JOINT COMMENTS ON THE PAPERS: N. J. KALTON AND G. LANCIEN, A SOLUTION TO THE PROBLEM OF L^p-MAXIMAL REGULARITY, MATH Z. 235, (2000), 559-568 AND N. J. KALTON AND G. LANCIEN, L^p-MAXIMAL REGULARITY ON BANACH SPACES WITH A SCHAUDER BASIS, ARCH. MATH. 78 (2002), 397-408.

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This short text contains joint comments on two papers. For simplicity, the paper "A solution to the problem of L^p-maximal regularity, Math Z. 235, (2000) 559-568" will be called Paper 1 in the sequel and "L^p-maximal regularity on Banach spaces with a Schauder basis, Arch. Math. 78 (2002), 397-408" will be referred to as Paper 2.

Let us first recall the basic facts and definitions on maximal regularity. This subject is about the study of the following abstract Cauchy problem:

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \leq t < T \\ u(0) = 0 \end{cases}$$

where $T \in (0, +\infty)$, B is a closed operator on a complex Banach space X with domain $D(B)$ dense in X and u and f are X-valued functions on $[0, T)$. Suppose $1 < p < \infty$. B is said to be L^p*−regular* if whenever f belongs to $L^p([0, T); X)$ the solution

$$u(t) = \int_0^t e^{-(t-s)B} f(s)\, ds$$

satisfies $u' \in L^p([0, T); X)$. In 1964, Sobolevskii showed in [15] that if B is L^p-regular, then $-B$ is the generator of an analytic semigroup. The problem of L^p-maximal regularity is the study of the validity of the converse of this statement. Namely, if $-B$ is the generator of an analytic semigroup on X, do we have that B is L^p-regular? By adding a multiple of the identity to B, we may assume that the analytic semigroup generated by B is bounded. Then, it is important to note that the L^p-regularity of B does not depend on $T > 0$, nor does it depend on $p \in (1, +\infty)$ (see [3], [5] and [15]). Finally, we shall say that a Banach space where the converse of Sobolevskii's result is true (i.e. B is L^p-regular whenever $-B$ is the generator of a bounded analytic semigroup) has the maximal regularity property (MRP). The first fundamental result is due to De Simon [3] who proved that any Hilbert space H has (MRP). The proof is based on the use of the Fourier-Plancherel transform on $L^2(H)$. The natural question, asked by Brézis in the early 80's was to describe the Banach spaces with (MRP). Then, large classes of counterexamples have been found. Coulhon and Lamberton showed in [2] that $X = L^2(\mathbb{R}, E)$ fails (MRP), whenever X is not a UMD space (recall that E is UMD if and only if the Hilbert transform is bounded on $L^2(\mathbb{R}, E)$). Later, Le Merdy [9] showed that $L^1(\mathbb{T})$, $C(\mathbb{T})$ and $K(\ell_2)$ also fail (MRP). All these results relied heavily on the non boundedness of the Hilbert transform or of an analogue of it. On the other hand, very important results on the closedness of the sum of two operators with bounded imaginary powers on a UMD space were obtained by Dore and Venni in [4]. So, before this work, the main question was: does every UMD Banach space have (MRP) and more importantly, does L^q have (MRP) when $1 < q \neq 2 < \infty$?

Let us also recall the definition of a sectorial operator. A closed densely defined operator B on a Banach space X is said to be sectorial of type ω, where $0 \leq \omega < \pi$, if the spectrum of B is included in Σ_ω, where $\Sigma_\omega = \{z \in \mathbb{C} \setminus \{0\} : \ |\mathrm{Arg}(z)| \leq \omega\} \cup \{0\}$ and for every $\theta \in (\omega, \pi)$ there exists $C_\theta > 0$ so that for any $\lambda \in \mathbb{C} \setminus \Sigma_\theta$ we have $\|(\lambda - B)^{-1}\| \leq C_\theta |\lambda|^{-1}$. It will be very useful to note that $-B$ generates a bounded analytic semigroup on X if and only if B is sectorial of type ω, for some $\omega < \frac{\pi}{2}$.

Let us first comment on Paper 1. The main result of this paper is that a Banach space with an unconditional basis has (MRP) if and only if it is isomorphic to a Hilbert space.

