A remark on least energy solutions in $\mathbf{R}^N$

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Abstract: We study a mountain pass characterization of least energy solutions of the following nonlinear scalar field equation in $\mathbf{R}^N$:

$$-\Delta u = g(u),\ u \in H^1(\mathbf{R}^N),$$

where $N \geq 2$. Without the assumption of the monotonicity of $t \mapsto \frac{g(t)}{t}$, we show that the Mountain Pass value gives the least energy level.

0. Introduction

In this note we study the following nonlinear scalar field equations in $\mathbf{R}^N$:

$$-\Delta u = g(u),\ u \in H^1(\mathbf{R}^N), \tag{0.1}$$

where $N \geq 2$. In particular, our aim is to enlighten a mountain pass characterization of least energy solutions. We recall that a solution $\omega(x)$ of (0.1) is said to be a *least energy solution* if and only if

$$I(\omega) = m,\ \text{ where } m = \inf\{I(u);\ u \in H^1(\mathbf{R}^N) \setminus \{0\} \text{ is a solution of (0.1)}\}. \tag{0.2}$$

Here $I : H^1(\mathbf{R}^N) \to \mathbf{R}$ is the *natural* functional corresponding to (0.1)

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \int_{\mathbf{R}^N} G(u)\, dx, \tag{0.3}$$

where $G(s) = \int_0^s g(\tau)\, d\tau$.

In the fundamental papers [**BL**] and [**BGK**] the authors establish the existence of least energy solutions through the minimization problems :

$$\text{Minimize } \left\{\int_{\mathbf{R}^N} |\nabla u|^2\, dx;\ \int_{\mathbf{R}^N} G(u)\, dx = 1\right\} \quad \text{for } N \geq 3,$$

$$\text{Minimize } \left\{\int_{\mathbf{R}^2} |\nabla u|^2\, dx;\ \int_{\mathbf{R}^2} G(u)\, dx = 0\right\} \quad \text{for } N = 2.$$

Precisely, they show

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Theorem 0.1. *([**BL**] for $N \geq 3$, [**BGK**] for $N = 2$). Assume*

(g0) $g(s) \in C(\mathbf{R}, \mathbf{R})$ *is continuous and odd.*

(g1) $-\infty < \liminf_{s \to 0} \frac{g(s)}{s} \leq \limsup_{s \to 0} \frac{g(s)}{s} = -\nu < 0$ *for* $N \geq 3$,

$\lim_{s \to 0} \frac{g(s)}{s} = -\nu \in (-\infty, 0)$ *for* $N = 2$.

(g2) *When* $N \geq 3$, $\lim_{s \to \infty} \frac{|g(s)|}{s^{\frac{N+2}{N-2}}} = 0$.

When $N = 2$, *for any* $\alpha > 0$ *there exists* $C_\alpha > 0$ *such that*

$$|g(s)| \leq C_\alpha e^{\alpha s^2} \quad \text{for all } s \geq 0.$$

(g3) *There exists* $\xi_0 > 0$ *such that* $G(\xi_0) > 0$, *where* $G(s) = \int_0^s g(\tau)\, d\tau$.

Then $m > 0$ *and there exists a least energy solution* $\omega_0(x)$ *of* (0.1) *satisfying* $\omega_0(x) > 0$ *for all* $x \in \mathbf{R}^N$ *and, as any solution* $u(x) \in H^1(\mathbf{R}^N)$ *of* (0.1), *the Pohozaev identity :*

$$\frac{N-2}{2} \int_{\mathbf{R}^N} |\nabla \omega_0|^2\, dx = N \int_{\mathbf{R}^N} G(\omega_0)\, dx. \tag{0.4}$$

Under the conditions (g0)–(g2), it is shown in [**BL**, **BGK**] that $I(u)$ is well-defined on $H^1(\mathbf{R}^N)$ and of class C^1. In Lemma 1.1, we show that $I(u)$ has a *Mountain Pass Geometry.* Indeed it has the following properties :

