AN APPROACH TO MINIMIZATION UNDER A CONSTRAINT: THE ADDED MASS TECHNIQUE

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ABSTRACT. For a class of minimization problems, where the functionals are weakly lower semicontinuous, we present, through the treatment of some semi-linear or quasi-linear type problems, techniques to show the existence of a minimizer.

1. INTRODUCTION

Let H be a reflexive Banach space of functions defined on $\mathbb{R}^N$ $(N \geq 1)$ with value in $\mathbb{R}^m$ $(m \geq 1)$ and let J, G be functionals defined on H of the type

$$J(u) = \int_{\mathbb{R}^N} j(x, u, |\nabla u|)dx, \quad G(u) = \int_{\mathbb{R}^N} g(u)dx,$$

where $j(x, s, t)$ and $g(s)$ are positive real-valued functions defined on $\mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}$ and $\mathbb{R}^m$ respectively. For a fixed $c \in \mathbb{R}^+$, we consider the problem

$$\text{minimize } J \text{ on the functions } u \in H \text{ with } G(u) = c. \tag{1.1}$$

Assuming that

$$m(c) = \inf\{J(u) : u \in H \text{ with } G(u) = c\} > -\infty$$

and that it holds

(H0) There exists a minimizing sequence $(u_n) \subset H$ and a $u \in H$ such that $u_n \rightharpoonup u$ with

$$J(u) \leq \liminf_{n\to\infty} J(u_n) \quad \text{and} \quad G(u) \leq c,$$

we shall, through the treatment of selected examples, present some ways to solve (1.1).

Condition (H0) somehow defines the class of minimization problems. Under (H0), solving (1.1), corresponds to show that $G(u) = c$.

Over the last twenty five years the Compactness by Concentration of P.L. Lions [11] has had a deep influence on the problem of minimizing a functional under a given constraint. Assume that a problem at infinity can be associated to (1.1). The limit of $j(x, u, |\nabla u|)$, as $|x| \to \infty$, is denoted $j_\infty(u, |\nabla u|)$ and, accordingly, we define

$$J_\infty(u) = \int_{\mathbb{R}^N} j_\infty(u, |\nabla u|)dx$$

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and, in turn,

$$m_\infty(c) = \inf\{J_\infty(u) : u \in H \text{ with } G(u) = c\}.$$

In [11], heuristic arguments are given that all minimizing sequences for (1.1) are compact if, and only if, the following strict inequality holds

$$m(c) < m(\lambda) + m_\infty(c-\lambda), \quad \forall \lambda \in [0, c[. \tag{1.2}$$

It is also explained (see pages 113-114) that the large inequalities

$$m(c) \le m(\lambda) + m_\infty(c-\lambda), \quad \forall c > 0, \quad \forall \lambda \in [0, c[\tag{1.3}$$

are expected to hold under very weak assumptions. In the setting defined by (H0), the results and considerations of [11] give precious indications. First observe that if the function $\mathbb{R}^+ \ni \lambda \mapsto m(\lambda)$ is strictly decreasing, then, for any fixed $c \in \mathbb{R}^+$, the value $m(c)$ is reached. Indeed, let $c \in \mathbb{R}^+$ be fixed. By (H0) we get that $J(u) \le m(c)$. Thus, necessarily, $m(G(u)) \le m(c)$ and so, if it was $G(u) < c$, we would get a contradiction with the assumption that $\lambda \mapsto m(\lambda)$ is strictly decreasing.

Observe that, if (1.3) holds and $m_\infty(d) < 0$ for any $d \in [0, c[$, the function $\lambda \mapsto m(\lambda)$ is strictly decreasing. Thus $m(c)$ is reached in this case. In some situations, however, the condition $m_\infty(d) < 0$ for any $d \in [0, c[$, is either difficult to check or fails to hold. On the contrary, proving that $m_\infty(d) \le 0$ for any $d \in [0, c[$, is often much easier. Assuming that (1.3) hold, we can then still deduce that $\lambda \mapsto m(\lambda)$ is non increasing. This information often proves very useful. Indeed, by (H0), there exist a $u \in H$ such that

$$J(u) \le m(c), \qquad \text{with} \quad G(u) \le c.$$

If $G(u) = c$ we are done. Thus we can assume, by contradiction, that $G(u) < c$. At this point if we can find a $v \in H$ such that $G(u + v) \le c$ and $J(u + v) < J(u)$ we immediately reach a contradiction. This way to conclude, by "adding mass", that is to increase the value of G, while strictly decreasing the value of the functional J is developed in several of the problems that we have treated.

In the problems we consider in this paper we do not prove directly that (1.3) (nor (1.2)), holds, preferring to give more direct proofs. Nevertheless the ideas of [11], as presented above, have acted as a source of inspiration for us.

In Section 2 we state the results we have obtained on four classes of constrained semi-linear and quasi-linear elliptic problems. More precisely, see the Subsections 2.2, 2.3, 2.4 and 2.1 respectively. Finally, in Section 3 we provide the proofs of the results stated in Section 2. See, Subsections 3.2, 3.3, 3.4 and 3.1, respectively.

Notations.

(1) For $N \ge 1$, we denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^N$.
(2) $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) is the set of positive (resp. negative) real values.
(3) For $p > 1$ we denote by $L^p(\mathbb{R}^N)$ the space of measurable functions u such that $\int_{\mathbb{R}^N} |u|^p dx < \infty$. The norm $(\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$ in $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$.
(4) We denote by $L^\infty(\mathbb{R}^N)$ the set of bounded measurable functions endowed with the standard essential supremum norm $\|\cdot\|_\infty$.

(5) For $s \in \mathbb{N}$, we denote by $H^s(\mathbb{R}^N)$ the Sobolev space of functions u in $L^2(\mathbb{R}^N)$ having generalized partial derivatives $\partial_i^k u$ in $L^2(\mathbb{R}^N)$ for all $i = 1, \dots, N$ and any $0 \le k \le s$.
(6) The norm $(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$ in $H^1(\mathbb{R}^N)$ is denoted by $\|\cdot\|$ and more generally, the norm in H^s is denoted by $\|\cdot\|_{H^s}$.
(7) We denote by $C_0^\infty(\mathbb{R}^N)$ the set of smooth and compactly supported functions in $\mathbb{R}^N$.
(8) We denote by $B(x_0, R)$ a ball in $\mathbb{R}^N$ of center x_0 and radius $R > 0$.

2. Statements of the main results

In this section we present the four problems that we have treated and the results obtained.

2.1. **A problem studied by Badiale-Rolando.** Let $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ with $N > k \ge 2$ and set

$$H := \Big\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^2} dx < \infty \Big\}$$

$$H_s := \Big\{ u \in H : \ u(y,z) = u(|y|, z) \Big\}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies, for $F(t) := \int_0^t f(s) ds$,
(f_0) $F(t_0) > 0$ for some $t_0 > 0$.
(f_1) there exists $q > 2$ such that

$$\lim_{t \to 0^+} \frac{f(t)}{|t|^{q-1}} = 0,$$

and fulfills, in addition, one of the following assumptions:
(f_2) $f(\beta) = 0$ for some $\beta > \beta_0 := \inf\{t > 0, F(t) > 0\}$.
(f_3) there exists $p \in]2, 2 + \frac{4}{N}[$ such that

$$\lim_{t \to +\infty} \frac{f(t)}{|t|^{p-1}} = 0.$$

Under these assumptions we have

Theorem 2.1. *Let $N > k \ge 2$ and $\mu > 0$. Assume that $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $(f_0), (f_1)$ and at least one of the hypotheses (f_2) and (f_3). Then there exists $\rho_0 > 0$ such that for all $\rho > \rho_0$ the minimization problem*

$$\inf_{u \in H_s, \, \|u\|_2^2 = \rho} \Big(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\mu}{2} \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^2} dx - \int_{\mathbb{R}^N} F(u) dx \Big) \tag{2.1}$$

admits a solution $u(y,z) = u(|y|, |z|) \ge 0$ which is non increasing in $|z|$.

Theorem 2.1 was originally proved in [1]. It is the central part of [1] in which is establish the existence of standing waves with non zero angular momentum for a class of Klein-Gordon equations. We refer to [1] for a detailed presentation of the problem and of its physical motivations. Here we concentrate on giving an alternative shorter proof of this result. The original proof in [1] is based on the full machinery of the Concentration Compactness Principle and the central issue is to rule out the dichotomy case. Here we observe that, exploiting the symmetry of (2.1), it is possible to choose a minimizing sequence such that (H0) holds. Then a simple scaling argument permits to conclude.

2.2. **A Choquard type problem in $\mathbb{R}^3$.** We consider a variant of the classical Choquard Problem (cf. [9, 13]). Precisely, we minimize the functional $J : H \to \mathbb{R}$ defined by

$$J(u) = \int_{\mathbb{R}^3} j(u, |\nabla u|)dx - \iint_{\mathbb{R}^6} \frac{u^2(x)u^2(y)}{|x-y|}dxdy \quad \text{over } \|u\|^2_{L^2(\mathbb{R}^3)} = c, \tag{2.2}$$

where c is a fixed positive number. Here H is given by $H^1(\mathbb{R}^3)$, and we assume that

$$j : \mathbb{R} \times [0, \infty[\to \mathbb{R}^+,$$

is continuous, convex and increasing with respect to the second argument and that there exists $\nu > 0$ such that

$$j(s, |\xi|) \geq \nu|\xi|^2, \quad \text{for all } s \in \mathbb{R}^+ \text{ and all } \xi \in \mathbb{R}^3. \tag{2.3}$$

Moreover there exists a positive constant C such that

$$j(s, |\xi|) \leq C|s|^6 + C|\xi|^2, \quad \text{for all } s \in \mathbb{R}^+ \text{ and all } \xi \in \mathbb{R}^3. \tag{2.4}$$

Finally, we assume that

$$j(|s|, |\xi|) \leq j(s, |\xi|), \quad \text{for all } s \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^3. \tag{2.5}$$

For all $c > 0$, let us set

$$m(c) = \min_{\|u\|^2_{L^2(\mathbb{R}^3)}=c} J(u).$$

Our result is the following

Proposition 2.2. *Under the assumptions* (2.3)-(2.5)*, $m(c)$ is reached for all $c > 0$.*

If one wants to treat this minimization problem using directly the Compactness Concentration Principle of [11] one faces the problem of checking the strict inequalities (1.2). To achieve this, one usually establishes (see Lemma II.1 of [11]) that

$$m(\theta\lambda) < \theta m(\lambda), \quad \text{for all } \lambda \in]0, c[\text{ and } \theta \in]1, c/\lambda]. \tag{2.6}$$

Under our assumptions on the Lagrangian $j(s, |\xi|)$ there is no reason for inequality (2.6) to be true. However we prove that (H0) holds and, using the fact that $m_\infty(\lambda) = m(\lambda) < 0$ for any $\lambda \in]0, c]$, we are able to conclude that $m(c)$ is reached. The trick of scaling by dilation, used in Subsection 2.1, does not apply anymore and we develop a contradiction argument based on the use of functions with disjoints supports. In order to check (H0) we choose a minimizing sequence consisting of Schwarz symmetric functions. The possibility to take a minimizing sequence of this type, for general $j(s, |\xi|)$, has recently been established in [4] for even weaker growth assumptions on j.

