Homoclinic orbits for a non periodic Hamiltonian system

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Abstract

In this paper we prove the existence and multiplicity of homoclinic orbits for first order Hamiltonian systems of the form

$$\dot{z} = \mathcal{J} H_z(t, z)$$

where H_z is asymptotically linear at ∞ and is not assumed to be periodic.

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1 Introduction and main results

Consider the first order Hamiltonian system

$$\text{(HS)} \qquad \dot{z} = \mathcal{J} H_z(t, z),$$

where $z = (p, q) \in \mathbb{R}^{2N}$, $\mathcal{J} := \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. Here $H \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ has the form

$$H(t, z) = \frac{1}{2} L(t) z \cdot z + R(t, z)$$

with $L(t)$ a continuous symmetric $2N \times 2N$-matrix valued function, $R_z(t, z) = o(|z|)$ as $z \to 0$ and asymptotically linear as $|z| \to \infty$. A solution z of (HS) is a homoclinic orbit if $z(t) \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$. In this paper we study the existence and multiplicity of homoclinic orbits *without assuming periodicity conditions*.

In the last years, existence and multiplicity of homoclinic orbits for the first order systems (HS) were studied extensively by means of critical point theory, and many results were obtained under the assumption that $H(t, z)$ *depends periodically on* t and L and R satisfy various hypotheses. In [6] Coti-Zelati, Ekeland and Séré assume that L is constant with 0 a hyperbolic point of the Hamiltonian operator $A := -\left(\mathcal{J} \frac{d}{dt} + L\right)$, $R(t, z)$ strictly convex in z and satisfying the Ambrosetti-Rabinowitz growth condition, that is, there is $\mu > 2$ such that

$$\text{(1.1)} \qquad 0 < \mu R(t, z) \leq R_z(t, z) z \quad \text{whenever } z \neq 0.$$

They prove the existence and multiplicity of homoclinic orbits of (HS). This result was deepened in [16, 17] when Séré established the existence of infinitely many homoclinic orbits. In these papers the convexity condition on R allows the authors to use a Mountain-Pass argument. Independently, Hofer and Wysocki [11], using Fredholm operator theory and a linking argument, and Tanaka [20], passing through a subharmonic approach, managed to remove the convexity assumption to get one homoclinic orbit. Later linking type arguments were used in [7, 9, 2] to show the existence and multiplicity of homoclinic orbits of (HS) when L depend periodically on t and certain symmetries on $R(t, z)$ are assumed for the multiplicity. See also [19] for a periodic setting but with different nonlinearities, in particular asymptotically linear ones.

Without assumptions of periodicity the problem is quite different in nature and there is not much work done so far. For describing our results, we use the $2N \times 2N$ matrix $\mathcal{J}_0 := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, and the notation

$$\tilde{R}(t, z) := \frac{1}{2} R_z(t, z) z - R(t, z).$$

Also given a $2N \times 2N$ matrix M, we say that $M \geq 0$ if and only if

$$\min_{\xi \in \mathbb{R}^{2N}, \, |\xi|=1} M\xi \cdot \xi \geq 0$$

and that $M < 0$ if and only if $M \geq 0$ does not hold. Also letting I_{2N} be the identity matrix in $\mathbb{R}^{2N}$ and $q \in \mathbb{R}$, we denote the matrix qI_{2N} by q.

We make the following assumptions:

(R_0) There is $b > 0$ such that the set $\Lambda^b := \{t \in \mathbb{R} : \ \mathcal{J}_0 L(t) < b\}$ is nonempty and has finite measure;

(R_1) $R(t,z) \geq 0$ and $R_z(t,z) = o(|z|)$ as $z \to 0$ uniformly in t;

(R_2) $R_z(t,z) = M(t)z + r_z(t,z)$, with M a bounded, continuous symmetric $2N \times 2N$-matrix valued function and $r_z(t,z) = o(|z|)$ uniformly in t as $|z| \to \infty$;

(R_3) $m_0 := \inf_{t\in\mathbb{R}} \left[\inf_{(\xi\in\mathbb{R}^{2N}, \, |\xi|=1)} M(t)\xi \cdot \xi\right] > \inf \sigma(A) \cap (0, \infty)$;

(R_4) either (i) $0 \notin \sigma(A - M)$ or (ii) $\tilde{R}(t,z) \geq 0$ for all (t,z) and $\tilde{R}(t,z) \geq \delta_0$ for some $\delta_0 > 0$ and all (t,z) with $|z|$ large enough;

(R_5) $\gamma < b_{\max}$, where $\gamma := \sup_{|t|\geq t_0, z\neq 0} |R_z(t,z)|/|z|$ for some $t_0 \geq 0$, and $b_{\max} := \sup\{b : \ |\Lambda^b| < \infty\}$.

We will show that the set $\sigma(A) \cap (0, b_{\max})$ consists only of eigenvalues of finite multiplicity. From the definitions of m_0 and γ we have $m_0 < \gamma < b_{max}$. Let ℓ denote the number of eigenfunctions with corresponding eigenvalues lying in $(0, m_0)$.

Theorem 1.1. *Let* $(R_0) - (R_5)$ *be satisfied. Then* (HS) *has at least one homoclinic orbit. If in addition* $R(t,z)$ *is even in* z*, then* (HS) *has at least* ℓ *pairs of homoclinic orbits.*

In the works where $H(t,z)$ is periodic, the periodicity is used to control the lack of compactness due to the fact that (HS) is set on all $\mathbb{R}$. In our situation we manage to recover sufficient compactness by imposing a control on the *size* of $R(t,z)$ with respect to the behavior of $L(t)$ at infinity in t, see condition (R_5). For related arguments we refer to [8, 13, 18].

We now give some examples.

Remark 1.2. Let $q \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfy

(q_0) There is $b > 0$ such that $0 < |Q^b| < \infty$ where $Q^b := \{t \in \mathbb{R} : \ q(t) < b\}$.

Then $L(t) = q(t)\mathcal{J}_0$ satisfies (R_0).

Remark 1.3. The following functions satisfy $(R_2) - (R_4)$ provided that $\inf a(t) > 0$:

*Ex*1. $R(t,z) := a(t)|z|^2 \left(1 - \frac{1}{\ln(e+|z|)}\right)$.

*Ex*2. $R_z(t,z) = h(t,|z|)z$, where $h(t,s)$ is increasing for $s \in [0,\infty)$, and $h(t,s) \to 0$ as $s \to 0$, $h(t,s) \to a(t)$ uniformly in t as $s \to \infty$.