(i) $I(0) = 0$. (0.5)

(ii) There exist $\rho_0 > 0$, $\delta_0 > 0$ such that

$$I(u) \geq \delta_0 \quad \text{for all } \|u\|_{H^1(\mathbf{R}^N)} = \rho_0. \tag{0.6}$$

(iii) There exists $u_0 \in H^1(\mathbf{R}^N)$ such that

$$\|u_0\|_{H^1(\mathbf{R}^N)} > \rho_0 \quad \text{and} \quad I(u_0) < 0. \tag{0.7}$$

Thus if we define the following minimax value (Mountain Pass value, MP value for short)

$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{0.8}$$

where

$$\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbf{R}^N));\ \gamma(0) = 0,\ I(\gamma(1)) < 0\}, \tag{0.9}$$

we have $b > 0$. At this point it is natural to ask if b is a critical value and whether the corresponding critical points are least energy solutions, that is, if $b = m$ holds or not.

Our main result is the following theorem which gives a positive answer :

Theorem 0.2. *Assume (g0)–(g3). Then it holds*

$$b = m,$$

where m, $b > 0$ *are defined in* (0.2) *and* (0.8)*. That is, the Mountain Pass value gives the least energy level. Moreover, for any least energy solution* $\omega(x)$ *of* (0.1)*, there exists a path* $\gamma \in \Gamma$ *such that* $\omega(x) \in \gamma([0,1])$ *and*

$$\max_{t\in[0,1]} I(\gamma(t)) = I(\omega).$$

Remark 0.3. In the case where a least energy solution $\omega(x)$ of (0.1) satisfies $\omega(x) > 0$ for all $x \in \mathbf{R}^N$, the path $\gamma \in \Gamma$ of Theorem 0.2 can be chosen such that $\gamma(t)(x) > 0$, for all $x \in \mathbf{R}^N$, $\forall t \in (0,1]$ (see Lemma 2.1).

In many non-autonomous semi-linear elliptic problems, it turns out that information on the least energy level of an associated autonomous problem is crucial in these years. The least energy level often appears as the first level of possible loss of compactness. Consider, for example, a problem of the type

$$-\Delta u = g(x,u),\ u \in H^1(\mathbf{R}^N), \tag{0.10}$$

where $g(x,u) \to g^\infty(u)$ as $|x| \to \infty$. We assume the following functionals $J(u)$, $J^\infty(u)$ are C^1 on $H^1(\mathbf{R}^N)$ and have a MP geometry.

$$J(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx,\ J^\infty(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{\mathbf{R}^N} G^\infty(u)\,dx.$$

Here $G(x,s) = \int_0^s g(x,\tau)\,d\tau$ and $G^\infty(s) = \int_0^s g^\infty(\tau)\,d\tau$. We denote the corresponding MP values by c and c^∞. Suppose in addition that $J(u)$ has a bounded PS sequence at the level c. Then, from the work of P. L. Lions on concentration-compactness [**L**], it is well known that $J(u)$ has a critical point at the level c, if $c < m^\infty$. Here

$$m^\infty = \inf\{J^\infty(u);\ u \in H^1(\mathbf{R}^N) \setminus \{0\} \text{ is a solution of (0.10)}\}.$$

Thus if one knows that $c^\infty = m^\infty$, to get a critical point, it is sufficient to show that $c < c^\infty$. Checking this inequality is easier than proving directly that $c < m^\infty$ because of the minimax characterizations of c and c^∞. To insure that $c^\infty = m^\infty$, the standard way so far is to assume that

$$s \mapsto \frac{g(s)}{s} :\ (0,\infty) \to \mathbf{R} \text{ is non decreasing.} \tag{0.11}$$

This property enables to make use of the *Nehari manifold*: $\mathcal{M} = \{u \in H^1(\mathbf{R}^N) \setminus \{0\};\ J'^\infty(u)u = 0\}$. Under (0.11), any non-zero critical point of J^∞ lies on $\mathcal{M}$ and the least energy level m^∞ is characterized as

$$m^\infty = \inf_{u\in\mathcal{M}} J^\infty(u).$$

This readily implies that $c^\infty = m^\infty$. What our Theorem 0.2 is saying is that the equality $c^\infty = m^\infty$ always holds without the assumption (0.11).

