(x, \mathcal{M}_\varepsilon^{2\beta} \cup \mathcal{Z}_\varepsilon^\beta)) \tag{61}$$

for some constants $C, c > 0$. Indeed from (f1) and (60) we see that

$$\lim_{\varepsilon\to 0} ||f(u_\varepsilon)/u_\varepsilon||_{L^\infty(\mathbf{R}^N\setminus \mathcal{M}_\varepsilon^{2\beta}\cup \mathcal{Z}_\varepsilon^{\beta})} = 0.$$

Also $\inf\{V(x)|x \notin \mathcal{M}_\varepsilon^{2\beta} \cup \mathcal{Z}_\varepsilon^{\beta}\} > 0$. Thus, we obtain the decay estimate (60) by applying standard comparison principles (see [31]) to (58). □

If $\mathcal{Z} \neq \emptyset$ we shall need, in addition, an estimate for u_ε on $\mathcal{Z}_\varepsilon^{2\beta}$. Let $\{H_\varepsilon^i\}_{i\in I}$ be the connected components of $\mathrm{int}(\mathcal{Z}_\varepsilon^{3\delta})$ for some index set I. Note that $\mathcal{Z} \subset \cup_{i\in I} H_\varepsilon^i$ and $\mathcal{Z}$ is compact. Thus, the set I is finite. For each $i \in I$, let (ϕ^i, λ_1^i) be a pair of first positive eigenfunction and eigenvalue of $-\Delta$ on H_ε^i with Dirichlet boundary condition. From now we fix an arbitrary $i \in I$. By elliptic estimates [20, Theorem 9.20] and using the fact that $\{Q_\varepsilon(u_\varepsilon)\}$ is bounded we see that for some constant $C > 0$

$$\|u_\varepsilon\|_{L^\infty(H_\varepsilon^i)} \le C\varepsilon^{3/\mu}. \tag{62}$$

Thus, from (f1) we have, for some $C > 0$

$$||f(u_\varepsilon)/u_\varepsilon||_{L^\infty(H_\varepsilon^i)} \le C\varepsilon^3.$$

Denote $\phi_\varepsilon^i(x) = \phi^i(\varepsilon x)$. Then, for sufficiently small $\varepsilon > 0$, we deduce that for $x \in \mathrm{int}(H_\varepsilon^i)$,

$$\Delta\phi_\varepsilon^i(x) - V_\varepsilon(x)\phi_\varepsilon^i(x) + \frac{f(u_\varepsilon(x))}{u_\varepsilon(x)}\phi_\varepsilon^i(x) \le \Big(C\varepsilon^3 - \lambda_1\varepsilon^2\Big)\phi_\varepsilon^i \le 0. \tag{63}$$

Now, since $\mathrm{dist}(\partial\mathcal{Z}_\varepsilon^{2\beta}, \mathcal{Z}_\varepsilon^{\beta}) = \beta/\varepsilon$, we see from (61) that for some constants $C, c > 0$,

$$||u_\varepsilon||_{L^\infty(\partial\mathcal{Z}_\varepsilon^{2\beta})} \le C\exp(-c/\varepsilon). \tag{64}$$

We normalize ϕ^i requiring that

$$\inf\{\phi_\varepsilon^i(x)\,|x \in H_\varepsilon^i \cap \partial\mathcal{Z}_\varepsilon^{2\delta}\} = C\exp(-c/\varepsilon) \tag{65}$$

for the same $C, c > 0$ as in (65). Then, we see that for some $D > 0$,

$$\phi_\varepsilon^i(x) \le DC\exp(-c/\varepsilon), x \in H_\varepsilon^i \cap \mathcal{Z}_\varepsilon^{2\beta}.$$

Now we deduce, using (62), (63), (64), (65) and [33, B.6 Theorem] that for each $i \in I$, $u_\varepsilon \leq \phi_\varepsilon^i$ on $H_\varepsilon^i \cap \mathcal{Z}_\varepsilon^{2\beta}$. Therefore

$$u_\varepsilon(x) \leq C \exp(-c/\varepsilon) \text{ on } \mathcal{Z}_\varepsilon^{2\delta} \tag{66}$$

for some $C, c > 0$. Now (61) and (66) implies that $Q_\varepsilon(u_\varepsilon) = 0$ for $\varepsilon > 0$ sufficiently small and thus u_ε satisfies (6). Finally let x_ε be a maximum point of u_ε. By Propositions 1 and 5, we readily deduce that $\varepsilon x_\varepsilon \to z$ for some $z \in \mathcal{M}$ as $\varepsilon \to 0$, and that for some $C, c > 0$,

$$u_\varepsilon(x) \leq C \exp(-c|x - x_\varepsilon|).$$

This completes the proof. □

Acknowledgements. This work of the first author was supported by the Korea Research Foundation Grant(KRF-2006-013-C00072). The third author is supported in part by Grant-in-Aid for Scientific Research (C)(2)(No. 17540205) of Japan Society for the Promotion of Science.

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