Note that in both examples, $m_0 = \inf a(t)$ and $\gamma = \sup_{|t|\geq t_0} a(t)$ for an arbitrarily fixed $t_0 > 0$.

The paper is organized as follows. In Section 2 we first study the spectrum of the operator A showing, thanks to (R_0), that the essential spectrum $\sigma_e(A) \subset \mathbb{R} \setminus (-b_{\max}, b_{\max})$. Based on the description on $\sigma(A)$, we derive a variational setting for (HS) and represent the associated functional in the form $\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}} R(t,z)$ with Φ being defined on the Hilbert space $E = \mathcal{D}(|A|^{1/2}) \hookrightarrow H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$ with decomposition $E = E^- \oplus E^0 \oplus E^+, z = z^- + z^0 + z^+$, $\dim E^{\pm} = \infty$. Our existence and multiplicity result is obtained using some critical point theorems recently developed that we present at the end of the Section. In Section 3 we show the linking structure of Φ, that is, $\inf \Phi(E^+ \cap \partial B_\rho) > 0$ for some $\rho > 0$ and there are finite dimensional subspaces $Y \subset E^+$ such that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ in $E_Y := E^- \oplus E^0 \oplus Y$. In Section 4 we show that the Cerami condition for Φ hold. Because of the lack of (1.1) and since E^0 maybe nontrivial this require some care. Finally, in Section 5, we give the proof of Theorem 1.1.

Notation: Throughout the paper we shall denote by $c > 0$ various positive constants which may vary from lines to lines and are not essential to the problem.

2 Variational setting

In order to establish a variational setting for the system (HS) we study the spectrum of the associated Hamiltonian operator.

Recall that $A = -\left(\mathcal{J}\frac{d}{dt} + L\right)$ is selfadjoint on $L^2(\mathbb{R}, \mathbb{R}^{2N})$ with domain $\mathcal{D}(A) = H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is bounded and $\mathcal{D}(A) \subset H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is unbounded. Let $\sigma(A)$, $\sigma_d(A)$ and $\sigma_e(A)$ denote, respectively, the spectrum, the eigenvalues of finite multiplicity, and the essential spectrum of A. Set

$$\mu_e^- := \sup\left(\sigma_e(A) \cap (-\infty, 0]\right), \quad \mu_e^+ := \inf\left(\sigma_e(A) \cap [0,\infty)\right). \tag{2.1}$$

In what follows by $|\cdot|_q$ we denote the usual L^q-norm, and by $(\cdot,\cdot)_2$ the usual L^2-inner product.

Proposition 2.1. *Assume* (R_0) *is satisfied. Then* $\sigma_e(A) \subset \mathbb{R}\backslash(-b_{\max},\, b_{\max})$*, that is,* $\mu_e^- \leq -b_{\max}$ *and* $\mu_e^+ \geq b_{\max}$*.*

Proof. Let $b > 0$ be such that $|\Lambda^b| < \infty$. Set

$$(\mathcal{J}_0 L(t) - b)^+ := \begin{cases} \mathcal{J}_0 L(t) - b & \text{if } \mathcal{J}_0 L(t) - b \geq 0 \\ 0 & \text{if } \mathcal{J}_0 L(t) - b < 0 \end{cases}$$

and $(\mathcal{J}_0 L(t) - b)^- := (\mathcal{J}_0 L(t) - b) - (\mathcal{J}_0 L(t) - b)^+$. We have, since $\mathcal{J}_0^2 = I$, $A = A_1 - \mathcal{J}_0(\mathcal{J}_0 L(t) - b)^-$ where

$$A_1 = -\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) - b\mathcal{J}_0.$$

Observe that $\mathcal{J}_0\mathcal{J} = -\mathcal{J}\mathcal{J}_0$. Thus, for $z \in \mathcal{D}(A)$,

$$\begin{aligned}
(A_1 z, A_1 z)_2 =& |A_1 z|_2^2 = \left|\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) z + b\mathcal{J}_0 z\right|_2^2 \\
=& \left|\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) z\right|_2^2 + b^2|z|_2^2 \\
& + (\mathcal{J}\dot{z},\, b\mathcal{J}_0 z)_2 + (b\mathcal{J}_0 z,\, \mathcal{J}\dot{z})_2 \\
& + \big(\mathcal{J}_0(\mathcal{J}_0 L - b)^+ z,\, b\mathcal{J}_0 z\big)_2 + \big(b\mathcal{J}_0 z,\, \mathcal{J}_0(\mathcal{J}_0 L - b)^+ z\big)_2 \\
=& \left|\left(\mathcal{J}\frac{d}{dt} + \mathcal{J}_0(\mathcal{J}_0 L - b)^+\right) z\right|_2^2 + b^2|z|_2^2 \\
& + 2b((\mathcal{J}_0 L - b)^+ z,\ z)_2 \\
\geq& b^2|z|_2^2\,.
\end{aligned} \tag{2.2}$$

Here we have used the fact that $(\mathcal{J}\dot{z},\, b\mathcal{J}_0 z)_2 + (b\mathcal{J}_0 z,\, \mathcal{J}\dot{z})_2 = 0$. Indeed for $z = (u, v) \in \mathcal{C}_0^\infty$ one has

$$\begin{aligned}
&(\mathcal{J}\dot{z},\, b\mathcal{J}_0 z)_2 + (b\mathcal{J}_0 z,\, \mathcal{J}\dot{z})_2 \\
&= 2b\int_{\mathbb{R}} (\dot{u}u - \dot{v}v) = b\int_{\mathbb{R}} \frac{d}{dt}(u^2(t) - v^2(t))) \\
&= b\lim_{t\to\infty} (|u(t)|^2 - |u(-t)|^2 - |v(t)|^2 + |v(-t)|^2) = 0.
\end{aligned}$$

Thus, by density, we get the result. Now (2.2) implies that $\sigma(A_1) \subset \mathbb{R} \setminus (-b, b)$.