2.3. **A general class of quasi-linear problems.** We study a general problem of minimization that goes back to the work of Stuart [15] and has recently undergone new developments [4]. Let

$$T_c = \inf\left\{J(u):\ u \in \mathcal{C}\right\}, \tag{2.7}$$

where we have set, for a fixed $c > 0$,

$$\mathcal{C} = \Big\{u \in H : G_k(u_k),\, j_k(u_k, |\nabla u_k|) \in L^1(\mathbb{R}^N) \text{ for any } k \text{ and } \sum_{k=1}^{m}\int_{\mathbb{R}^N} G_k(u_k)dx = c\Big\},$$

for $m \geq 1$ and $H = W^{1,p}(\mathbb{R}^N, \mathbb{R}^m)$. Here J is a functional defined, for any function $u = (u_1, \dots, u_m) \in \mathcal{C}$, by

$$J(u) = \sum_{k=1}^{m}\int_{\mathbb{R}^N} j_k(u_k, |\nabla u_k|)dx - \int_{\mathbb{R}^N} F(|x|, u_1, \dots, u_m)dx.$$

We collect below the assumptions on j_k, F, G.

• **Assumptions on j_k.** For $m \geq 1$, $N \geq 1$, $1 < p < N$, let

$$j_k : \mathbb{R} \times [0, \infty[\to \mathbb{R}^+, \quad \text{for } k = 1, \dots, m$$

be continuous, increasing in the first argument, convex, increasing and p-homogeneous in the second argument and such that there exists $\nu > 0$ with, for $k = 1, \dots, m$,

$$\nu|\xi|^p \leq j_k(s, |\xi|), \quad \text{for all } s \in \mathbb{R}^+ \text{ and all } \xi \in \mathbb{R}^N. \tag{2.8}$$

Moreover there exists a continuous increasing function $\beta_k : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\beta_k(s) \leq C(1 + |s|^{p^*-p}), \quad \text{for all } s \in \mathbb{R}^+, \quad p^* = \frac{pN}{N-p}, \tag{2.9}$$

such that

$$j_k(s, |\xi|) \leq \beta_k(s)|\xi|^p, \quad \text{for all } s \in \mathbb{R}^+ \text{ and all } \xi \in \mathbb{R}^N. \tag{2.10}$$

Finally we require, for $k = 1, \dots, m$,

$$j_k(|s|, |\xi|) \leq j_k(s, |\xi|), \quad \text{for all } s \in \mathbb{R} \text{ and all } \xi \in \mathbb{R}^N. \tag{2.11}$$

• **Assumptions on F.** Let us consider a function

$$F : [0, \infty[\times \mathbb{R}^m \to \mathbb{R},$$

of variables $(r, s_1, \dots, s_m)$, measurable and bounded with respect r and continuous with respect to $(s_1, \dots, s_m) \in \mathbb{R}^N$ with $F(r, 0, \dots, 0) = 0$ for any $r \in \mathbb{R}^+$. We assume that

$$F(r, s + he_i + ke_j) + F(r, s) \geq F(r, s + he_i) + F(r, s + ke_j), \tag{2.12}$$

$$F(r_1, s + he_i) + F(r_0, s) \leq F(r_1, s) + F(r_0, s + he_i), \tag{2.13}$$

for every $i \neq j$, $i, j = 1, \dots, m$ where e_i denotes the i-th standard basis vector in $\mathbb{R}^m$, $r > 0$, for all $h, k > 0$, $s = (s_1, \dots, s_m)$ and r_0, r_1 such that $0 < r_0 < r_1$.

Conditions (2.12)-(2.13) are known as cooperativity conditions. If F is smooth, (2.12) yields $\partial^2_{ij}F(r, s_1, \dots, s_m) \geq 0$ for $i \neq j$. In general, (2.12)-(2.13) are necessary for rearrangement inequalities to hold (see [16] for some indications). Moreover, we assume that

$$\limsup_{(s_1,\dots,s_m)\to(0,\dots,0)^+} \frac{F(r,s_1,\dots,s_m)}{\sum_{k=1}^m s_k^p} = 0, \tag{2.14}$$

$$\lim_{|(s_1,\dots,s_m)|\to\infty} \frac{F(r,s_1,\dots,s_m)}{\sum_{k=1}^m s_k^{p+\frac{p^2}{N}}} = 0, \tag{2.15}$$

uniformly with respect to r.

For a $j\in\{1,\dots,m\}$, there exist $\delta>0$, $\mu>0$, and $\sigma\in[0,\frac{p^2}{N}[$ such that

$$F(r,0,\dots,s_j,\dots,0)\geq \mu s_j^{\sigma+p} \text{ for } s_j\in[0,\delta] \text{ and all } r\geq 0. \tag{2.16}$$

Finally, we require:

$$F(r,s_1,\dots,s_m)\leq F(r,|s_1|,\dots,|s_m|), \quad \text{for all } r>0 \text{ and } (s_1,\dots,s_m)\in\mathbb{R}^m. \tag{2.17}$$

• **Assumptions on G_k.** Consider $m\geq 1$ continuous, increasing and even functions

$$G_k:\mathbb{R}\to\mathbb{R}^+, \quad G_k(0)=0, \quad \text{for } k=1,\dots,m$$

such that there exists $\gamma>0$ with

$$G_k(s)\geq \gamma|s|^p, \quad \text{for all } s\in\mathbb{R}. \tag{2.18}$$

We also require

$$G_j \quad \text{is } p\text{-homogeneous where } j\in\{1,\dots,m\} \text{ is defined in (2.16)}. \tag{2.19}$$

Under the assumptions (2.8)-(2.19), we prove the following

Theorem 2.3. *Assume that (2.8)-(2.19) hold. Then problem (2.7) admits a radially symmetric and radially decreasing nonnegative solution.*

In problem (2.7), (H0) holds since we can choose a suitable minimizing sequence consisting of Schwarz symmetric functions as in Subsection 2.2. Denoting by u the weak limit we develop an argument by contradiction to show that $u\in\mathcal{C}$. Here the problem is not invariant by translation but it is still possible to apply our method of adding functions with disjoints supports, presented in Subsection 2.2. However it is more technically involved, in particular because of our weak regularity assumptions.

Remark 2.4. In [4], in order to prove that the weak limit u satisfies the constraint, the growth of j_k is related to the one of $F(|x|,s_1,\dots,s_m)$. More precisely, it is assumed that there exists $\alpha\geq p$ such that, for all $k\in\{1,\cdots,m\}$

$$j_k(ts,t|\xi|)\leq t^\alpha j_k(s,|\xi|), \quad \text{for all } t\geq 1,\ s\in\mathbb{R}^+ \text{ and } \xi\in\mathbb{R}^N \tag{2.20}$$

and

$$F(r,ts_1,\dots,ts_m)\geq t^\alpha F(r,s_1,\dots,s_m), \tag{2.21}$$

for all $r>0$, $t\geq 1$ and $(s_1,\dots,s_m)\in\mathbb{R}^m$. Such global conditions are not needed here anymore.

Remark 2.5. Take $\beta \geq 0$, $\sigma \in [0, \frac{p^2}{N}[$ and consider a continuous and decreasing function $a : [0, \infty[\to [0, \infty[$ such that $a(|x|)$ converges to $a_0 > 0$ as $|x| \to \infty$. Then the function

$$F(|x|, s_1, \ldots, s_m) = \frac{a(|x|)}{p+\sigma} \sum_{k=1}^{m} |s_k|^{p+\sigma} + \frac{2\beta a(|x|)}{p+\sigma} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} |s_i|^{\frac{p+\sigma}{2}} |s_j|^{\frac{p+\sigma}{2}}$$

satisfies all the required assumptions.

