Introduction to linear programming

Lycée Jules Haag, Besançon, April 12th 2012

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How many boxes of each kind should they make in order to maximize the profit ?

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And also $x \geq 0$ and $y \geq 0$.

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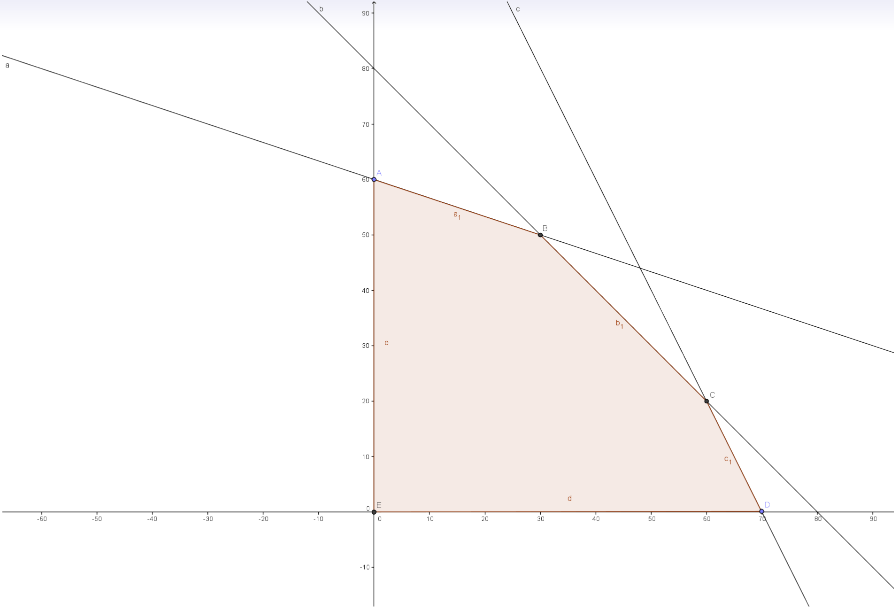
This will be called the feasible region.


a
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Can we conjecture the statement of the general theorem ?

Conjecture : The maximum of a linear objective function in a polygonal feasible region is attained at a corner point (or vertex) of the feasible region.

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Step 1 The maximum of the objective function on a segment is attained at an endpoint of the segment.

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Therefore

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We obtain
$x_M = (1 - t)x_A + tx_B \quad \text{and} \quad y_M = (1 - t)y_A + ty_B.$

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And then, after a few computations...
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This yields the conclusion of Step 1.

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End of proof.

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Why should we prove it ? ? ?