Among other applications of our mountain pass characterization of the least energy solutions of (0.1) we mention singular perturbation problems, i.e., the search of peak solutions. For this subject we refer, for example, to Ni-Takagi [**NT**] and del Pino-Felmer [**DF**]. An autonomous problem of the type of (0.1) appears in these problems through a scaling argument. Precise estimates are necessary on its least energy level in order to get peak solutions. Usually the condition (0.11) is required for these estimates. In [**JT**] we present some results on this topic which relies on our Theorem 0.2.

To give a proof of Theorem 0.2, we make use of properties of the dilation $u_t(x) = u(x/t)$ $(t > 0)$ as in [**BL**, **BGK**]. Actually, for any least energy solution $\omega(x)$ of (0.1), we construct, in Lemma 2.1, a path $\gamma \in \Gamma$ such that

$$\omega \in \gamma([0,1]) \quad \text{and} \quad \max_{t\in[0,1]} I(\gamma(t)) = m.$$

The existence of such paths implies that $b \le m$. To show that $b \ge m$, we introduce the set $\mathcal{P}$ of non-trivial functions satisfying Pohozaev identity (0.4) :

$$\mathcal{P} = \{u \in H^1(\mathbf{R}^N) \setminus \{0\};\ \frac{N-2}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - N\int_{\mathbf{R}^N} G(u)\,dx = 0\}.$$

We will show, in Lemma 3.1, that

$$m = \inf_{u\in\mathcal{P}} I(u)$$

and, in Lemma 4.1, that

$$\gamma([0,1]) \cap \mathcal{P} \neq \emptyset \quad \text{for all } \gamma \in \Gamma.$$

This directly leads to $b \ge m$.

Notation : We will use the following notation :

$$\begin{aligned} \|u\|_p &= \left(\int_{\mathbf{R}^N} |u|^p\,dx\right)^{1/p} \quad \text{for } p \in [1,\infty), \\ \|u\|_\infty &= \operatorname*{ess\,sup}_{x\in\mathbf{R}^N} |u(x)|, \\ \|u\|_{H^1} &= (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}. \end{aligned}$$

1. Mountain Pass Geometry

We observe here that under (g0)–(g3) the functional $I(u)$ defined in (0.3) has a mountain pass geometry.

Lemma 1.1. *Assume (g0)–(g2). Then $I(u)$ satisfies* (0.5)–(0.6).

Proof. We deal with (0.6). (0.5) trivially holds. First we prove (0.6) for $N \ge 3$. By the assumption (g1)–(g2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$-g(s) \ge (\nu - \varepsilon)s - C_\varepsilon s^{\frac{N+2}{N-2}} \quad \text{for all } s \ge 0.$$

Thus, recalling that $g(s)$ is an odd function, we have, for a $C'_\varepsilon > 0$

$$-G(u) \geq \frac{1}{2}(\nu-\varepsilon)s^2 - C'_\varepsilon |s|^{\frac{2N}{N-2}} \quad \text{for all } s \in \mathbf{R}\,.$$

It follows from the embedding $H^1(\mathbf{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbf{R}^N)$, that for a $C''_\varepsilon > 0$

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx + \frac{\nu-\varepsilon}{2}\int_{\mathbf{R}^N} |u|^2\,dx - C'_\varepsilon \int_{\mathbf{R}^N} |u|^{\frac{2N}{N-2}}\,dx \\
&\geq \frac{1}{2}\min\{1,\nu-\varepsilon\}\|u\|_{H^1}^2 - C'_\varepsilon \|u\|_{\frac{2N}{N-2}}^{\frac{2N}{N-2}} \\
&\geq \frac{1}{2}\min\{1,\nu-\varepsilon\}\|u\|_{H^1}^2 - C''_\varepsilon \|u\|_{H^1}^{\frac{2N}{N-2}} \qquad \text{for all } u \in H^1(\mathbf{R}^N).
\end{aligned}$$

Therefore choosing $\rho_0 > 0$ small, we can see that (0.6) holds.