We start by describing a very elementary tool for the construction of sectorial operators on a Banach space. Indeed, assume that $(E_n)_{n=1}^\infty$ is a Schauder decomposition of a Banach space X and that $(\lambda_n)_{n=1}^\infty$ is a sequence in $\mathbb{C}$. An elementary Abel transform shows that whenever the sequence $(\lambda_n)_{n=1}^\infty$ is of bounded variation, the multiplier by $(\lambda_n)_{n=1}^\infty$ on $(E_n)_{n=1}^\infty$ is a bounded operator on X. Then it is easy to see that if $0 < b_1 < .. < b_n < ..$, then the (possibly unbounded) multiplier by $(b_n)_{n=1}^\infty$ on a Schauder decomposition $(E_n)_{n=1}^\infty$ of X is an invertible sectorial operator of type 0 on X. This idea had already been used in order to build examples of commuting sectorial operators whose sum is not closed on a Hilbert space (see [1]), or on $L^p(H)$ even if one of them has bounded imaginary powers (see [8]).

One other tool is Proposition 2.1 of Paper 1, which states that if $-B$ is an invertible generator of a bounded analytic semigroup that is L^2 regular on X, then the Fourier multiplier with (operator valued) symbol $(in(in+B)^{-1})_{n\in\mathbb{Z}}$ is bounded on $L^2(\mathbb{T}, X)$.

The key idea of Paper 1 is to use a theorem of Lindenstrauss and Tzafriri [10] which is a classic from Banach space theory. It asserts that if X is a Banach space with an unconditional basis (x_n) such that for every permutation π of $\mathbb{N}$ we have that any block basis of $(x_{\pi(n)})$ spans a complemented subspace of X, then (x_n) is equivalent to the canonical basis of ℓ_p $(1 \leq p < \infty)$ or c_0. If (u_n) is a block basis of $(e_n) = (x_{\pi(n)})$ and u_n is supported by $e_{r_n+1}, .., e_{r_{n+1}}$, let X_n be the linear span of $e_{r_n+1}, .., e_{r_{n+1}}$, $E_{2n} = \mathbb{C}u_n$ and E_{2n-1} be the kernel of a contractive projection from X_n onto E_{2n}. Then (E_n) is a Schauder decomposition of X. The next idea is to define two sectorial multipliers on (E_n) in such a way that if they are assumed to be L^2-regular, then Proposition 2.1 and the unconditinality of (x_n) will imply that the closed linear span of (u_n) is complemented in X. When this is done, we already know that a Banach space with an unconditional basis and with (MRP) can only be ℓ_p with $1 \leq p < \infty$ or c_0. In fact, we also know that X cannot be ℓ_p with $1 < p \neq 2 < \infty$, because this space admits an unconditional basis that is not equivalent to its canonical basis. Indeed, A. Pełczyński proved in [13], that for $p \in (1, +\infty)$, ℓ_p is isomorphic to $(\sum_{n=1}^{\infty} \ell_2^n)_{\ell_p}$. Finally, one can settle that c_0 and ℓ_1 fail (MRP) by considering similar multipliers or their duals together with the summing basis of c_0.

The proof that I briefly described yields the main result of Paper 1: a Banach space with an unconditional basis has (MRP) if and only if it is isomorphic to ℓ_2. In particular, the L^p spaces, for $1 < p \neq 2 < \infty$ fail (MRP). This characterization extends to order continuous or separable Banach lattices. It is also interesting to note that using the same ideas, the result of Coulhon and Lamberton can be improved: $L^p(\mathbb{R}, E)$ has (MRP) if and only if $p = 2$ and E is isomorphic to a Hilbert space.

This work was done while Nigel Kalton was a visiting professor at the Université de Franche-Comté in 1999 where he came to work on uniform homeomorphisms between Banach spaces. One afternoon, we became tired of our lack of success with these questions and I started to describe the maximal regularity problem to Nigel. At the beginning he only showed a polite interest and pretended he did not know anything about PDE's and semigroups of operators. Finally, I decided to speak in his favorite language and said "consider a multiplier on a Schauder basis.." It was just the spark he needed. After that, it took him an amazingly short time to realize that we should relate this to Lindenstrauss and Tzafriri's result. Once this idea was in our hands, it was only a technical matter to conclude. This is one of many examples of Nigel's ability to view analysis as a whole and to the see the deep links between problems coming from seemingly distant mathematical communities.