We claim that $\sigma_e(A) \cap (-b,b) = \emptyset$. Assume by contradiction that there is $\lambda \in \sigma_e(A)$ with $|\lambda| < b$. Let $(z_n) \subset \mathcal{D}(A)$ with $|z_n|_2 = 1$, $z_n \rightharpoonup 0$ in L^2 and $|(A-\lambda)z_n|_2 \to 0$. Then $\|z_n\|_{H^1}$ stays bounded, hence $|\mathcal{J}_0(\mathcal{J}_0 L - b)^- z_n|_2 \to 0$. We get

$$\begin{aligned} o(1) = |(A-\lambda)z_n|_2 &= |A_1 z_n - \lambda z_n - \mathcal{J}_0(\mathcal{J}_0 L - b)^- z_n|_2 \\ &\geq |A_1 z_n|_2 - |\lambda| - o(1) \\ &\geq b - |\lambda| - o(1) \end{aligned}$$

which implies that $0 < b - |\lambda| \leq 0$, a contradiction. Since the claim is true for any $b > 0$ with $|\Lambda^b| < \infty$, one sees that $\sigma_e(A) \subset \mathbb{R} \setminus (-b_{\max},\ b_{\max})$. $\square$

Since 0 may belong to $\sigma(A)$, some care is necessary for getting the suitable variational framework. Observe that $\mathcal{D}(A)$ is a Hilbert space with the graph inner product

$$(z,w)_A := (Az, Aw)_2 + (z,w)_2$$

and the induced norm $|z|_A := (z,z)_A^{1/2}$. Let $(F_\lambda)_{\lambda \in \mathbb{R}}$ denotes the spectral family and $|A|$ the absolute value of A. A has the polar decomposition $A = U|A|$ with $U = 1 - F_0 - F_{-0}$. Proposition 2.1 induces an orthogonal decomposition of $L^2 := L^2(\mathbb{R}, \mathbb{R}^{2N})$

$$L^2 = L^- \oplus L^0 \oplus L^+, \quad z = z^- + z^0 + z^+$$

so that A is negative definite on L^-, positive definite on L^+ and $L^0 = \ker A$. In fact, $L^\pm = \{u \in L^2 : Uu = \pm u\}$ and $L_0 = \{u \in L^2 : Uu = 0\}$ (see Theorem IV, 3.3 in [10]). Note that Proposition 2.1 also implies that $\dim(L^0) < \infty$. Let $P^0 : L^2 \to L^0$ denote the associated projector. Then P^0 commutes with A and $|A|$. On $\mathcal{D}(A)$ we introduce the inner product

$$\langle z,w\rangle_A := (Az, Aw)_2 + (P^0 z, P^0 w)_2 = (|A|z, |A|w)_2 + (P^0 z, w)_2$$

whose induced norm will be denoted by $||z||_A$. Since 0 is at most an isolated eigenvalue of finite multiplicity, it is clear that $|\cdot|_A$ and $||\cdot||_A$ are equivalent norms on $\mathcal{D}(A)$: $d_1|z|_A \leq ||z||_A \leq d_2|z|_A$, for all $z \in \mathcal{D}(A)$. Define

$$\tilde{A} := |A| + P^0.$$

Then $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Noting that $P^0|A| = |A|P^0 = 0$ we have for $z, w \in \mathcal{D}(A)$,

$$\begin{aligned} (\tilde{A}z, \tilde{A}w)_2 &= (|A|z, |A|w)_2 + (|A|z, P^0 w)_2 + (P^0 z, |A|w)_2 + (P^0 z, P^0 w)_2 \\ &= (|A|z, |A|w)_2 + (P^0 z, P^0 w)_2 = \langle z,w\rangle_A, \end{aligned}$$

hence,

$$d_1|z|_A \le ||z||_A = |\tilde{A}z|_2 \le d_2|z|_A, \text{ for all } z \in \mathcal{D}(A). \tag{2.3}$$

Let $E := \mathcal{D}(|A|^{1/2})$ be the domain of the self-adjoint operator $|A|^{1/2}$ which is a Hilbert space equipped with the inner product

$$(z,w) = \big(|A|^{1/2}z,\, |A|^{1/2}w\big)_2 + (P^0 z, P^0 w)_2$$

and the induced norm $\|z\| = (z,z)^{1/2}$. E has the following decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad \text{where } E^\pm = E \cap L^\pm \text{ and } E^0 = L^0$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$ inner products. In fact, the $(\cdot,\cdot)_2$ orthogonality follows from the decomposition of L^2. To show the $(\cdot,\cdot)$ orthogonality, observe that, for $z^\pm \in L^\pm \cap \mathcal{D}(A)$,

$$\begin{aligned}(z^+, z^-) &= (|A|^{1/2}z^+, |A|^{1/2}z^-)_2 = (|A|z^+, z^-)_2 = (|A|Uz^+, z^-)_2 \\ &= (Az^+, z^-)_2 = (z^+, Az^-)_2 = (z^+, |A|Uz^-)_2 = -(z^+, |A|z^-)_2 \\ &= -(|A|^{1/2}z^+, |A|^{1/2}z^-)_2 = -(z^+, z^-),\end{aligned}$$

hence $(z^+, z^-) = 0$. Since $\mathcal{D}(A)$ is dense in E, one sees that E^+ and E^- are orthogonal in $(\cdot,\cdot)$. Similarly one checks that $E^\pm$ are orthogonal to E^0 in $(\cdot,\cdot)$. Observe that for all $z \in \mathcal{D}(A)$ and $w \in \mathcal{D}(|A|^{1/2})$

$$\begin{aligned}(\tilde{A}^{1/2}z, \tilde{A}^{1/2}w)_2 &= (\tilde{A}z, w)_2 = ((|A| + P^0)z, w)_2 = (|A|z, w)_2 + (P^0 z, w)_2 \\ &= (|A|^{1/2}z, |A|^{1/2}w)_2 + (P^0 z, P^0 w)_2 = (z, w).\end{aligned}$$

Consequently, since $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$ is a core of $\tilde{A}^{1/2}$ we have

$$(z,w) = (\tilde{A}^{1/2}z, \tilde{A}^{1/2}w)_2 \text{ for all } z, w \in \mathcal{D}(|A|^{1/2})$$

which implies in particular that

$$\|z\| = |\tilde{A}^{1/2}z|_2 \quad \text{for all } z \in E. \tag{2.4}$$

The self adjoint operator $A_0 = \mathcal{J}\frac{d}{dt} + \mathcal{J}_0$ acts on L^2 with $\mathcal{D}(A_0) = H^1 := H^1(\mathbb{R}, \mathbb{R}^{2N})$. Then $A_0^2 = -\frac{d^2}{dt^2} + 1$ and, letting $|\tilde{A}_0|$ denote the absolute value of A_0, we have for all $z \in H^1$,

$$||A_0|z|_2^2 = |A_0 z|_2^2 = (A_0 z, A_0 z)_2 = (A_0^2 z, z)_2 = ||z||_{H^1}^2. \tag{2.5}$$

Lemma 2.2. *The condition $\mathcal{D}(A) \subset H^1$ implies that*

$$||z||_{H^1} = ||A_0|z|_2 \le d_3|\tilde{A}z|_2 \quad \textit{for all } z \in \mathcal{D}(A). \tag{2.6}$$