2.4. **A Stuart's type problem.** Assume that $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and consider the problem

$$\text{minimize} \quad I \quad \text{on} \quad \|u\|_{L^2}^2 = c \tag{2.22}$$

where $c > 0$ and $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(x, u) dx.$$

We set

$$m(c) = \inf\{I(u) : \|u\|_2^2 = c\},$$

and discuss problem (2.22) under the assumptions:

$$\limsup_{s \to 0^+} \frac{F(x,s)}{s^2} < \infty \quad \text{and} \quad \lim_{s \to \infty} \frac{F(x,s)}{s^{2+\frac{4}{N}}} = 0, \text{ uniformly for } x \in \mathbb{R}^N. \tag{2.23}$$

$$\lim_{|x| \to \infty} F(x,s) = 0, \quad \text{uniformly in } s \in \mathbb{R}. \tag{2.24}$$

$$F(x,s) \leq F(x,|s|), \quad \text{for all } x \in \mathbb{R}^N \text{ and } s \in \mathbb{R}. \tag{2.25}$$

We also require: there exists a $\delta > 0$ such that $F : \mathbb{R}^N \times [0, \delta] \to \mathbb{R}^+$ and

$$\begin{cases} N \geq 1 \text{ and there exist } r_0, A > 0,\ d \in]0, 2[\text{ and } \alpha \in]0, \frac{2(2-d)}{N}[\text{ with} \\ F(x,s) \geq A(1+|x|)^{-d} s^{2+\alpha}, \quad \text{for all } s \in [0,\delta] \text{ and } |x| \geq r_0, \textit{or} \\ N = 1 \text{ and there exist } r_0 > 0 \text{ and } \alpha \in]0, 2[\text{ with} \\ F(x,s) \geq r(x) s^{2+\alpha}, \quad \text{for all } s \in [0,\delta] \text{ and } |x| \geq r_0, \end{cases} \tag{2.26}$$

where $r \in L^\infty(\mathbb{R}),\ r \geq 0$ and

$$\int_{\mathbb{R} \setminus [-r_0, r_0]} r(x) dx > 0,$$

the value $+\infty$ being admissible.

Assumptions (2.23)-(2.26) are classical assumptions, first introduced in [15], under which I is well defined and continuous. Also (H0) holds for (2.22), because of (2.24). In order to get a minimizer for (2.22), it is assumed in [15] that

$$F(x, ts) \geq t^2 F(x,s) \tag{2.27}$$

for all $x \in \mathbb{R}^N$, $t \geq 1$ and $s \in \mathbb{R}$. The condition (2.27) is also present in all the subsequent works on (2.22), see Remark 2.4. Under (2.27), and since $m(c) < 0$ for any $c > 0$, one readily has, that

$$m(\lambda c) \leq \lambda^2 m(c) < 0, \quad \text{for any } c > 0 \text{ and } \lambda \geq 1.$$

In particular it implies that $c \mapsto m(c)$ is strictly decreasing. Here our purpose is to remove the global condition (2.27).

Remark 2.6. If we consider problem (2.22) within the formalism of [11] we see that, because of (2.24), the associated "problem at infinity" is

$$\text{minimize} \quad I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \text{on } \|u\|_2^2 = c,$$

and setting

$$m_\infty(c) = \inf\{I_\infty(u) : u \in H^1(\mathbb{R}^N) \text{ with } \|u\|_2^2 = c\},$$

we have $m_\infty(c) = 0$. Thus (1.3) is equivalent to the fact that $\lambda \to m(\lambda)$ is non increasing and (1.2) that it is strictly decreasing.

To derive our existence result we shall crucially use the following information

Proposition 2.7. *Assume that (2.23)-(2.26) hold. Then $m(c) < 0$ for all $c > 0$ and $c \mapsto m(c)$ is non increasing.*

Our first existence result is

Proposition 2.8. *Assume that (2.23)-(2.26) hold. In addition assume $N = 1$ and that there exists $\delta > 0$ such that, for any $x \in \mathbb{R}$,*

$$s \mapsto F(x,s) \text{ is strictly increasing for } s \in [0,\delta]. \tag{2.28}$$

Then $m(c)$ is reached.

Proposition 2.8 only deals with $N = 1$ since we use geometric properties of the graph of elements of $H^1(\mathbb{R})$. It is an open question if adding (2.28) suffices to guarantee that $m(c)$ is reached when $N \geq 2$.

For a result in higher dimension we require additional regularity of the nonlinearity $F(x,s)$. We assume that the derivative $f(x,s) = F_s(x,s)$ of $F(x,s)$ with respect to $s \in \mathbb{R}$ exists, that $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Carathéodory function and satisfies

$$\limsup_{s\to 0^+} \frac{f(x,s)}{s} < +\infty \quad \text{and} \quad \lim_{s\to+\infty} \frac{f(x,s)}{s^{1+\frac{4}{N}}} = 0, \tag{2.29}$$

uniformly with respect to $x \in \mathbb{R}^N$. These assumptions were already made in [15]. We also ask that $f(x,s) > 0$ for $s > 0$ and $x \in \mathbb{R}^N$ and replace (2.26) by

$$\begin{cases} N < 5 \text{ and there exist } r_0, A > 0,\ d \in]0,2[\text{ and } \alpha \in]0, \frac{2(2-d)}{N}[\text{ with} \\ f(x,s) \geq A(1+|x|)^{-d} s^{1+\alpha}, \quad \text{for all } s \in \mathbb{R}^+ \text{ and } |x| \geq r_0, \\ N \geq 5 \text{ and there exist } r_0, A > 0,\ d \in]0,2[\text{ and } \alpha \in]0, \frac{2-d}{N-2}[\text{ with} \\ f(x,s) \geq A(1+|x|)^{-d} s^{1+\alpha}, \quad \text{for all } s \in \mathbb{R}^+ \text{ and } |x| \geq r_0. \end{cases} \tag{2.30}$$

Proposition 2.9. *Assume that* (2.24)-(2.25) *and* (2.29)-(2.30) *hold. Then* $m(c)$ *is reached.*

In this last problem we cannot use the idea of compensating the possible lost of mass anymore, namely the fact that the weak limit in (H0) may satisfies $\|u\|_2^2 < c$, by adding functions with disjoints supports as in problems (2.2) and (2.22). However, exploiting systematically the fact that $c \to m(c)$ is non increasing, in a way which seems new to us, permits to derive the existence of minimizer. More generally, as it will be apparent from the proofs of Propositions 2.8 and 2.9, we obtain that any weakly converging minimizing sequence for (2.22) is strongly convergent. We also point out that our treatment of (2.22) have application to bifurcation issues for the associated Euler-Lagrange equation, see Remark 3.33.

3. Proofs of the main results

In this section we prove the results announced in Section 2.

3.1. Proof of Theorem 2.1.

We start with some preliminaries following closely [1].

We equip the Sobolev spaces H and H_s with the Hilbert norm

$$\|u\| := \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^2} dx + \int_{\mathbb{R}^N} |u|^2 dx \Big)^{\frac{1}{2}}, \quad \text{for all } u \in H. \tag{3.1}$$

Clearly $H_s \subset H \subset H^1(\mathbb{R}^N)$ and thus $H \subset L^p(\mathbb{R}^N)$, for $2 \leq p \leq \frac{2N}{N-2}$. To simplify the notation it is useful to denote

$$\|u\|_X := \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^2} dx \Big)^{\frac{1}{2}}.$$

Also observe that, for any function $f \in C(\mathbb{R}, \mathbb{R})$ satisfying (f_1) and (f_2) or (f_3), we have

$$|F(t)| \leq M(|t|^p + |t|^q), \quad \text{for all } t \in \mathbb{R} \tag{3.2}$$

with $p, q \in]2, 2 + \frac{4}{N}[$ and for some positive constant M. Now it is a standard fact that, under inequality (3.2), the functional $J : H \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2}\|u\|_X^2 - \int_{\mathbb{R}^N} F(u) dx$$

is well defined and continuous on H. Finally, to study the minimization problem (2.1), for any $\rho > 0$, we set

$$\mathcal{M}_\rho := \Big\{ u \in H_s : \int_{\mathbb{R}^N} |u|^2 dx = \rho \Big\} \quad \text{and} \quad m_\rho := \inf_{u \in \mathcal{M}_\rho} J(u).$$

We recall, from [1] the following two lemmas, which hold true under the assumptions of Theorem 2.1.

Lemma 3.1. *There exists a* $\rho_0 > 0$ *such that* $m_\rho < 0$ *for any* $\rho > \rho_0$.

Proof. This follows directly from [1, Proposition 3.1 and Corollary 3.1]. □

The next result is exactly Lemma 4.2 of [1].

Lemma 3.2. *For every $\rho > 0$, problem (2.1) admits bounded minimizing sequences (u_n) such that $u_n(y,z) = u_n(|y|,|z|) \geq 0$ is non increasing in $|z|$. Moreover, if any such sequence satisfies*

$$\inf_{n\in\mathbb{N}} \int_{B(x_n,R)} |u_n|^2 dx > 0, \qquad \text{for some } R > 0 \text{ and } (x_n) \subset \mathbb{R}^N, \tag{3.3}$$

then the sequence (x_n) is bounded.

Now we conclude our proof of Theorem 2.1 with the following lemma.

Lemma 3.3. *Let $\rho > 0$ be such that $m_\rho < 0$ and $(u_n) \subset H_s$ be a minimizing sequence as given by Lemma 3.2. Then, up to a subsequence, $u_n \rightharpoonup u$ with $J(u) \leq m_\rho$ and $\|u\|_2^2 = \rho$.*

Proof. Taking a minimizing sequence as given in Lemma 3.2, we can assume that $u_n \rightharpoonup u$ in H_s as $n \to \infty$. Also, from the second part of Lemma 3.2, we see that, for any $\varepsilon > 0$, there exists a radius $R(\varepsilon) > 0$ such that

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^N\setminus B(0,R(\varepsilon))} \int_{B(x,1)} |u_n|^2 dx \leq \varepsilon. \tag{3.4}$$

Following the proof of [12, Lemma I.1], we thus have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B(0,R(\varepsilon))} |u_n|^p dx \leq C(\varepsilon), \qquad \text{for any } 2 < p < \frac{2N}{N-2}, \tag{3.5}$$

where $C(\varepsilon) \to 0$ provided that $\varepsilon \to 0$. Now, we fix an arbitrary $\varepsilon > 0$. Because of the compact embedding $H \subset L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \leq p < \frac{2N}{N-2}$, using (3.2), as $n \to \infty$ we obtain

$$\int_{B(0,R(\varepsilon))} F(u_n)dx \to \int_{B(0,R(\varepsilon))} F(u)dx. \tag{3.6}$$