Next we prove (0.6) for $N = 2$. By the assumptions (g1)–(g2), for any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$-g(s) \geq \frac{1}{2}\nu s - C_\alpha s^4 e^{\alpha s^2} \quad \text{for all } s \geq 0.$$

Since $\int_0^s \tau^4 e^{\alpha\tau^2}\,d\tau = \frac{1}{2\alpha}s^3(e^{\alpha s^2}-1) - \frac{3}{2\alpha}\int_0^s \tau^2(e^{\alpha s^2}-1)\,d\tau \leq \frac{1}{2\alpha}s^3(e^{\alpha s^2}-1)$, we have

$$-G(u) \geq \frac{\nu}{4}s^2 - \frac{C_\alpha}{2\alpha}s^3(e^{\alpha s^2}-1) \quad \text{for all } s \in \mathbf{R}$$

and thus, for a $C'_\alpha > 0$,

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{\nu}{4}\|u\|_2^2 - \frac{C_\alpha}{2\alpha}\int_{\mathbf{R}^2} u^3(e^{\alpha s^2}-1)\,dx \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{\nu}{4}\|u\|_2^2 - \frac{C_\alpha}{2\alpha}\|u\|_6^3 \sqrt{\int_{\mathbf{R}^2} (e^{\alpha s^2}-1)^2\,dx} \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{\nu}{4}\|u\|_2^2 - \frac{C'_\alpha}{2\alpha}\|u\|_{H^1}^3 \sqrt{\int_{\mathbf{R}^2} (e^{2\alpha s^2}-1)\,dx}.
\end{aligned}$$

Also, from the Moser-Trudinger inequality (c.f. Adachi-Tanaka [**AT**] and references therein), there exist $\sigma_0 > 0$, $M > 0$ such that

$$\int_{\mathbf{R}^2} (e^{\sigma_0 u^2}-1)\,dx \leq M \quad \text{for all } \|u\|_{H^1} \leq 1.$$

Thus for any $c > 0$ it holds that

$$\int_{\mathbf{R}^2} (e^{\frac{\sigma_0}{c^2}u^2}-1)\,dx \leq M \quad \text{for all } \|u\|_{H^1} \leq c$$

and choosing $\rho_0 > 0$ small, we can see that (0.6) holds. ∎

Remark 1.2. Actually we see, from the proof of Lemma 1.1, that

$$I(u) > 0 \quad \text{for all } 0 < \|u\|_{H^1(\mathbf{R}^N)} \leq \rho_0.$$

where ρ_0 is given in (0.6).

Remark 1.3. Modifying slightly the arguments of the proof of Lemma 1.1, it is possible to show that, for $N \geq 3$, there exists $\rho_0 > 0$ such that

$$\frac{N-2}{2}\|\nabla u\|_2^2 - N\int_{\mathbf{R}^N} G(u)\,dx > 0 \quad \text{for all } 0 < \|u\|_{H^1(\mathbf{R}^N)} \leq \rho_0.$$

Lemma 1.4. *Assume (g0)–(g3). Then* (0.5)–(0.7) *hold. In particular $I(u)$ has a mountain pass geometry and the MP value b in* (0.8)–(0.9) *is well-defined.*

Proof. We know, from Lemma 1.1 that (0.5)–(0.6) hold. Also, since $I(0) = 0$, we see from Remark 1.2, that proving (0.7) is equivalent to show that $\Gamma \neq \emptyset$. This will be done in Lemma 2.1. ∎

As stated in the Introduction, the proof of Theorem 0.2 consists of 3 steps

Step 1 : Construction of a path $\gamma \in \Gamma$ such that

$$\omega \in \gamma([0,1]), \tag{1.1}$$

$$\max_{t\in[0,1]} I(\gamma(t)) = m, \tag{1.2}$$

where $\omega(x)$ is a given least energy solution of (0.1).