Proof. Let A_r be the restriction of A_0 to $\mathcal{D}(A)$. A_r is a linear operator from $\mathcal{D}(A)$ to L^2. We claim that A_r is closed. Indeed, let $z_n \stackrel{|\cdot|_A}{\rightarrow} z$ and $A_r z_n \stackrel{|\cdot|_2}{\rightarrow} w$. Then $z \in \mathcal{D}(A)$, and since A_0 is closed, $A_r z_n = A_0 z_n \rightarrow A_0 z = A_r z$, hence the claim. Now the Closed Graph Theorem implies that $A_r : \mathcal{D}(A) \rightarrow L^2$ is a bounded linear operator, so $|A_0 z|_2 = |A_r z|_2 \leq d_4 |z|_A$ for all $z \in \mathcal{D}(A)$. This together with (2.3) and (2.5), implies (2.6). $\square$

By interpolation theory we have that $H^{1/2} = [H^1, L^2]_{1/2}$ (see Theorem 2.4.1 of [15]). Noting that $\mathcal{D}(|A_0|^0) = L^2$ one has by (2.5),

$$H^{1/2} = [\mathcal{D}(|A_0|), \mathcal{D}(|A_0|^0)]_{1/2}$$

with equivalent norms. It then follows from Theorem 1.18.10 of [15] that

$$H^{1/2} = [\mathcal{D}(|A_0|), \mathcal{D}(|A_0|^0)]_{1/2} = \mathcal{D}(|A_0|^0)]_{1/2},$$

hence $||z||_{H^{1/2}}$ and $||A_0|^{1/2} z|_2$ are equivalent norm on $H^{1/2}$:

$$d_5 ||z||_{H^{1/2}} \leq ||A_0|^{1/2} z|_2 \leq d_6 ||z||_{H^{1/2}} \text{ for all } z \in H^{1/2}. \tag{2.7}$$

Lemma 2.3. *E embeds continuously into $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$, hence, E embeds continuously into L^p for all $p \geq 2$ and compactly into L^p_{loc} for all $p \geq 1$.*

Proof. By (2.6),

$$||A_0| z|_2 \leq d_3 |\tilde{A} z|_2 = |(d_3 \tilde{A}) z|_2$$

for all $z \in \mathcal{D}(A)$. Thus $(|A_0| z, z)_2 \leq (d_3 \tilde{A} z, z)_2$ for all $z \in \mathcal{D}(A)$ (see Proposition III 8.11 of [10]). This implies

$$||A_0|^{1/2} z|_2^2 = (|A_0| z, z)_2 \leq (d_3 \tilde{A} z, z)_2 = d_3 |\tilde{A}^{1/2} z|_2^2$$

for all $z \in \mathcal{D}(A)$ (see Proposition III 8.12 of [10]). Since $\mathcal{D}(A)$ is a core of $\tilde{A}^{1/2}$ we obtain that $||A_0|^{1/2} z|_2^2 \leq d_3 |\tilde{A}^{1/2} z|_2^2$ for all $z \in E$. This, jointly with (2.4), shows that

$$||A_0|^{1/2} z|_2^2 \leq d_3 ||z||^2 \text{ for all } z \in E$$

which, together with (2.7), implies that

$$\|z\|_{H^{1/2}} \leq d_6 \|z\| \quad \text{for all } z \in E$$

ending the proof. $\square$

From now on we fix a number b with

$$\gamma < b < \mathrm{b}_{\max} \tag{2.8}$$

where γ appears in (R_5). Let k be the number of the eigenfunctions with corresponding eigenvalues lying in $[-b, b]$. We write $f_i\,(1 \le i \le k)$ for the eigenfunctions. Setting

$$L^d := \operatorname{span}\{f_1, \cdots, f_k\},$$

we have another orthogonal decomposition

$$L^2 = L^d \oplus L^e, \quad u = u^d + u^e.$$

Correspondingly, E has the decomposition:

$$E = E^d \oplus E^e \ \text{ with } \ E^d = L^d \text{ and } E^e = E \cap L^e, \tag{2.9}$$

orthogonal with respect to both the inner products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$. Remark that by Proposition 2.1

$$b|z|_2^2 \le \|z\|^2 \quad \text{for all } z \in E^e. \tag{2.10}$$

Now, note that, using A, the system (HS) can be rewritten as

$$Az = R_z(t, z). \tag{2.11}$$

On E we define the functional

$$\Phi(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi(z) \ \text{ where } \ \Psi(z) = \int_{\mathbb{R}} R(t, z). \tag{2.12}$$

Our hypotheses on $H(t, z)$ imply that $\Phi \in \mathcal{C}^1(E, \mathbb{R})$ and a standard argument shows that critical points of Φ are homoclinic orbits of (HS) (cf. [9]). We write Φ' for the derivative of Φ.

In order to study the critical points of Φ, we now recall some abstract critical point theory developed recently in [5]; see also [3] and [14] for earlier results on that direction.

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$ and corresponding projections P_X, P_Y onto X, Y, respectively. For a functional $\Phi \in \mathcal{C}^1(E, \mathbb{R})$ we write $\Phi_a = \{z \in E : \Phi(z) \ge a\}$, $\Phi^b = \{z \in E : \Phi(z) \le b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Recall that Φ is said to be weakly sequentially lower semi-continuous if for any $z_n \rightharpoonup z$ in E one has $\Phi(z) \le \liminf_{n\to\infty} \Phi(z_n)$, and Φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty} \Phi'(z_n)w = \Phi'(z)w$ for each $w \in E$. A sequence $(z_n) \subset E$ is said to be a $(C)_c$-sequence if $\Phi(z_n) \to c$ and $(1 + \|z_n\|)\Phi'(z_n) \to 0$. Φ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence.

From now on we assume that X is separable and reflexive, and we fix a countable dense subset $\mathcal{S} \subset X^*$. For each $s \in \mathcal{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(z) = |s(x)| + \|y\| \quad \text{for } z = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the induced topology. Let w^* denote the weak*-topology on E^*.

Suppose:

(Φ_0) For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous.

(Φ_1) For any $c > 0$, there exists $\zeta > 0$ such that $\|z\| < \zeta\|P_Y z\|$ for all $z \in \Phi_c$.

(Φ_2) There exists $\rho > 0$ with $\kappa := \inf \Phi(S_\rho Y) > 0$ where $S_\rho Y := \{z \in Y : \|z\| = \rho\}$.

The following theorem is a special case of the Theorem 4.4 of [5] (see also [4]).