Gathering (3.5) and (3.6), since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_{\mathbb{R}^N} F(u_n)dx \to \int_{\mathbb{R}^N} F(u)dx,$$

as $n \to \infty$. Also, because $\|\cdot\|_X$ is a norm, $\|u\|_X^2 \leq \liminf_{n\to\infty} \|u_n\|_X^2$. Thus we do have

$$J(u) \leq \liminf_{n\to\infty} J(u_n) = m_\rho.$$

Namely (H0) holds. Now if $\|u\|_2^2 = \rho$ we are done. Consequently we assume, by contradiction, that $\|u\|_2^2 < \rho$. Since $J(u) \leq m_\rho < 0$, $u = 0$ is impossible. Thus $0 < \|u\|_2^2 = \lambda$ and we consider the scaling $v(x) = u(t^{-\frac{1}{N}}x)$ for $t > 1$. Clearly for $t = \frac{\rho}{\lambda} > 1$ we have $\|v\|_2^2 = \rho$. Now, since $t > 1$ and $J(u) < 0$,

$$\begin{aligned} J(v) &= \frac{1}{2}t^{1-\frac{2}{N}}\|u\|_X^2 - t\int_{\mathbb{R}^N} F(u) \\ &= t\Big[\frac{1}{2}t^{-\frac{2}{N}}\|u\|_X^2 - \int_{\mathbb{R}^N} F(u)\Big] < tJ(u) < m_\rho. \end{aligned}$$

Thus we reach a contradiction and the proof is complete. □

3.2. **Proof of Proposition 2.2.** We define the Coulomb energy in $\mathbb{R}^3$ by setting

$$\mathbb{D}(u) = \iint_{\mathbb{R}^6} \frac{u^2(x)u^2(y)}{|x-y|}dxdy,$$

for all $u \in H^1(\mathbb{R}^3)$. First we have the following

Lemma 3.4. *Let $u \in H^1(\mathbb{R}^3)$ with $\|u\|^2_{L^2(\mathbb{R}^3)} = c > 0$. There exists a positive constant C, depending only on c, such that*

$$\mathbb{D}(u) \leq C\|u\|_{H^1(\mathbb{R}^3)}.$$

Proof. Combining Hardy-Littlewood-Sobolev inequality (see e.g. Lieb-Loss, Thm 4.3, p.106) with Gagliardo-Nirenberg inequality, yields a positive constant C_0 such that

$$\mathbb{D}(u) \leq C_0\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \leq C_0\|u\|^3_{L^2(\mathbb{R}^3)}\,\|u\|_{H^1(\mathbb{R}^3)} = C_0c^{3/2}\,\|u\|_{H^1(\mathbb{R}^3)}, \tag{3.7}$$

which concludes the proof. □

Secondly, we need the following approximation result.

Lemma 3.5. *Assume that conditions (2.3)-(2.5) hold. Let $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ be given. Then, for any $\varepsilon > 0$ there exists $\tilde{u} \in C_0^\infty(\mathbb{R}^3)$ such that*

$$J(\tilde{u}) \leq J(u) + \varepsilon \quad \textit{and} \quad \|\tilde{u}\|^2_{L^2(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)}.$$

Proof. By density of $C_0^\infty(\mathbb{R}^3)$ into $H^1(\mathbb{R}^3)$ there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^3)$ with $u_n \to u$ in $H^1(\mathbb{R}^3)$, as $n \to \infty$. In particular $\|u\|_{L^2(\mathbb{R}^3)}/\|u_n\|_{L^2(\mathbb{R}^3)} \to 1$, as $n \to \infty$. Thus

$$\left\| u - \frac{\|u\|_{L^2(\mathbb{R}^3)}}{\|u_n\|_{L^2(\mathbb{R}^3)}}u_n \right\|_{H^1(\mathbb{R}^3)} \leq \|u - u_n\|_{H^1(\mathbb{R}^3)} + \left| 1 - \frac{\|u\|_{L^2(\mathbb{R}^3)}}{\|u_n\|_{L^2(\mathbb{R}^3)}} \right| \|u_n\|_{H^1(\mathbb{R}^3)} \to 0,$$

as $n \to \infty$. This proves that there exists a sequence $(\tilde{u}_n) \subset C_0^\infty(\mathbb{R}^3)$ with $\|\tilde{u}_n\|_{L^2(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)}$ such that $\tilde{u}_n \to u$ in $H^1(\mathbb{R}^3)$, as $n \to \infty$. To conclude we just need to prove that $J(\tilde{u}_n) \to J(u)$, as $n \to \infty$. Clearly, by Lemma 3.4, $\mathbb{D}(\tilde{u}_n) \to \mathbb{D}(u)$ (see e.g. estimate (3.10) hereafter). Now, in light of the growth condition (2.4), by the Lebesgue Theorem, we readily get that $\int_{\mathbb{R}^3} j(\tilde{u}_n, |\nabla \tilde{u}_n|)dx \to \int_{\mathbb{R}^3} j(u, |\nabla u|)dx$, as $n \to \infty$. □

Let us now introduce the two variable functional

$$\mathbb{D}(v, w) := \iint_{\mathbb{R}^6} \frac{v^2(x)w^2(y)}{|x-y|}dxdy,$$

for all $v, w \in H^1(\mathbb{R}^3)$. The following inequality holds (see e.g. Lieb-Loss, Thm 9.8, p.250)

$$\mathbb{D}(v, w)^2 \leq \mathbb{D}(v, v)\,\mathbb{D}(w, w), \quad \text{for all } v, w \in H^1(\mathbb{R}^3). \tag{3.8}$$

We can now give the proof of Proposition 2.2.

Proof. Let us fix a positive number c and let $(u_n) \subset H^1(\mathbb{R}^3)$ be a minimizing sequence for $m(c)$, namely $\|u_n\|_2^2 = c$, for all $h \geq 1$, and

$$\int_{\mathbb{R}^3} j(u_n, |\nabla u_n|)dx = m(c) + \mathbb{D}(u_n) + o(1), \quad \text{as } n \to \infty.$$

By virtue of Lemma 3.4 and assumption (2.3), as $n \to \infty$, we have

$$\min\{\nu, 1\}\|u_n\|^2_{H^1(\mathbb{R}^3)} \leq \nu\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)} + c \leq m(c) + C\|u_n\|_{H^1(\mathbb{R}^3)} + c + o(1),$$

so that (u_n) is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, (u_n) weakly converges to some function u in $H^1(\mathbb{R}^3)$. Observe now that, if u_n^* denotes the symmetrically decreasing rearrangement of u_n, for all $n \geq 1$,

$$\iint_{\mathbb{R}^6} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy \leq \iint_{\mathbb{R}^6} \frac{(u_n^2)^*(x)(u_n^2)^*(y)}{|x-y|}dxdy = \iint_{\mathbb{R}^6} \frac{(u_n^*)^2(x)(u_n^*)^2(y)}{|x-y|}dxdy,$$

where we have used the fact that $(u_n^*)^2 = (u_n^2)^*$. For this rearrangement inequality, started with the work of Lieb [9], we refer for instance to [3]. In turn, by taking into account that by [4, Corollary 3.3] we have

$$\int_{\mathbb{R}^3} j(u_n^*, |\nabla u_n^*|)dx \leq \int_{\mathbb{R}^3} j(u_n, |\nabla u_n|)dx,$$

we conclude that $J(u_n^*) \leq J(u_n)$, for all $n \geq 1$. Hence, we may assume that (u_n^*) is a positive (since $J(|v|) \leq J(v)$, for all $v \in H^1(\mathbb{R}^3)$) minimizing sequence for J, which is radially symmetric and radially decreasing. In what follows, we denote it again by (u_n). Taking into account that (u_n) is bounded in $L^2(\mathbb{R}^3)$, it follows that (see [2, Lemma A.IV]) $u_n(x) \leq M|x|^{-3/2}$ for all $x \in \mathbb{R}^3 \setminus \{0\}$ and $n \in \mathbb{N}$, for some constant $M > 0$ and hence (u_n) turns out to be strongly convergent to u in $L^q(\mathbb{R}^3)$ for any $2 < q < 6$. In particular, we have the strong limit

$$u_n \to u \quad \text{in } L^{\frac{12}{5}}(\mathbb{R}^3), \text{ as } n \to \infty. \tag{3.9}$$

We want to show that

$$\mathbb{D}(u_n) \to \mathbb{D}(u), \quad \text{as } n \to \infty.$$

To this end, we use that the Coulomb potential $|x|^{-1}$ is even and write

$$|\mathbb{D}(u_n) - \mathbb{D}(u)| \leq \mathbb{D}(||u_n|^2 - |u|^2|^{1/2}, (|u_n|^2 + |u|^2)^{1/2}).$$

Now, by means of Hardy-Littlewood-Sobolev inequality (see the first line of (3.7)) as well as Hölder's inequality, it follows that (just use inequality (3.8) with $v = v_n = ||u_n|^2 - |u|^2|^{1/2}$ and $w = w_n = (|u_n|^2 + |u|^2)^{1/2}$ for all $n \geq 1$) there exists a constant C with

$$\begin{aligned} |\mathbb{D}(u_n) - \mathbb{D}(u)|^2 &\leq C\| \, ||u_n|^2 - |u|^2|^{1/2}\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \| \, (|u_n|^2 + |u|^2)^{1/2}\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \\ &\leq C\|u_n - u\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)}. \end{aligned} \tag{3.10}$$

This implies, via (3.9), the desired convergence of $\mathbb{D}(u_n)$ to $\mathbb{D}(u)$. Also as $j(s,t)$ is positive, convex and increasing in the second argument (and thus $\xi \mapsto j(s, |\xi|)$ is convex), by well known lower semicontinuity results (cf. [5, 6]) it follows that

$$\int_{\mathbb{R}^N} j(u, |\nabla u|)dx \leq \liminf_n \int_{\mathbb{R}^N} j(u_n, |\nabla u_n|)dx, \tag{3.11}$$

and we can conclude that

$$J(u) \leq \liminf_{n\to\infty} J(u_n).$$

Therefore, condition (H0) is fulfilled.