Step 2 : $\min_{u\in\mathcal{P}} I(u) = m.$ (1.3)

Step 3 : $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ for all $\gamma \in \Gamma$. (1.4)

Step 1 implies $b \leq m$ and Steps 2 and 3 imply $b \geq m$.

2. A path $\gamma \in \Gamma$ satisfying (1.1)–(1.2)

Let $\omega(x)$ be an arbitrary least energy solution of (0.1)

Lemma 2.1. *Under the assumptions (g0)–(g3), there exists a path $\gamma \in \Gamma$ satisfying* (1.1)–(1.2).

Proof of Lemma 2.1. We will find a curve $\gamma(t) : [0, L] \to H^1(\mathbf{R}^N)$ such that

$$\gamma(0) = 0,\ I(\gamma(L)) < 0, \tag{2.1}$$

$$\omega \in \gamma([0, L]), \tag{2.2}$$

$$\max_{t\in[0,L]} I(\gamma(t)) = m. \tag{2.3}$$

After a suitable scale change in t, we can get the desired path $\gamma \in \Gamma$.

When $N \geq 3$, our construction is rather simple. Setting

$$\gamma(t)(x) = \begin{cases} \omega(x/t) & \text{for } t > 0, \\ 0 & \text{for } t = 0, \end{cases}$$

we can see that

1. $\|\gamma(t)\|_{H^1}^2 = t^{N-2}\|\nabla\omega\|_2^2 + t^N\|\omega\|_2^N$.
2. $I(\gamma(t)) = \dfrac{t^{N-2}}{2}\|\nabla\omega\|_2^2 - t^N\displaystyle\int_{\mathbf{R}^N} G(\omega)\,dx$.

Thus $\gamma(t) \in C([0,\infty), H^1(\mathbf{R}^N))$. Moreover, Pohozaev identity (0.4) implies $\int_{\mathbf{R}^N} G(\omega)\,dx > 0$, $\frac{d}{dt}I(\gamma(t)) > 0$ for $t \in (0,1)$ and $\frac{d}{dt}I(\gamma(t)) < 0$ for $t > 1$. Thus for sufficiently large $L > 1$ our path $\gamma(t)$ satisfies (2.1)–(2.3).

When $N = 2$, our construction is more complicated. We choose $t_0 \in (0,1)$, $t_1 \in (1,\infty)$ and $\theta_1 > 1$ so that a curve γ, constituted of the three pieces defined below, gives a desired path :

$$[0,1] \;\to H^1(\mathbf{R}^2);\; \theta \mapsto \theta\omega_{t_0}, \tag{2.4}$$
$$[t_0,t_1] \to H^1(\mathbf{R}^2);\; t \mapsto \omega_t, \tag{2.5}$$
$$[1,\theta_1] \;\to H^1(\mathbf{R}^2);\; \theta \mapsto \theta\omega_{t_1}. \tag{2.6}$$

Here $\omega_t(x) = \omega(x/t)$.