Let us now comment shortly on Paper 2. It must first be noted that (MRP) does not characterize Hilbert spaces in general. This follows from a result of H. Lotz [11] which shows that every strongly continuous semigroup on L^∞ is uniformly continuous and therefore L^p-regular. The unconditionality of the basis was crucial in Paper 1 and Paper 2 investigates the more difficult question of maximal regularity in Banach spaces with a Schauder basis or a finite dimensional Schauder decomposition (FDD). Theorem 3.1 is a general result that we will not restate. It is applied to show that under rather general conditions, a Banach space with an (FDD) and (MRP) is isomorphic to an ℓ_2 sum of finite dimensional spaces (Theorems 3.3 and 3.4). We will just comment a bit more on Theorem 3.4, which states that a UMD Banach space X with an (FDD) and satifying (MRP) is isomorphic to an ℓ_2 sum of finite dimensional spaces. The idea of the proof is to show that there is a blocking of the (FDD) satisfying an upper 2-estimate and to conclude by duality. What is interesting is that the proof uses weakly null martingale difference trees in $L^2(X)$ together with renorming techniques that are closely related to the behavior of the Szlenk index of a space and of its dual (see [12] for a nice survey on these notions). The link between (MRP) and techniques inspired by the study of the Szlenk index is a pleasant surprise. This notion was invented by W. Szlenk in 1968 [16] to solve universality problems for separable reflexive Banach spaces, then used in renorming theory and turned out to provide nice non linear invariants for Banach spaces. It is interesting to also feel its flavor, even remotely, in maximal regularity problems. It is not impossible that some of the ideas from section 3 of Paper 2 could still be useful to specialists in the subject. To the best of my knowledge, the question whether a Banach space satisfying the assumptions of Theorem 3.4 is isomorphic to a Hilbert space is still open.

In the last section of Paper 2, the result on the failure of (MRP) for L^p $(1 \leq p \neq 2 < \infty)$ is improved. For r and s in $[1,\infty)$, we say that (r,s) is a *regularity pair* if whenever $-B$ is the infinitesimal generator of a bounded analytic semigroup on $L^s = L^s([0,1])$ and $f \in L^2([0,T);L^s)$, the solution u of

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \leq t < T \\ u(0) = 0 \end{cases}$$

satisfies $u' \in L^2([0,T);L^r)$.
Then the result (Theorem 4.2) is that (r,s) is a regularity pair if and only if $r \leq s = 2$.

The main ingredient is to refine the ideas in Paper 1 and apply them to the Haar basis of L^s. Then it is possible to show that, when (r, s) is a regularity pair, the Haar system has to satisfy some lower 2-estimates in L^s. Classical properties of the Haar basis then yield the conclusion.

Approximately at the time the counterexamples on maximal regularity were constructed, Lutz Weis obtained a beautiful characterization of L^p-regular operators on UMD spaces using the notion of R-bounded families of operators and R-sectorial operators [17]. Then Lutz and Nigel started a collaboration that can be considered as a revolution for the subject (see for instance the fundamental paper [7]). We must also mention the existence of an unpublished preprint by N. Kalton and L. Weis on what they called "Euclidean structures". This paper is the deepest piece of work on the subject. Most of the specialists have a version of it, but it would be nice if the community could collaborate to have it properly published. Since then, there has been an impressive amount of positive results obtained about the sums of closed operators. We will not attempt to describe it in this note. On the more precise subject of counterexamples to maximal regularity, let us just mention a couple of results. In [14], P. Portal has obtained analogues of the results of Paper 1 for discrete time analytic semigroups. More recently, S. Fackler gave in [6] a more explicit proof of the result in Paper 1. Moreover, he showed that maximal regularity does not extrapolate by constructing consistent holomorphic semigroups on $L^p(\mathbb{R})$, for $p \in (1, \infty)$, that are L^p-regular only for $p = 2$.

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