Now, given a function $w \in C_0^\infty(\mathbb{R}^3)$ with $\|w\|_2^2 = c$, and considering the rescaling $\{t \mapsto w_t\}$ with $w_t(x) = t^{3/2}w(tx)$, we have $\|w_t\|_2^2 = c$ for all $t > 0$ and

$$\mathbb{D}(w_t) = \iint_{\mathbb{R}^6} \frac{w_t^2(x)w_t^2(y)}{|x-y|}dxdy = t^6 \iint_{\mathbb{R}^6} \frac{w^2(tx)w^2(ty)}{|x-y|}dxdy = t\mathbb{D}(w).$$

Hence, taking into account the growth condition (2.4), we conclude

$$\begin{aligned} m(c) &\leq \int_{\mathbb{R}^3} j(w_t, |\nabla w_t|)dx - \mathbb{D}(w_t) \\ &\leq C\int_{\mathbb{R}^3} |w_t|^6 dx + C\int_{\mathbb{R}^3} |\nabla w_t|^2 dx - t\mathbb{D}(w) \\ &= Ct^6 \int_{\mathbb{R}^3} |w|^6 dx + Ct^2 \int_{\mathbb{R}^3} |\nabla w|^2 dx - t\mathbb{D}(w) < 0, \end{aligned}$$

for $t > 0$ sufficiently small. In turn, we have $J(u) \leq m(c) < 0$, which also yields $u \neq 0$. Now, if it was $\|u\|^2_{L^2(\mathbb{R}^3)} = c$, the proof would be over. Otherwise we assume, by contradiction, that $\|u\|^2_{L^2(\mathbb{R}^3)} = \lambda$ with $0 < \lambda < c$. Following the proof that $m(c) < 0$, we see that there exists a function $v \in C_0^\infty(\mathbb{R}^3)$ such that $\|v\|^2_{L^2(\mathbb{R}^3)} = c - \lambda > 0$ and $J(v) < 0$. Also, by Lemma 3.5, it is possible to find a $\tilde{u} \in C_0^\infty(\mathbb{R}^3)$ with $\|\tilde{u}\|^2_{L^2(\mathbb{R}^3)} = \lambda$ and $J(\tilde{u}) \leq J(u) + |J(v)|/2$. Since translating the support of v leaves $J(v)$ unchanged, we can assume that v and $\tilde{u}$ have disjoint supports. Thus

$$\|v + \tilde{u}\|^2_{L^2(\mathbb{R}^3)} = \|v\|^2_{L^2(\mathbb{R}^3)} + \|\tilde{u}\|^2_{L^2(\mathbb{R}^3)} = (c - \lambda) + \lambda = c,$$

as well as

$$J(v + \tilde{u}) \leq J(v) + J(\tilde{u}) \leq J(v) + J(u) - \frac{J(v)}{2} \leq J(u) + \frac{J(v)}{2} < J(u),$$

where, to achieve the first inequality, we have also exploited the fact that

$$\begin{aligned} \mathbb{D}(v + \tilde{u}) &= \iint_{\mathbb{R}^6} \frac{(v+\tilde{u})^2(x)(v+\tilde{u})^2(y)}{|x-y|}dxdy \\ &\geq \iint_{(K\times K)\cup(Q\times Q)} \frac{(v+\tilde{u})^2(x)(v+\tilde{u})^2(y)}{|x-y|}dxdy \\ &= \iint_{K\times K} \frac{v^2(x)v^2(y)}{|x-y|}dxdy + \iint_{Q\times Q} \frac{\tilde{u}^2(x)\tilde{u}^2(y)}{|x-y|}dxdy = \mathbb{D}(v) + \mathbb{D}(\tilde{u}), \end{aligned}$$

where $K \subset \mathbb{R}^3$ and $Q \subset \mathbb{R}^3$ denote the (disjoint) supports of v and $\tilde{u}$, respectively. The last equality is justified since (for v supported on K, similarly for $\tilde{u}$ supported on Q)

$$\begin{aligned}\mathbb{D}(v) &= \int_{\mathbb{R}^3\setminus K}\int_K \frac{v^2(x)v^2(y)}{|x-y|}dxdy + \int_K\int_{\mathbb{R}^3\setminus K} \frac{v^2(x)v^2(y)}{|x-y|}dxdy \\ &+ \int_{\mathbb{R}^3\setminus K}\int_{\mathbb{R}^3\setminus K} \frac{v^2(x)v^2(y)}{|x-y|}dxdy + \int_K\int_K \frac{v^2(x)v^2(y)}{|x-y|}dxdy \\ &= \iint_{K\times K} \frac{v^2(x)v^2(y)}{|x-y|}dxdy.\end{aligned}$$

With the above conclusions, the proof is now complete. □

3.3. **Proof of Theorem 2.3.** We divide the proof into three steps. The first part of the proof (Step I), aiming to prove that condition (H0) holds, follows the pattern of the proof of [4, Theorem 4.5]. For the sake of completeness we report here some of the arguments. Instead, the last part of the proof (Steps II and III) contains the main elements of novelty and improvement with respect to [4, Theorem 4.5].

Step I. [Verification of (H0)] Let $u^h = (u_1^h, \dots, u_m^h) \subset \mathcal{C}$ be a minimizing sequence for the functional J. Then

$$\lim_h \Big(\sum_{k=1}^m \int_{\mathbb{R}^N} j_k(u_k^h, |\nabla u_k^h|)dx - \int_{\mathbb{R}^N} F(|x|, u_1^h, \dots, u_m^h)dx \Big) = T_c, \tag{3.12}$$

$$G_k(u_k^h),\ j_k(u_k^h, |\nabla u_k^h|) \in L^1(\mathbb{R}^N), \quad \sum_{k=1}^m \int_{\mathbb{R}^N} G_k(u_k^h)dx = c, \quad \text{for all } h \in \mathbb{N}.$$

In light of (2.11) and (2.17), we obtain $J(|u_1^h|, \dots, |u_m^h|) \le J(u_1^h, \dots, u_m^h)$ for all $h \in \mathbb{N}$, so we may assume, without loss of generality, that $u_k^h \ge 0$ a.e. in $\mathbb{R}^N$, for all $k = 1, \dots, m$ and $h \in \mathbb{N}$. For any $k = 1, \dots, m$ and $h \in \mathbb{N}$, we denote by u_k^{*h} the Schwarz symmetric rearrangement of u_k^h. By means of [3, Theorem 1], we have

$$\int_{\mathbb{R}^N} F(|x|, u_1^h, \dots, u_m^h)dx \le \int_{\mathbb{R}^N} F(|x|, u_1^{*h}, \dots, u_m^{*h})dx.$$

Moreover, by [4, Corollary 3.3], we know that

$$\int_{\mathbb{R}^N} j_k(u_k^{*h}, |\nabla u_k^{*h}|)dx \le \int_{\mathbb{R}^N} j_k(u_k^h, |\nabla u_k^h|)dx.$$

Finally, $u^{*h} \in \mathcal{C}$. Hence, since $J(u^{*h}) \le J(u^h)$, $u^{*h} \in \mathcal{C}$, for $h \in \mathbb{N}$, it follows that $u^{*h} = (u_1^{*h}, \dots, u_m^{*h})$ is a positive minimizing sequence for $J|_{\mathcal{C}}$, which is radially symmetric and radially decreasing. In what follows, we denote it again $u^h = (u_1^h, \dots, u_m^h)$. Let us now prove that (u^h) is bounded in $W^{1,p}(\mathbb{R}^N, \mathbb{R}^m)$. Since $(u^h) \subset \mathcal{C}$, by assumption (2.18) on G_k, the sequence (u^h) is bounded in $L^p(\mathbb{R}^N)$. Now we see, from (2.14)-(2.15), that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with

$$F(r, s_1, \dots, s_m) \le C_\varepsilon \sum_{k=1}^m s_k^p + \varepsilon \sum_{k=1}^m s_k^{p+\frac{p^2}{N}}, \quad \text{for all } r, s_1, \dots, s_m \in]0, \infty[. \tag{3.13}$$

Therefore, in view of the Gagliardo-Nirenberg inequality

$$\|u_k^h\|_{L^{p+\frac{p^2}{N}}(\mathbb{R}^N)}^{p+\frac{p^2}{N}} \leq C\|u_k^h\|_{L^p(\mathbb{R}^N)}^{\frac{p^2}{N}}\|\nabla u_k^h\|_{L^p(\mathbb{R}^N)}^p, \tag{3.14}$$

the boundedness of (u^h) in $W^{1,p}(\mathbb{R}^N,\mathbb{R}^m)$ follows from (2.8) and (3.12).
Now, after extracting a subsequence, still denoted by (u^h), for any $k=1,\dots,m$, we have

$$u_k^h \rightharpoonup u_k \text{ in } L^{p^*}(\mathbb{R}^N),\ \nabla u_k^h \rightharpoonup \nabla u_k \text{ in } L^p(\mathbb{R}^N),\ u_k^h(x)\to u_k(x) \quad \text{a.e. } x\in\mathbb{R}^N. \tag{3.15}$$

Taking into account that u_k^h is bounded in $L^p(\mathbb{R}^N)$, it follows that (see [2, Lemma A.IV]) $u_k^h(x)\leq c_k|x|^{-N/p}$ for all $x\in\mathbb{R}^N\setminus\{0\}$ and $h\in\mathbb{N}$, for a positive constant c_k, independent of h. In turn, by virtue of condition (2.14), for all $\varepsilon>0$ there exists $\rho_\varepsilon>0$ such that

$$|F(|x|,u_1^h(|x|),\dots,u_m^h(|x|))|\leq \varepsilon\sum_{k=1}^m |u_k^h(|x|)|^p, \quad \text{for all } x\in\mathbb{R}^N \text{ with } |x|\geq\rho_\varepsilon.$$