First we remark that since $\omega(x)$ satisfies (0.1),

$$\int_{\mathbf{R}^2} g(\omega)\omega\,dx = \|\nabla\omega\|_2^2 > 0.$$

Thus we can find $\theta_1 > 1$ such that

$$\int_{\mathbf{R}^2} g(\theta\omega)\omega\,dx > 0 \quad \text{for all } \theta \in [1,\theta_1]. \tag{2.7}$$

Next we set $\varphi(s) = g(s)/s$. By the assumption (g1) we have $\varphi(s) \in C(\mathbf{R},\mathbf{R})$. With this notation (2.7) becomes

$$\int_{\mathbf{R}^2} \varphi(\theta\omega)\omega^2\,dx > 0 \quad \text{for all } \theta \in [1,\theta_1]. \tag{2.8}$$

Now we compute $\frac{d}{d\theta}I(\theta\omega_t)$:

$$\begin{aligned}\frac{d}{d\theta}I(\theta\omega_t) &= I'(\theta\omega_t)\omega_t\\ &= \theta\left(\|\nabla\omega_t\|_2^2 - \int_{\mathbf{R}^2}\varphi(\theta\omega_t)\omega_t^2\,dx\right)\\ &= \theta\left(\|\nabla\omega\|_2^2 - t^2\int_{\mathbf{R}^2}\varphi(\theta\omega)\omega^2\,dx\right).\end{aligned}$$

Choosing $t_0 \in (0,1)$ sufficiently small, we have

$$\|\nabla\omega\|_2^2 - t_0^2\int_{\mathbf{R}^2}\varphi(\theta\omega)\omega^2\,dx > 0 \quad \text{for all } \theta \in [0,1]. \tag{2.9}$$

By (2.8), we can also choose $t_1 > 1$ such that

$$\|\nabla\omega\|_2^2 - t_1^2 \int_{\mathbf{R}^2} \varphi(\theta\omega)\omega^2\,dx \le -\frac{1}{\theta_1 - 1}\|\nabla\omega\|_2^2 \quad \text{for all } \theta \in [1, \theta_1]. \tag{2.10}$$

Thus we can see by (2.9) that along the line (2.4), $I(\theta\omega_{t_0})$ increases and takes its maximal at $\theta = 1$. Since $\int_{\mathbf{R}^2} G(\omega)\,dx = 0$ by Pohozaev identity (0.4), we have $I(\omega_t) = I(\omega) = \frac{1}{2}\|\nabla\omega\|_2^2$ along the curve (2.5). Next by (2.10), $I(\theta\omega_{t_1})$ decreases along the line (2.6) and we have

$$\begin{aligned} I(\theta_1\omega_{t_1}) &= I(\omega_{t_1}) + \int_1^{\theta_1} \frac{d}{d\theta} I(\theta\omega_{t_1})\,d\theta \\ &\le \frac{1}{2}\|\nabla\omega\|_2^2 - \int_1^{\theta_1} \frac{1}{\theta_1 - 1}\|\nabla\omega\|_2^2\,d\theta \\ &< -\frac{1}{2}\|\nabla\omega\|_2^2 < 0. \end{aligned}$$

Therefore we get the desired curve. ∎

As a corollary to Lemma 2.1, we have

Corollary 2.2. $b \le m$. ∎

3. Proof of (1.3)

In this section we give a proof of (1.3). Namely we show :

Lemma 3.1. $m = \inf_{u \in \mathcal{P}} I(u)$.

Proof. We argue for the cases $N \ge 3$ and $N = 2$ separately.

For $N \ge 3$ we use an idea from Coleman-Glazer-Martin [**CGM**] as in [**BL**]. We introduce a set

$$\mathcal{S} = \{u \in H^1(\mathbf{R}^N);\ \int_{\mathbf{R}^N} G(u)\,dx = 1\}.$$

There is a one-to-one correspondence $\Phi : \mathcal{S} \to \mathcal{P}$ between $\mathcal{S}$ and $\mathcal{P}$:

$$(\Phi(u))(x) = u(x/t_u), \quad \text{where } t_u = \sqrt{\frac{N-2}{2N}}\|\nabla u\|_2.$$