Theorem 2.4. *Let* (Φ_0)–(Φ_2) *be satisfied and suppose there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \le \kappa$ *where* $Q = \{z = x + te : t \ge 0, x \in X, \|z\| < R\}$. *If* Φ *satisfies the* $(C)_c$*-condition for all* $c \le \bar{c} := \sup \Phi(Q)$ *then* Φ *has a critical point* z *with* $\kappa \le \Phi(z) \le \bar{c}$.

For our next result on multiple critical points we assume:

(Φ_3) There is a finite-dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that we have for $E_0 := X \oplus Y_0$ and $B_0 := \{z \in E_0 : \|z\| \le R\}$ that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus B_0) < \inf \Phi(B_\rho \cap Y)$.

A special case of Theorem 4.6 of [5] is

Theorem 2.5. *If* Φ *is even, satisfies* (Φ_0), (Φ_2), (Φ_3) *and the* $(C)_c$ *condition for all* $c \in [\kappa, \bar{c}]$, *then it has at least* $m := \dim Y_0$ *pairs of critical points with critical values less or equal to* $\bar{c}$.

3 Linking structure

We now study the linking structure of Φ. Remark that under $(R_1)-(R_2)$, given $p \geq 2$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|R_z(t,z)| \leq \varepsilon|z| + C_\varepsilon|z|^{p-1} \tag{3.1}$$

and

$$R(t,z) \leq \varepsilon|z|^2 + C_\varepsilon|z|^p \tag{3.2}$$

for all (t,z). First we have the following lemma.

Lemma 3.1. *Let $(R_0)-(R_2)$ be satisfied. Then there is $\rho > 0$ such that $\kappa := \inf \Phi(S_\rho^+) > 0$ where $S_\rho^+ = \partial B_\rho \cap E^+$.*

Proof. Choose $p > 2$ such that (3.2) holds for any $\varepsilon > 0$. This yields

$$\Psi(z) \leq \varepsilon|z|_2^2 + C_\varepsilon|z|_p^p \leq C(\varepsilon\|z\|^2 + C_\varepsilon\|z\|^p)$$

for all $z \in E$. Now the lemma follows from the form of Φ (see (2.12)). □

In the following, we arrange all the eigenvalues (counted with multiplicity) of A in $(0, m_0)$ by $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_\ell < m_0$ and let e_j denote the corresponding eigenfunctions: $Ae_j = \mu_j e_j$ for $j = 1, ..., \ell$. Set $Y_0 := \text{span}\{e_1, ..., e_\ell\}$. Note that

$$\mu_1|w|_2^2 \leq \|w\|^2 \leq \mu_\ell|w|_2^2 \quad \text{for all } w \in Y_0. \tag{3.3}$$

For any finite dimensional subspace W of Y_0 set $E_W = E^- \oplus E^0 \oplus W$.

Lemma 3.2. *Let $(R_0)-(R_2)$ be satisfied and $\rho > 0$ be given by Lemma 3.1. Then for any subspace W of Y_0, $\sup \Phi(E_W) < \infty$, and there is $R_W > 0$ such that $\Phi(z) < \inf \Phi(B_\rho \cap E^+)$ for all $z \in E_W$ with $\|z\| \geq R_W$.*

Proof. It is sufficient to show that $\Phi(z) \to -\infty$ as $z \in E_W, \|z\| \to \infty$. Arguing indirectly we assume that for some sequence $(z_j) \subset E_W$ with $\|z_j\| \to \infty$, there is $a > 0$ such that $\Phi(z_j) \geq -a$ for all j. Then, setting $w_j = z_j/\|z_j\|$, we have $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^0 \to w^0$, $w_j^+ \to w^+ \in Y$ and

$$-\frac{a}{\|z_j\|^2} \leq \frac{\Phi(z_j)}{\|z_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R}} \frac{R(t,z_j)}{\|z_j\|^2}. \tag{3.4}$$

We claim that $w^+ \neq 0$. Indeed, if not it follows from (3.4) and (R1) that $\|w_j^-\| \to 0$ and thus $w_j \to w = w^0$. Also $\int_{\mathbb{R}} \frac{R(t,z_j)}{\|z_j\|^2} \to 0$.

Recall that $R(t,z) = \frac{1}{2}M(t)z\cdot z + r(t,z)$ and $r(t,z)/|z|^2 \to 0$ uniformly in t as $|z| \to \infty$. Thus, since $|z_j(t)| \to \infty$ if $w(t) \neq 0$,

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{r(t,z_j)}{\|z_j\|^2} &= \int_{\mathbb{R}} \frac{r(t,z_j)}{|z_j|^2}|w_j|^2 \\
&\le \int_{\mathbb{R}} \frac{|r(t,z_j)|}{|z_j|^2}|w_j - w|^2 + \int_{\mathbb{R}} \frac{|r(t,z_j)|}{|z_j|^2}|w|^2 \\
&= o(1) + \int_{w(t)\neq 0} \frac{|r(t,z_j)|}{|z_j|^2}|w|^2 = o(1).
\end{aligned}
\tag{3.5}
$$

Also, by (R_3),

$$
\frac{1}{2}\int_{\mathbb{R}} \frac{M(t)z_j\cdot z_j}{\|z_j\|^2} = \frac{1}{2}\int_{\mathbb{R}} \frac{M(t)z_j\cdot z_j}{|z_j|^2}|w_j|^2 \ge \frac{m_0}{2}|w_j|_2^2. \tag{3.6}
$$

From (3.5)-(3.6) and since $\int_{\mathbb{R}} \frac{R(t,z_j)}{\|z_j\|^2} \to 0$ it follows that $|w_j|_2 \to 0$. Then $1 = \|w_j\| \to 0$ and this contradiction implies that $w^+ \neq 0$. Now since

$$
\begin{aligned}
\|w^+\|^2 - \|w^-\|^2 - \int_{\mathbb{R}} M(t)w\cdot w &\le \|w^+\|^2 - \|w^-\|^2 - m_0|w|_2^2 \\
&\le -\big((m_0-\mu_\ell)|w^+|_2^2 + \|w^-\|^2 + m_0|w^0|_2^2\big) < 0,
\end{aligned}
$$

there is $a > 0$ such that

$$
\|w^+\|^2 - \|w^-\|^2 - \int_{-a}^{a} M(t)w\cdot w \; < \; 0. \tag{3.7}
$$

As in (3.5) it follows from the fact $|w_j - w|_{L^2(-a,a)} \to 0$ that

$$
\lim_{j\to\infty} \int_{-a}^{a} \frac{r(t,z_j)}{\|z_j\|^2} = \lim_{j\to\infty} \int_{-a}^{a} \frac{r(t,z_j)|w_j|^2}{|z_j|^2} = 0.
$$

Thus (3.4) and (3.7) imply that

$$
\begin{aligned}
0 &\le \lim_{j\to\infty}\left(\frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{-a}^{a}\frac{R(t,z_j)}{\|z_j\|^2}\right) \\
&\le \frac{1}{2}\left(\|w^+\|^2 - \|w^-\|^2 - \int_{-a}^{a} M(t)w\cdot w\right) \; < \; 0,
\end{aligned}
$$

a contradiction. $\square$

As a special case we have

Lemma 3.3. *Let* $(R_0) - (R_2)$ *be satisfied and* $\kappa > 0$ *be given by Lemma* 3.1. *Then, letting* $e \in Y_0$ *with* $\|e\| = 1$, *there is* $r_0 > 0$ *such that* $\sup \Phi(\partial Q) \le \kappa$ *where* $Q := \{u = u^- + u^0 + se : \; u^- + u^0 \in E^- \oplus E^0, s \ge 0, \|u\| \le r_0\}$.