Hence, it is easy to see that

$$\int_{\mathbb{R}^N\setminus B(0,\rho_\varepsilon)} F(|x|,u_1^h,\dots,u_m^h)dx\leq\varepsilon C, \quad \int_{\mathbb{R}^N\setminus B(0,\rho_\varepsilon)} F(|x|,u_1,\dots,u_m)dx\leq\varepsilon C.$$

In turn, one readily obtains

$$\lim_h \int_{\mathbb{R}^N} F(|x|,u_1^h,\dots,u_m^h)dx = \int_{\mathbb{R}^N} F(|x|,u_1,\dots,u_m)dx. \tag{3.16}$$

Recalling (3.12) and (3.16) and since

$$\int_{\mathbb{R}^N} j_k(u_k,|\nabla u_k|)dx\leq \liminf_h \int_{\mathbb{R}^N} j_k(u_k^h,|\nabla u_k^h|)dx, \tag{3.17}$$

we have

$$j_k(u_k,|\nabla u_k|)\in L^1(\mathbb{R}^N), \quad \text{for any } k. \tag{3.18}$$

From (3.16) and (3.17) it follows

$$J(u)\leq\liminf_h J(u^h)=\lim_h J(u^h)=T_c. \tag{3.19}$$

Finally, by Fatou's lemma, we have

$$\sum_{k=1}^m\int_{\mathbb{R}^N} G_k(u_k)dx\leq\liminf_{h\to\infty}\sum_{k=1}^m\int_{\mathbb{R}^N} G_k(u_k^h)dx=c.$$

In particular $G_k(u_k)\in L^1(\mathbb{R}^N)$. At this point (H0) is established.

Step II. $[T_c<0]$ Let, say, $\gamma:\mathbb{R}^+\to\mathbb{R}^+$ be given by

$$\gamma(r)=\begin{cases}1 & \text{if } r\in[0,1]\\ 2-r & \text{if } r\in[1,2]\\ 0 & \text{if } r\in[2,\infty[.\end{cases}$$

Because of (2.18) we have $\int G_1 \circ \gamma(|x|)dx > 0$. Thus, setting $q = [\int G_1 \circ \gamma(|x|)]^{-1/N}$, the function

$$\Upsilon^c(x) = \gamma\Big(\frac{|x|}{\sigma_c}\Big), \qquad \text{with } \sigma_c = qc^{1/N} \tag{3.20}$$

satisfies

$$\int_{\mathbb{R}^N} G_1(\Upsilon^c)dx = c. \tag{3.21}$$

Let us now consider the scaling $\Upsilon^c_\theta(x) := \theta^{N/p}\Upsilon^c(\theta x)$. We may assume, without loss of generality, that $j = 1$ in (2.16), namely that there exist $\delta > 0, \mu > 0$ and $\sigma \in [0, \frac{p^2}{N}[$ with

$$F(r, s_1, 0, \dots, 0) \geq \mu s_1^{\sigma+p}, \quad \text{for any } s_1 \in [0, \delta]. \tag{3.22}$$

Since G_1 is a p-homogeneous function, for all $\theta \in]0, 1]$, we get

$$\int_{\mathbb{R}^N} G_1(\Upsilon^c_\theta(x))dx = \theta^N \int_{\mathbb{R}^N} G_1(\Upsilon^c(\theta x))dx = \int_{\mathbb{R}^N} G_1(\Upsilon^c)dx = c.$$

Taking $\theta > 0$ small enough so that $\theta^{N/p}\Upsilon^c(\theta x) \leq \theta^{N/p} \leq \delta$ (recall that $\gamma(r) \leq 1$) for all $x \in \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} F(|x|, \theta^{N/p}\Upsilon^c(\theta x), 0, \dots, 0)dx \geq \mu \int_{\mathbb{R}^N} \theta^{N(\sigma+p)/p}|\Upsilon^c(\theta x)|^{\sigma+p}dx = \mu\theta^{N\sigma/p} \int_{\mathbb{R}^N} |\Upsilon^c|^{\sigma+p}dx.$$

In turn, taking into account that $\{s \mapsto j_1(s, |\xi|)\}$ is increasing, that $\theta^{N/p}\Upsilon^c(\theta x) \leq \Upsilon^c(\theta x)$ for all $\theta \in]0, 1]$, and the p-homogeneity of the map $\{t \mapsto j(s, t)\}$, we obtain

$$\begin{aligned}
&J(\Upsilon^c_\theta(x), 0, \dots, 0) \\
&= \int_{\mathbb{R}^N} j_1(\Upsilon^c_\theta(x), |\nabla\Upsilon^c_\theta(x)|)dx - \int_{\mathbb{R}^N} F(|x|, \Upsilon^c_\theta(x), 0, \dots, 0)dx \\
&= \int_{\mathbb{R}^N} j_1(\theta^{N/p}\Upsilon^c(\theta x), \theta^{N/p+1}|(\nabla\Upsilon^c)(\theta x)|)dx - \int_{\mathbb{R}^N} F(|x|, \theta^{N/p}\Upsilon^c(\theta x), 0, \dots, 0)dx \\
&\leq \theta^{N+p} \int_{\mathbb{R}^N} j_1(\Upsilon^c(\theta x), |(\nabla\Upsilon^c)(\theta x)|)dx - \mu\theta^{N\sigma/p} \int_{\mathbb{R}^N} |\Upsilon^c|^{\sigma+p}dx \\
&= \theta^p C_1(c) - \theta^{N\sigma/p}C_2(c) = \theta^p[C_1(c) - \theta^{(N\sigma-p^2)/p}C_2(c)],
\end{aligned}$$

where we have set

$$C_1(c) := \int_{\mathbb{R}^N} j_1(\Upsilon^c, |\nabla\Upsilon^c|)dx, \qquad C_2(c) := \mu \int_{\mathbb{R}^N} |\Upsilon^c|^{\sigma+p}dx. \tag{3.23}$$

Now, fixing $\theta > 0$ sufficiently small (depending upon c), we have

$$\theta^p[C_1(c) - \theta^{(N\sigma-p^2)/p}C_2(c)] := -\delta, \tag{3.24}$$

with $\delta > 0$ (depending upon c) and, setting $v := (\Upsilon^c_\theta, 0, \dots, 0)$, we conclude that

$$\sum_{k=1}^m \int_{\mathbb{R}^N} G_k(v_k) = \int_{\mathbb{R}^N} G_1(\Upsilon^c_\theta)dx = c, \qquad J(v) \leq -\delta. \tag{3.25}$$

At this point the fact that $T_c < 0$ is established. Notice that $(u_1, \ldots, u_m) \neq (0, \ldots, 0)$, otherwise we would immediately get a contradiction combining (3.19) and $T_c < 0$. We now define

$$\zeta := \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k)dx \in]0, c].$$

If it was $\zeta = c$, then $(u_1, \ldots, u_m)$ would belong to the constraint $\mathcal{C}$ and we would be done. We thus assume that $\zeta < c$ and look for a contradiction.

Step III. [Conclusion] First we show that u can suitably be approximated by compactly supported functions. Let $\phi \in C_0^\infty(\mathbb{R}^+)$ be radial, decreasing, and such that $\phi(s) = 1$ on $0 \leq s \leq 1$, $\phi(s) = 0$ for $s \geq 2$ and $0 \leq \phi(s) \leq 1$ for all $s \in \mathbb{R}^+$. Let $\phi_n(x) = \phi(\frac{|x|}{n})$, for $n \in \mathbb{N}$, and consider the sequence $u_n = (u_1\phi_n, u_2\phi_n, \ldots, u_m\phi_n)$. We claim that, for any $k = 1, \ldots, m$,

$$\int_{\mathbb{R}^N} j_k(u_k\phi_n, |\nabla(u_k\phi_n)|)dx \to \int_{\mathbb{R}^N} j_k(u_k, |\nabla u_k|)dx.$$

Indeed, $u_k\phi_n \to u_k$ as well as $\nabla(u_k\phi_n) \to \nabla u_k$ as $n \to \infty$ a.e. on $\mathbb{R}^N$, for all $k = 1, \ldots, m$. Moreover, in light of (3.18),

$$j_k(u_k, |\nabla u_k|) \in L^1(\mathbb{R}^N), \quad \text{for any } k = 1, \ldots, m.$$

Then, in light of condition (2.10) (see also (2.9)), by the convexity, monotonicity (in both arguments) and the p-homogeneity assumptions on j_k, we obtain

$$\begin{aligned} j_k(u_k\phi_n, |\nabla(u_k\phi_n)|) &= j_k(u_k\phi_n, |\frac{1}{2}\nabla u_k 2\phi_n + \frac{1}{2}\nabla\phi_n 2u_k|) \\ &\leq \frac{1}{2} j_k(u_k, |2\phi_n \nabla u_k|) + \frac{1}{2} j_k(u_k, |2u_k \nabla\phi_n|) \\ &\leq C j_k(u_k, |\nabla u_k|) + C\beta_k(u_k)|u_k|^p \\ &\leq C j_k(u_k, |\nabla u_k|) + C(1 + |u_k|^{p^*-p})|u_k|^p \\ &\leq C j_k(u_k, |\nabla u_k|) + C|u_k|^p + C|u_k|^{p^*} \in L^1(\mathbb{R}^N), \end{aligned}$$

where the positive constant C can change from term to term. Thus Lebesgue's Theorem gives the claim. Similarly, from (2.14) and (2.15) (see e.g. formula (3.13)),

$$\int_{\mathbb{R}^N} F(|x|, u_1\phi_n, \ldots, u_m\phi_n)dx \to \int_{\mathbb{R}^N} F(|x|, u_1, \ldots, u_m)dx$$

and thus,

$$J(u_n) \to J(u). \tag{3.26}$$

Finally, since for each $k = 1, \ldots, m$, the functions G_k are increasing we have, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} G_k(u_k\phi_n)dx \leq \int_{\mathbb{R}^N} G_k(u_k)dx.$$