For $u \in \mathcal{S}$,

$$\begin{aligned} I(\Phi(u)) &= \frac{1}{2} t_u^{N-2}\|\nabla u\|_2^2 - t_u^N \int_{\mathbf{R}^N} G(u)\,dx \\ &= \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^N \end{aligned}$$

and thus

$$\inf_{u \in \mathcal{P}} I(u) = \inf_{u \in \mathcal{S}} I(\Phi(u)) = \inf_{u \in \mathcal{S}} \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^N.$$

It is observed in [**BL**] that $\inf_{u\in\mathcal{S}}\|\nabla u\|_2^2$ is achieved and that the corresponding $\Phi(u)$ is a least energy solution. Thus we have

$$m=\inf_{u\in\mathcal{P}} I(u).$$

For $N=2$, we have $\mathcal{P}=\{u\in H^1(\mathbf{R}^2)\setminus\{0\};\int_{\mathbf{R}^2}G(u)\,dx=0\}$. We remark that $I(u)=\frac{1}{2}\|\nabla u\|_2^2$ on $\mathcal{P}$. It is shown in [**BGK**] that $\inf_{u\in\mathcal{P}}\|\nabla u\|_2^2$ is achieved and the minimizer is a least energy solution of (0.1) after a suitable scale change $u(x/t)$. Thus $m=\inf_{u\in\mathcal{P}}I(u)$ also holds for $N=2$.

Therefore the proof of Lemma 3.1 is completed. ∎

4. Proof of (1.4)

In this section we prove the following intersection property :

Lemma 4.1. $\gamma([0,1])\cap\mathcal{P}\neq\emptyset$ *for all* $\gamma\in\Gamma$.

As a corollary to Lemmas 3.1 and 4.1, we have

Corollary 4.2. $b\geq m$. ∎

In the proof of Lemma 4.1 we use the notation :

$$\begin{aligned}P(u)&=\frac{N-2}{2}\|\nabla u\|_2^2-N\int_{\mathbf{R}^N}G(u)\,dx\\&=NI(u)-\|\nabla u\|_2^2.\end{aligned}$$

Proof of Lemma 4.1 for $N\geq 3$**.** The proof of Lemma 4.1 for $N\geq 3$ is straightforward. By Remark 1.3, there exists $\rho_0>0$ such that

$$0<\|u\|_{H^1}\leq\rho_0 \implies P(u)>0.$$

For any $\gamma\in\Gamma$ we have $\gamma(0)=0$ and $P(\gamma(1))\leq NI(\gamma(1))<0$. Thus there exists $t_0\in[0,1]$ such that

$$\begin{aligned}&\|\gamma(t_0)\|_{H^1}>\rho_0,\\&P(\gamma(t_0))=0.\end{aligned}$$

Since $\gamma(t_0)\in\gamma([0,1])\cap\mathcal{P}$ we have $\gamma([0,1])\cap\mathcal{P}\neq\emptyset$. ∎

When $N=2$, $P(u)=-2\int_{\mathbf{R}^2}G(u)\,dx$ and since our $P(u)$ does not have a $\|\nabla u\|_2^2$-component we can not argue as in Lemma 1.1 and Remark 1.3.

To prove Lemma 4.1 for N=2, we choose $\rho(x)\in C_0^\infty(\mathbf{R}^2)$ such that

$$\begin{aligned}&\rho(x)\geq 0\quad\text{for all } x\in\mathbf{R}^2,\\&\int_{\mathbf{R}^2}\rho(x)\,dx=1\end{aligned}$$

and for any given $\gamma\in\Gamma$ we set for $\varepsilon>0$

$$\gamma_\varepsilon(t)(x)=\int_{\mathbf{R}^2}\rho(\frac{x-y}{\varepsilon})\gamma(t)(y)\,dy.$$

Then it is easily proved that

(i) For any $\varepsilon>0$ and $t\in[0,1]$, $\gamma_\varepsilon(t)\in H^1(\mathbf{R}^2)\cap L^\infty(\mathbf{R}^2)$.
(ii) $\gamma_\varepsilon(t):\,[0,1]\to L^\infty(\mathbf{R}^2)$ is continuous.
(iii) $\max_{t\in[0,1]}\|\gamma_\varepsilon(t)-\gamma(t)\|_{H^1}\to 0$ as $\varepsilon\to 0$.