4 The (C)-sequences

Here we discuss the Cerami condition.

Lemma 4.1. *Let $(R_0)-(R_2)$ and $(R_4)-(R_5)$ be satisfied. Then any $(C)_c$-sequence is bounded.*

Proof. Let $(z_j) \subset E$ be such that

$$\Phi(z_j) \to c \quad \text{and} \quad (1+\|z_j\|)\Phi'(z_j) \to 0. \tag{4.1}$$

Then, for a $C_0 > 0$,

$$C_0 \geq \Phi(z_j) - \frac{1}{2}\Phi'(z_j)z_j = \int_{\mathbb{R}} \tilde{R}(t, z_j)\,. \tag{4.2}$$

To prove that (z_j) is bounded we develop a contradiction argument related to the one introduced in [12] (see also [13, 19]). We assume that, up to a subsequence, $\|z_j\| \to \infty$ and set $v_j = z_j/\|z_j\|$. Then $\|v_j\| = 1$, $|v_j|_s \leq \gamma_s\|v_j\| = \gamma_s$ for all $s \in [2, \infty)$, and passing to a subsequence if necessary, $v_j \rightharpoonup v$ in E, $v_j \to v$ in L^s_{loc} for all $s \geq 1$, $v_j(t) \to v(t)$ for a.e. $t \in \mathbb{R}$. Since, by (R_2), $|r_z(t,z)| = o(z)$ as $|z| \to \infty$ uniformly in t and $|z_j(t)| \to \infty$ if $v(t) \neq 0$, it is easy to see that

$$\int_{\mathbb{R}} \frac{R_z(t, z_j(t))\varphi(t)}{||z_j||} \to \int_{\mathbb{R}} M(t)v\varphi$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^{2N})$. From this we deduce, using (4.1), that

$$\mathcal{J}\frac{d}{dt}v + (L(t) + M(t))v = 0. \tag{4.3}$$

Multiplying (4.3) by $J^{-1} = -J$ we also get

$$\frac{d}{dt}v = \mathcal{J}(L(t) + M(t))v. \tag{4.4}$$

We claim that $v \neq 0$. Arguing by contradiction we assume that $v = 0$. Then $v_j^d \to 0$ in E and $v_j \to 0$ in L^s_{loc}. Set $I_0 := (-t_0, t_0)$ and $I_0^c := \mathbb{R} \setminus I_0$ where $t_0 > 0$ is the number given in (R_5). It follows from

$$\frac{\Phi'(z_j)(z_j^{e+} - z_j^{e-})}{\|z_j\|^2} = \|v_j^e\|^2 - \int_{\mathbb{R}} \frac{R_z(t, z_j)}{|z_j|}(v_j^{e+} - v_j^{e-})|v_j| \tag{4.5}$$

that

$$\begin{aligned}\|v_j^e\|^2 &= \int_{I_0} \frac{R_z(t,z_j)}{|z_j|}(v_j^{e+} - v_j^{e-})|v_j| \\ &\quad + \int_{I_0^c} \frac{R_z(t,z_j)}{|z_j|}(v_j^{e+} - v_j^{e-})|v_j| + o(1) \\ &\le c\int_{I_0} |v_j||v_j^{e+} - v_j^{e-}| + \gamma \int_{I_0^c} |v_j||v_j^{e+} - v_j^{e-}| + o(1) \\ &\le \gamma |v_j^e|_2^2 + o(1).\end{aligned}$$

By (2.10) one gets

$$\left(1 - \frac{\gamma}{b}\right) \|v_j^e\|^2 \le o(1),$$

which implies, by (2.8), that $\|v_j^e\|^2 \to 0$. Hence $1 = \|v_j\|^2 = \|v_j^d\|^2 + \|v_j^e\|^2 \to 0$, a contradiction.

Therefore, $v \ne 0$ which is impossible if (i) of (R_4) is satisfied. Thus we assume (ii) of (R_4). Let $\Omega_j(0,r) := \{t \in \mathbb{R} : \ |z_j(t)| < r\}$, $\Omega_j(r,\infty) := \{t \in \mathbb{R} : \ |z_j(t)| \ge r\}$, and set for $r \ge 0$

$$g(r) := \inf\left\{\tilde{R}(t,z) : \ t \in \mathbb{R} \text{ and } z \in \mathbb{R}^{2N} \text{ with } |z| \ge r\right\}.$$

By assumption there is $r_0 > 0$ such that $g(r_0) > 0$, hence one has by (4.2) that $|\Omega_j(r_0,\infty)| \le C_0/g(r_0)$. Set $\Omega := \{t : \ v(t) \ne 0\}$. Since v satisfies (4.4) it follows from Cauchy Uniqueness Principle that $\Omega = \mathbb{R}$. Indeed otherwise $v \equiv 0$ on $\mathbb{R}$ contradicting the fact that $v \ne 0$. Now since $|\Omega| = \infty$ there exists $\varepsilon > 0$ and $\omega \subset \Omega$ such that $|v(t)| \ge 2\varepsilon$ for $t \in \omega$ and $2C_0/g(r_0) \le |\omega| < \infty$. By an Egoroff's theorem we can find a set $\omega' \subset \omega$ with $|\omega'| > C_0/g(r_0)$ such that $v_j \to v$ uniformly on ω'. So for almost all j, $|v_j(t)| \ge \varepsilon$ and $|z_j(t)| \ge r$ in ω'. Then

$$\frac{C_0}{g(r_0)} < |\omega'| \le |\Omega_j(r,\infty)| \le \frac{C_0}{g(r_0)},$$

a contradiction. □

Let $(z_j) \subset E$ be an arbitrary $(C)_c$-sequence. By Lemma 4.1 it is bounded, hence, we may assume without loss of generality that $z_j \rightharpoonup z$ in E, $z_j \to z$ in L^q_{loc} for $q \ge 1$ and $z_j(t) \to z(t)$ a.e. in t. Plainly z is a critical point of Φ.