Also, by Fatou's Lemma,

$$\int_{\mathbb{R}^N} G_k(u_k)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} G_k(u_k\phi_n)dx$$

and thus, for each $k = 1, \dots, m$,

$$\int_{\mathbb{R}^N} G_k(u_k\phi_n)dx \to \int_{\mathbb{R}^N} G_k(u_k)dx.$$

In conclusion,

$$\sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k\phi_n)dx \to \zeta, \quad \text{as } n \to \infty. \tag{3.27}$$

Let us now show that, if a positive number d stays inside the close interval

$$Q_c = [\frac{c}{2} - \frac{\zeta}{2}, c - \frac{\zeta}{2}],$$

then the corresponding value of $\delta > 0$ given in (3.24) (where c is replaced by d) can be chosen independently of d. Indeed, the constants $C_1(d), C_2(d)$ can be given an explicit representation formula using (3.20) and scaling σ_d inside (3.23), namely

$$C_1(d) = \Big[q^{N-p} \int_{\mathbb{R}^N} j_1(\gamma(|x|), |\gamma'(|x|)|)dx\Big] d^{\frac{N-p}{N}}, \qquad C_2(d) = \Big[\mu q^N \int_{\mathbb{R}^N} |\gamma(|x|)|^{\sigma+p}dx\Big] d.$$

Then, $C_1(d), C_2(d)$ can be bounded above and below on Q_c, yielding (cf. formula (3.24))

$$\theta^p[C_1(d) - \theta^{(N\sigma-p^2)/p}C_2(d)] \le \theta^p[M_1(c) - \theta^{(N\sigma-p^2)/p}M_2(c)] := -\delta_c < 0,$$

for a fixed θ sufficiently small (depending merely upon c), where we have set

$$M_1(c) := \max_{d\in Q_c} C_1(d) > 0, \quad M_2(c) := \min_{d\in Q_c} C_2(d) > 0.$$

Now, by virtue of (3.26) and (3.27) we can fix an index $n_0 \in \mathbb{N}$ sufficiently large that

$$J(u_{n_0}) \le J(u) + \frac{\delta_c}{2} \quad \text{and} \quad \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k\phi_{n_0})dx \in [\frac{\zeta}{2}, \frac{\zeta}{2} + \frac{c}{2}] \tag{3.28}$$

Let

$$d_c := c - \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_k\phi_{n_0})dx \in [\frac{c}{2} - \frac{\zeta}{2}, c - \frac{\zeta}{2}]$$

and v be the corresponding function, depending on d_c, which satisfies (3.25). Translating v if necessary we can assume that $u_{n_0} = (u_1\phi_{n_0}, \dots, u_m\phi_{n_0})$ and $v = (v_1, 0, \dots, 0)$ have disjoints supports (namely the union of the supports of the $u_i\phi_{n_0}$'s for $i = 1, \dots, m$ is disjoint to the support of v_1). The translation is possible in light of assumption (2.16). Now, of course

$$\sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_{n_0} + v)dx = \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(u_{n_0})dx + \sum_{k=1}^{m} \int_{\mathbb{R}^N} G_k(v_k)dx = c$$

and

$$J(u_{n_0} + v) = J(u_{n_0}) + J(v) \le J(u) + \frac{\delta_c}{2} - \delta_c < J(u).$$

This contradiction concludes the proof.

3.4. **Proof of Propositions 2.7, 2.8 and 2.9.** First we state some known facts.

Lemma 3.6. *Assume that* (2.23)-(2.26) *hold. Then we have*

1) *Any minimizing sequence for* (2.22) *is bounded in* $H^1(\mathbb{R}^N)$.
2) *Any minimizing sequence satisfies, up to a subsequence,* (H0).
3) $m(d) < 0$ *for any* $d > 0$.

Proof. The proof of these statements can be found in [15], up to straightforward modifications at some places. We just outline here the main steps. Assertion 1) is a direct consequence of (2.23) combined with standard Hölder and Sobolev inequalities. Assertion 2) holds true because of the limit (2.24) (see, for instance, [15, Lemma 5.2] for such a result). Assertion 3) can be proved using suitable test functions and taking advantage that, under (2.26), $F(x, s)$ does not decrease too fast as $|x|$ goes to infinity (see [15, Theorem 5.4]). □

The proof of Proposition 2.7 relies on the following two lemmas.

Lemma 3.7. *Assume that* (2.23)-(2.26) *hold. Then, for any* $d > 0$*, any* $\varepsilon > 0$ *and all* $R_0 > 0$ *there exists a function* $v \in C_0^\infty(\mathbb{R}^N)$ *such that*

$$\|v\|_2^2 = d, \qquad \mathrm{supp}(v) \subset \mathbb{R}^N \setminus B(0, R_0), \qquad I(v) \le \varepsilon.$$

Proof. Take a positive function $u \in C_0^\infty(\mathbb{R}^N)$ such that $\|u\|_2^2 = d$. Then, considering the scaling $t \mapsto t^{\frac{N}{2}} u(tx) = u_t(x)$, for all $t > 0$, we get

$$\int_{\mathbb{R}^N} |u_t|^2 dx = d, \qquad \int_{\mathbb{R}^N} |\nabla u_t|^2 dx = t^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Since $\|u_t\|_\infty \to 0$ as $t \to 0^+$, given $\varepsilon > 0$, we can fix a value $t_0 > 0$ such that

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{t_0}|^2 dx \le \varepsilon \qquad \text{and} \qquad \|u_{t_0}\|_\infty \le \delta,$$

where $\delta > 0$ is the number which appears in condition (2.26). Translate now u_{t_0} into $\tilde{u}_{t_0}(\cdot) = u_{t_0}(\cdot + y)$ for a suitable $y \in \mathbb{R}^N$ in such a way that

$$\mathrm{supp}(\tilde{u}_{t_0}) \subset \mathbb{R}^N \setminus B(0, R_0).$$

Then, since in view of (2.26), $F(x, s) \ge 0$ for all $|x|$ sufficiently large and for $s \in [0, \delta]$, we obtain

$$\int_{\mathbb{R}^N} F(x, u_{t_0}) dx \ge 0.$$

Thus

$$I(\tilde{u}_{t_0}) \le \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_{t_0}|^2 dx \le \varepsilon,$$

and $v := \tilde{u}_{t_0}$ has all the desired properties. □

Lemma 3.8. *Assume that* (2.23)-(2.26) *hold and let* $u \in C_0^\infty(\mathbb{R}^N)$ *be such that* $\|u\|_2^2 < c$*. Then, for any* $\varepsilon > 0$*, there exists a function* $v \in C_0^\infty(\mathbb{R}^N)$ *such that*

$$I(u + v) \le I(u) + \varepsilon, \qquad \|u + v\|_2^2 = c.$$

Proof. Let $\varepsilon > 0$ be fixed. By Lemma 3.7 we learn that there exists a function $v \in C_0^\infty(\mathbb{R}^N)$ with $\|v\|_2^2 = c - \|u\|_2^2 > 0$ and such that (since the supports of u and v can be assumed to be disjoint)

$$\|u + v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 = c,$$

and

$$I(u + v) = I(u) + I(v) \le I(u) + \varepsilon.$$

This concludes the proof. □

We can now give the proof of Proposition 2.7.

Proof. We know by Lemma 3.6 that $m(c) < 0$ for any $c > 0$. Now assume by contradiction that there exist $0 < c_1 < c_2$ such that $m(c_1) < m(c_2)$ and set $m(c_2) - m(c_1) = \delta > 0$. By definition of $m(c_1)$ there exists a $u_{c_1} \in H^1(\mathbb{R}^N)$ such that $\|u_{c_1}\|_2^2 = c_1$ and $I(u_{c_1}) \le m(c_1) + \frac{\delta}{4}$. Arguing as in Lemma 3.5, where we can directly use the continuity of the functional I, we can assume that $u_{c_1} \in C_0^\infty(\mathbb{R}^N)$. Now, by Lemma 3.8, since $\|u_{c_1}\|_2^2 < c_2$, we can find a function $v \in C_0^\infty(\mathbb{R}^N)$ such that

$$I(u_{c_1} + v) \le I(u_{c_1}) + \frac{\delta}{4}$$

and $\|u_{c_1} + v\|_2^2 = c_2$. Then we get that

$$I(u_{c_1} + v) \le m(c_1) + \frac{\delta}{2} < m(c_2).$$

This contradiction proves Proposition 2.7. □

Remark 3.9. Assume that conditions (2.23)-(2.26) hold and let $u \in H^1(\mathbb{R}^N)$ be a function such that $\|u\|_2^2 \le c$ and $I(u) \le m(c) < 0$ (such a u comes from a weakly convergent minimizing sequence (u_n) over which the functional I is lower semicontinuous). Then $u \in H^1(\mathbb{R}^N)$ minimizes I on the constraint $d := \|u\|_2^2 > 0$. Indeed if there exists $v \in H^1(\mathbb{R}^N)$ with $\|v\|_2^2 = \|u\|_2^2 = d$ and $I(v) < I(u)$ we get a contradiction since, by Proposition 2.7, the map $\lambda \mapsto m(\lambda)$ is non increasing.

We now give the proof of Proposition 2.8, which covers the case $N = 1$.

Proof. Let $(u_n) \subset H^1(\mathbb{R})$ be a positive minimizing sequence for problem (2.22). This is possible by (2.25). From Lemma 3.6, we can assume that $u_n \rightharpoonup u$ with $u \ge 0$ and $I(u) \le m(c) < 0$. To conclude, we need to show that $\|u\|_2^2 = c$. Since $I(u) < 0$, we have $u \ne 0$. Thus assume by contradiction that $0 < \|u\|_2^2 < c$. We distinguish two cases according to the fact that there exists, or not, a point $x_0 \in \mathbb{R}$ such that $u(x_0) > 0$ and u is non-increasing over $[x_0, +\infty[$. We also recall that elements of $H^1(\mathbb{R})$ are continuous functions which vanish as $|x| \to \infty$.