We also remark that

Lemma 4.3. *Under (g1), there exists $\rho_0 > 0$ such that for $u \in H^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$*

$$0 < \|u\|_\infty \leq \rho_0 \implies P(u) > 0.$$

Proof. By (g1), there exists $\rho_0 > 0$ such that $-G(s) > 0$ for $0 < s \leq \rho_0$. Thus we have Lemma 4.3. ∎

Proof of Lemma 4.1 for $N = 2$. Let $\gamma \in \Gamma$ be given. First we remark that, because of (iii), we have $P(\gamma_\varepsilon(1)) \leq 2I(\gamma_\varepsilon(1)) < 0$ for any small $\varepsilon > 0$. Also, since $\gamma_\varepsilon(0) = 0$ for any $\varepsilon > 0$, by Lemma 4.3 and (ii) we have $P(\gamma_\varepsilon(t)) > 0$ for $t > 0$ sufficiently small. Thus, assuming $\varepsilon > 0$ small, we can find $t_\varepsilon \in [0, 1]$ such that

$$\begin{aligned} &\|\gamma_\varepsilon(t_\varepsilon)\|_\infty > \rho_0, \\ &P(\gamma_\varepsilon(t_\varepsilon)) = 0. \end{aligned}$$

In particular, $\gamma_\varepsilon(t_\varepsilon) \in \mathcal{P}$.

We extract a subsequence $\varepsilon_n \to 0$ such that $t_{\varepsilon_n} \to t_0$ as $n \to \infty$. From (ii)-(iii) it follows that

$$\begin{aligned} &\|\gamma_{\varepsilon_n}(t_{\varepsilon_n}) - \gamma(t_0)\|_{H^1} \to 0, \\ &P(\gamma(t_0)) = 0 \end{aligned} \tag{4.1}$$

and to conclude we just need to show

$$\gamma(t_0) \neq 0. \tag{4.2}$$

To establish (4.2), we recall a result of [**BGK**] saying that $\inf_{u \in \mathcal{P}} \|\nabla u\|_2^2 = 2m > 0$. Thus

$$\|u\|_{H^1} \geq \sqrt{2m} \quad \text{for all } u \in \mathcal{P}$$

and in particular $\|\gamma_{\varepsilon_n}(t_{\varepsilon_n})\|_{H^1} \geq \sqrt{2m}$ for all n. Therefore it follows from (4.1) that $\|\gamma(t_0)\|_{H^1} \geq \sqrt{2m} > 0$. Thus $\gamma(t_0) \in \gamma([0,1]) \cap \mathcal{P}$ and $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$. ∎

Remark 4.4. In the proof of Lemma 4.1 for $N = 2$, making use of the continuity of the path $\gamma(t)$ in $H^1(\mathbf{R}^2)$ is essential. We give an example. For $g(s) = -s + s^3$ we have $P(u) = \|u\|_2^2 - \frac{1}{2}\|u\|_4^4$. Now for any $u_0 \in H^1(\mathbf{R}^2)$ with $P(u_0) < 0$, the path $\gamma(t) = t^{-1/4}u_0(x/t) : [0,1] \to H^1(\mathbf{R}^2)$ is a continuous path in $L^2(\mathbf{R}^2) \cap L^4(\mathbf{R}^2)$ (but not in $H^1(\mathbf{R}^2)$) joining 0 and u_0. However $P(\gamma(t)) < 0$ for all $t \in (0, 1]$.

End of the proof of Theorem 0.2. Combining Corollaries 2.2 and 4.2, we get $b = m$. This is the desired result. ∎

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