Choose $p > 2$ such that $|R_z(t,z)| \le |z| + C_1|z|^{p-1}$ for all (t,z), and let q stands for either 2 or p. Set $I_a := [-a,a]$ for $a > 0$.

Lemma 4.2. *Let $q \ge 2$ and assume that $(R_0) - (R_2)$ and $(R_4) - (R_5)$ are satisfied. Along a subsequence, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that*

$$\limsup_{n\to\infty} \int_{I_n \setminus I_r} |z_{j_n}|^q \le \varepsilon \tag{4.6}$$

for all $r \geq r_\varepsilon$.

Proof. Note that, for each $n \in \mathbb{N}$, $\int_{I_n} |z_j|^q \to \int_{I_n} |z|^q$ as $j \to \infty$. There exists $i_n \in \mathbb{N}$ such that

$$\int_{I_n} (|z_j|^q - |z|^q) < \frac{1}{n} \quad \text{for all } j = i_n + m,\ m = 1, 2, 3, \ldots.$$

Without loss of generality we can assume $i_{n+1} \geq i_n$. In particular, for $j_n = i_n + n$ we have

$$\int_{I_n} (|z_{j_n}|^q - |z|^q) < \frac{1}{n}.$$

Observe that there is r_ε satisfying

$$\int_{\mathbb{R}\setminus I_r} |z|^q < \varepsilon \tag{4.7}$$

for all $r \geq r_\varepsilon$. Since

$$\begin{aligned}\int_{I_n\setminus I_r} |z_{j_n}|^q &= \int_{I_n} (|z_{j_n}|^q - |z|^q) + \int_{I_n\setminus I_r} |z|^q + \int_{I_r} (|z|^q - |z_{j_n}|^q) \\ &\leq \frac{1}{n} + \int_{\mathbb{R}\setminus I_r} |z|^q + \int_{I_r} (|z|^q - |z_{j_n}|^q),\end{aligned}$$

the lemma now follows easily. □

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \leq 1$, $\eta(s) = 0$ if $s \geq 2$. At this point we make use of techniques first developed in [1] (see also [8]). Define $\tilde{z}_n(t) = \eta(2|t|/n)z(t)$ and set $h_n := z - \tilde{z}_n$. Since z is a homoclinic orbit, we have by definition that $h_n \in H^1$ and

$$\|h_n\| \to 0 \ \text{ and } \ |h_n|_\infty \to 0 \ \text{ as } n \to \infty. \tag{4.8}$$

Lemma 4.3. *Assume that* $(R_0) - (R_2)$ *and* $(R_4) - (R_5)$ *are satisfied. Then* $\Phi'(z_{j_n} - \tilde{z}_n) \to 0$.

Proof. Observe that, for any $\varphi \in E$,

$$\begin{aligned}\Phi'(z_{j_n} - \tilde{z}_n)\varphi &= \Phi'(z_{j_n})\varphi - \Phi'(\tilde{z}_n)\varphi \\ &\quad + \int_{\mathbb{R}} \Big(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\Big)\varphi.\end{aligned}$$

Now, (4.7) and the compactness of Sobolev embeddings imply that, for any $r > 0$,

$$\lim_{n\to\infty} \left| \int_{I_r} (R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n))\, \varphi \right| = 0$$

uniformly in $\|\varphi\| \le 1$. For any $\varepsilon > 0$ let $r_\varepsilon > 0$ be so large that (4.6) and (4.7) hold. Then

$$\limsup_{n\to\infty} \int_{I_n \setminus I_r} |\tilde{z}_n|^q \le \int_{\mathbb{R}\setminus I_r} |z|^q \le \varepsilon$$

for all $r \ge r_\varepsilon$. Using (4.6) for $q = 2, p$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int_{\mathbb{R}} \left(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\right) \varphi \right| \\
&= \limsup_{n\to\infty} \left| \int_{I_n \setminus I_r} \left(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\right) \varphi \right| \\
&\le c_1 \limsup_{n\to\infty} \int_{I_n \setminus I_r} \left(|z_{j_n}| + |\tilde{z}_n|\right) |\varphi| \\
&\quad + c_2 \limsup_{n\to\infty} \int_{I_n \setminus I_r} \left(|z_{j_n}|^{p-1} + |\tilde{z}_n|^{p-1}\right) |\varphi| \\
&\le c_1 \limsup_{n\to\infty} \left(|z_{j_n}|_{L^2(I_n\setminus I_r)} + |\tilde{z}_n|_{L^2(I_n\setminus I_r)}\right) |\varphi|_2 \\
&\quad + c_2 \limsup_{n\to\infty} \left(|z_{j_n}|^{p-1}_{L^p(I_n\setminus I_r)} + |\tilde{z}_n|^{p-1}_{L^p(I_n\setminus I_r)}\right) |\varphi|_p \\
&\le c_3 \varepsilon^{1/2} + c_4 \varepsilon^{(p-1)/p}.
\end{aligned}$$

Thus we get

$$\lim_{n\to\infty} \int_{\mathbb{R}} \Big(R_z(t, z_{j_n}) - R_z(t, z_{j_n} - \tilde{z}_n) - R_z(t, \tilde{z}_n)\Big) \varphi = 0$$

uniformly in $\|\varphi\| \le 1$ and this proves the lemma. □

Lemma 4.4. *Let $(R_0) - (R_2)$ and $(R_4) - (R_5)$ be satisfied. Then Φ satisfies the $(C)_c$-condition.*

Proof. Let $(z_j) \subset E$ be an arbitrary $(C)_c$-sequence. The conclusions of Lemmas 4.2 and 4.3 apply to it. Now we use the decomposition $E = E^d \oplus E^e$ (see (2.9)). Recall that $\dim(E^d) < \infty$. Write

$$y_n := z_{j_n} - \tilde{z}_n = y_n^d + y_n^e.$$

Then $y_n^d = (z_{j_n}^d - z^d) + (z^d - \tilde{z}_n^d) \to 0$ and, by Lemma 4.3, $\Phi'(y_n) \to 0$. Set $\bar{y}_n^e = y_n^{e+} - y_n^{e-}$. Observe that

$$o(1) = \Phi'(y_n)\bar{y}_n^e = \|y_n^e\|^2 - \int_{\mathbb{R}} R_z(t, y_n)\bar{y}_n^e. \tag{4.9}$$