Case I. We assume that there exists a $x_0 \in \mathbb{R}$ such that $u(x_0) > 0$ and u is non-increasing over $[x_0, +\infty[$. Since $u(x) \to 0$ as $|x| \to \infty$, without loss of generality, we may assume that $u(x) \in [0, \delta]$, for all $x \in [x_0, +\infty[$. Now we define a function $w : \mathbb{R} \to \mathbb{R}$ by

$$w(x) := \begin{cases} u(x) & \text{if } x \in]-\infty, x_0], \\ u(x_0) & \text{if } x \in [x_0, x_0 + \mu], \\ u(x-\mu) & \text{if } x \in [x_0 + \mu, +\infty[. \end{cases}$$

Here $\mu > 0$ is chosen in order to have $\|w\|_2^2 = c$. Clearly $\|w'\|_2^2 = \|u'\|_2^2$. We now split the integral as

$$\int_{\mathbb{R}} F(x, w)dx = \int_{]-\infty, x_0[} F(x, u)dx + \int_{[x_0, +\infty[} F(x, w)dx.$$

By construction of w and using the fact that u is non-increasing on $[x_0, +\infty[$ we have $w \geq u_1$ on $[x_0, +\infty[$. Also, since $u(x) \to 0$ as $|x| \to \infty$ and is continuous we necessarily have that $w > u$ on a set of positive measure. Thus, from the monotonicity condition (2.28), we have that

$$\int_{[x_0, +\infty[} F(x, w)dx > \int_{[x_0, +\infty[} F(x, u)dx.$$

We then deduce that

$$\int_{\mathbb{R}} F(x, w)dx > \int_{\mathbb{R}} F(x, u)dx.$$

Thus $I(w) < I(u)$ and, since $\|w\|_2^2 = c$, we have reached a contradiction.

Case II. In this case there is no point $x_0 \in \mathbb{R}$ such that $u(x_0) > 0$ and u is non-increasing on $[x_0, +\infty[$. In this situation, necessarily, the following occurs: there exists $x_1, x_2 \in [x_0, +\infty[$ with $x_1 < x_2$ such that $u(x) < u(x_1) = u(x_2)$ for $x \in]x_1, x_2[$. Now we define $w : \mathbb{R} \to \mathbb{R}$ by setting

$$w(x) := \begin{cases} u(x) & \text{if } x \in]-\infty, x_1], \\ u(x_1) & \text{if } x \in [x_1, x_2], \\ u(x) & \text{if } x \in [x_2, +\infty[. \end{cases}$$

Then $w \in H^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} |w'|^2 dx < \int_{\mathbb{R}} |u'|^2 dx$$

and also, by (2.28),

$$\int_{\mathbb{R}} F(x, w)dx > \int_{\mathbb{R}} F(x, u)dx.$$

Now observe that the points x_1, x_2 can be chosen such that

$$\int_{[x_1, x_2]} |u(x_1)|^2 - |u(x)|^2 dx > 0$$

is smaller than $c - \|u\|_2^2 > 0$. Then $I(w) < I(u)$ and $\|w\|_2^2 = d < c$, so that the conclusion follows by Proposition 2.7. $\square$

Before proving Proposition 2.9 we show, under our additional regularity assumptions, that any minimizer satisfies a Euler-Lagrange equation and we discuss the value of the associated Lagrange parameter.

Lemma 3.10. *Assume that $f(x,s) = F_s(x,s)$ exists and that (2.24)-(2.25) and (2.29) hold. Then $I \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and we have*

i) *Any minimizer $v \in H^1(\mathbb{R}^N)$ of I on $\|v\|_2^2 = c$ satisfies*

$$-\Delta v - f(x,v) = \beta v, \qquad \text{with } \beta = \frac{I'(v)v}{\|v\|_2^2} \leq 0.$$

ii) *Let $(u_n) \subset H^1(\mathbb{R}^N)$ with $\|u_n\|_2^2 = c$ be such that $u_n \rightharpoonup u$ with $I(u) \leq m(c) < 0$ and $0 < \|u\|_2^2 < c$. Then u satisfies the equation*

$$-\Delta u - f(x,u) = 0. \tag{3.29}$$

Proof. Assuming that $f(x,s) = F_s(x,s)$ exists and under (2.24)-(2.25) and (2.29) it is classical to show that I is a C^1-functional (see [15]). Thus, by standard considerations, any minimizer of I on the constraint $\|v\|_2^2 = c$ satisfies

$$-\Delta v - f(x,v) = \beta v, \quad \text{where } \beta \text{ is given by } \beta = \frac{I'(v)v}{\|v\|_2^2}. \tag{3.30}$$

Now assume by contradiction that $\beta > 0$. Then $I'(v)v = \beta\|v\|_2^2 > 0$ and thus, since one has,

$$I((1-t)v) = m(c) - t(I'(v)v + o(1)) \quad \text{as } t \to 0, \tag{3.31}$$

we can fix a small $t_0 > 0$ such that $v_0 = (1-t_0)v$ satisfies $I(v_0) < m(c)$. Since $\|v_0\|_2^2 < c$ we have a contradiction with Proposition 2.7 which says that $\lambda \to m(\lambda)$ is non increasing. This proves i). Now assume that the assumptions of ii) hold. By Remark 3.9 the weak limit $u \in H^1(\mathbb{R}^N)$ minimizes I on the constraint $\|u\|_2^2 := d < c$ (and $m(d) = m(c)$). Also, by Part i) we know that the associated Lagrange multiplier $\beta \in \mathbb{R}$ satisfies $\beta \leq 0$. Let us prove that $\beta < 0$ is impossible. If we assume, by contradiction, that $\beta < 0$ then $I'(u)u < 0$ and since one has

$$I((1+t)u) = m(c) + t(I'(u)u + o(1)) \quad \text{as } t \to 0, \tag{3.32}$$

we can fix a small $t_0 > 0$ such that $u_0 = (1+t_0)v$ satisfies both $I(u_0) < m(c)$ and $\|u_0\|_2^2 < c$. Here again this provides a contradiction with the fact that $\lambda \to m(\lambda)$ is non increasing. □

We can now give the proof of Proposition 2.9.

Proof. Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a positive minimizing sequence for (2.22). Choosing a positive minimizing sequence is possible since, using (2.25), we have

$$\int_{\mathbb{R}^N} F(x,v)dx \leq \int_{\mathbb{R}^N} F(x,|v|)dx$$

for any $v \in H^1(\mathbb{R}^N)$ and thus also $I(|v|) \leq I(v)$. From Lemma 3.6 we can assume that $u_n \rightharpoonup u$ with $u \geq 0$ and $I(u) \leq m(c) < 0$. To conclude we need to show that $\|u\|_2^2 = c$. Since $I(u) < 0$ we have $u \neq 0$. Thus assume by contradiction that $0 < \|u\|_2^2 < c$. In turn, from Part ii) of Lemma 3.10, we learn that $u \in H^1(\mathbb{R}^N)$ satisfies equation (3.29). Therefore, taking into account (2.30), we see that u is a weak solution of the variational inequality

$$-\Delta u \geq b(x)u^{1+\alpha} \quad \text{in } \mathbb{R}^N,$$

where $b : \mathbb{R}^N \to \mathbb{R}^+$ is defined by

$$b(x) = \begin{cases} \frac{f(x,u(x))}{u^{1+\alpha}(x)} & \text{if } |x| \leq r_0 \text{ and } u(x) > 0, \\ A(1+|x|)^{-d} & \text{if } |x| \geq r_0 \text{ and } u(x) > 0, \\ 1 & \text{if } u(x) = 0, \end{cases}$$

being r_0, d and α the positive numbers appearing in (2.30). Now, from the Liouville type theorem [14, Theorem 3.1, Chapter I], we know that $u \equiv 0$ under the restrictions on the values of α given in condition (2.30) (notice that only the behaviour of $b(x)$ for large values of $|x|$, and hence the behaviour of the weight $|x|^{-d}$, determines the validity of the result from [14] (see [14, formulas (3.4) and (3.5)]). This immediately provides us a contradiction, since $u \not\equiv 0$. □

Remark 3.11. From our study of (2.22) we can derive bifurcation results for the equation

$$-\Delta u + \beta u = f(x,u), \quad u \in H^1(\mathbb{R}^N), \ \ \beta \in \mathbb{R}. \tag{3.33}$$

We recall that $\beta = 0$ is a bifurcation point if there exists a sequence $(\beta_n, u_n) \subset \mathbb{R} \times H^1(\mathbb{R}^N)\backslash\{0\}$ of solutions of (3.33) such that $\beta_n \to 0$ and $\|u_n\|_{H^1(\mathbb{R}^N)} \to 0$ as $n \to \infty$.

Let $(c_n) \subset]0, +\infty[$ be such that $c_n \to 0$. Under the assumptions that $f(x,s)$ exists and that (2.24)-(2.26) and (2.29) hold, we immediately derive, from Remark 3.9 and Part i) of Lemma 3.10, the existence of a sequence $(\beta_n, u_n) \subset [0, +\infty[\times H^1(\mathbb{R}^N)\backslash\{0\}$ such that (β_n, u_n) satisfies (3.33) with $0 < \|u_n\|_2^2 \leq c_n$. From this it is standard to show that $\beta_n \to 0$ and $\|u_n\|_{H^1(\mathbb{R}^N)} \to 0$ as $n \to \infty$ (see [15]). If instead of (2.26) we require assumption (2.30), we know, in addition, that $(\beta_n) \subset]0, +\infty[$ and that $\|u_n\|_2^2 = c_n$. The fact that $\|u_n\|_2^2 = c_n$ follows directly from Proposition 2.9 and Part i) of Lemma 3.10. To exclude the possibility that that $\beta_n = 0$ (thus showing that the bifurcation occurs by regular values) one can argue as in the proof of Proposition 2.9. These bifurcations results are obtained under conditions that we believe nearly optimal and which should be compared to the ones of [15], [7] and [8].

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