Thus it follows that, for $I = [-t_0, t_0]$ where $t_0 \geq 0$ is defined in (R_5),

$$\begin{aligned}
\|y_n^e\|^2 &\leq o(1) + \int_{I_0} \frac{|R_z(t, y_n)|}{|y_n|} |y_n||\bar{y}_n^e| + \int_{I_0^c} \frac{|R_z(t, y_n)|}{|y_n|} |y_n||\bar{y}_n^e| \\
&\leq o(1) + c \int_{I_0} |y_n||\bar{y}_n^e| + \gamma \int_{I_0^c} |y_n||\bar{y}_n^e| \\
&\leq o(1) + \gamma |y_n^e|_2^2 \leq o(1) + \frac{\gamma}{b} \|y_n^e\|^2.
\end{aligned}$$

Hence $(1 - \frac{\gamma}{b})\|y_n^e\|^2 \to 0$, and so $\|y_n\| \to 0$. Remark that $z_{j_n} - z = y_n + (\tilde{z}_n - z)$, hence $\|z_{j_n} - z\| \to 0$. This ends the proof. $\square$

5 Proof of Theorem 1.1

In order to apply the abstract Theorems 2.4 and 2.5 to Φ, we choose $X = E^- \oplus E^0$ and $Y = E^+$. X is separable and reflexive and let $\mathcal{S}$ be a countable dense subset of X^*. First we have

Lemma 5.1. Φ *satisfies* (Φ_0).

Proof. We first show that Φ_a is $\mathcal{T}_{\mathcal{S}}$-closed for every $a \in \mathbb{R}$. Consider a sequence (z_n) in Φ_a which $\mathcal{T}_{\mathcal{S}}$-converges to $z \in E$, and write $z_n = z_n^- + z_n^0 + z_n^+$, $z = z^- + z^0 + z^+$. Observe that (z_n^+) converges to z^+ in norm. Since Ψ is bounded from below it follows from

$$\frac{1}{2}\|z_n^-\|^2 = \frac{1}{2}\|z_n^+\|^2 - \Phi(z_n) - \Psi(z_n) \leq C$$

that (z_n^-) is bounded, hence it converges weakly towards z^-. Since $\dim E^0 < \infty$, the $\mathcal{T}_{\mathcal{S}}$-convergence coincides with the weak convergence. Therefore $z_n \rightharpoonup z$. It is standard to show that Ψ is weakly sequentially lower semi-continuous. Thus, from the form of Φ it follows that $\Phi(z) \geq \liminf \Phi(z_n) \geq a$, so $z \in \Phi_a$. Next we show that $\Phi' : (\Phi_a, \mathcal{T}_{\mathcal{S}}) \to (E^*, \mathcal{T}_{w^*})$ is continuous. Suppose (z_n) $\mathcal{T}_{\mathcal{S}}$-converges towards z in Φ_a. As above it follows that (z_n) is bounded and converges weakly towards z. Then, since clearly Ψ' is weakly sequentially continuous, $\Phi'(z_n) \xrightarrow{w^*} \Phi'(z)$. $\square$

Lemma 5.2. *Under* $(R_0) - (R_2)$, *for any* $c > 0$, *there is* $\zeta > 0$ *such that :*

$$\|z\| < \zeta \|z^+\| \quad \text{for all } z \in \Phi_c.$$

Proof. We assume by contradiction that for some $c > 0$ there is a sequence (z_n) with $\Phi(z_n) \ge c$ and $\|z_n\|^2 \ge n\|z_n^+\|^2$. The form of Φ implies

$$\|z_n^- + z_n^0\|^2 \ge (n-1)\|z^+\|^2 \ge (n-1)\left(2c + \|z_n^-\|^2 + 2\int_{\mathbb{R}} R(t, z_n)\right),$$

or

$$\|z_n^0\|^2 \ge (n-1)2c + (n-2)\|z_n^-\|^2 + 2(n-1)\int_{\mathbb{R}} R(t, z_n).$$

Since $c > 0$ and $R(t, z) \ge 0$, it follows that $\|z_n^0\| \to \infty$, hence $\|z_n\| \to \infty$. Set $w_n = z_n/\|z_n\|$. We have $\|w_n^+\|^2 \le 1/n \to 0$. By

$$1 \ge \|w_n^0\|^2 \ge \frac{(n-1)2c}{\|z_n\|^2} + (n-2)\|w_n^-\|^2 + 2(n-1)\int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2},$$

we also have $\|w_n^-\|^2 \le 1/(n-2) \to 0$. Therefore, $w_n \to w = w^0$ in E and $\|w^0\| = 1$. Recall that $R(t, z) = \frac{1}{2}M(t)z \cdot z + r(t, z)$ with $|r(t, z)|/|z|^2 \to 0$ as $|z| \to \infty$. Therefore, since $|z_n(t)| \to \infty$ for $w(t) \ne 0$,

$$\begin{aligned}\int_{\mathbb{R}} \frac{r(t, z_n)}{\|z_n\|^2} &= \int_{w(t)\ne 0} \frac{r(t, z_n)}{|z_n|^2}|w_n|^2 + \int_{w(t)=0} \frac{r(t, z_n)}{|z_n|^2}|w_n - w|^2 \\ &\le 2\int_{w(t)\ne 0} \frac{|r(t, z_n)|}{|z_n|^2}|w|^2 + c|w_n - w|_2^2 \to 0.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{2(n-1)} &\ge \int_{\mathbb{R}} \frac{R(t, z_n)}{\|z_n\|^2} = \frac{1}{2}\int_{\mathbb{R}} M(t)w_n \cdot w_n + \int_{\mathbb{R}} \frac{r(t, z_n)}{\|z_n\|^2} \\ &\ge \frac{m_0}{2}|w_n|_2^2 + o(1),\end{aligned}$$

consequently, $w^0 = 0$, a contradiction. □

Proof of Theorem 1.1 *(Existence).* With $X = E_- \oplus E^0$ and $Y = E_+$ the condition (Φ_0) holds by Lemma 5.1 and (Φ_1) holds by Lemma 5.2. Lemma 3.1 implies (Φ_2). Lemma 3.3 shows that Φ possesses the linking structure of Theorem 2.4. Finally, Φ satisfies the $(C)_c$-condition by virtue of Lemma 4.4. Therefore, Φ has at least one critical point z with $\Phi(z) \ge \kappa > 0$.

(Multiplicity) Assume moreover that $R(t, z)$ is even in z. Then Φ is even. Lemma 3.2 says that Φ satisfies (Φ_3) with $\dim Y = \ell$. Therefore, Φ has at least ℓ pairs of nontrivial critical points by Theorem 2.5.

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