Equations aux dérivées partielles non-linéaires avec conditions aux limites périodiques pour la modélisation des peignes de fréquences optiques

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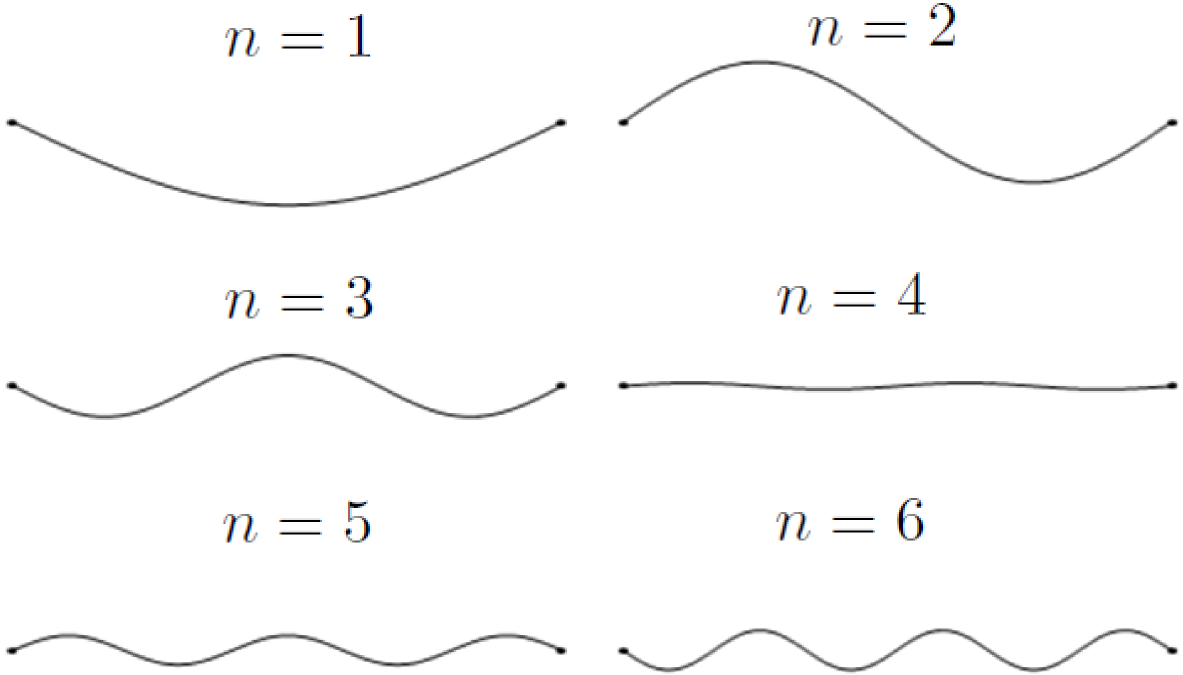
CNRS & FEMTO-ST Institute, Besançon

2ème Séminaire croisé LMB/FEMTO-ST
Besançon, le 2 juillet 2013.

What is a frequency comb?

The example of the vibrating string

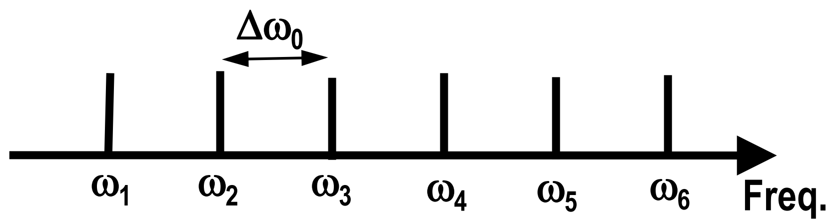


$$L = n\,\frac{\lambda}{2}$$

Resonance condition

$$\omega_n = \frac{2\pi c}{\lambda} = n\,\Delta\omega_0$$

with $\Delta\omega_0 = \frac{\pi c}{L}$



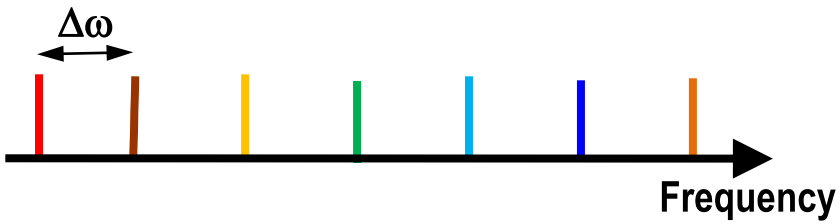
Acoustic frequency comb
(Frequencies ~ 1 kHz)

Can we obtain such combs with optical frequencies (~100 THz)?

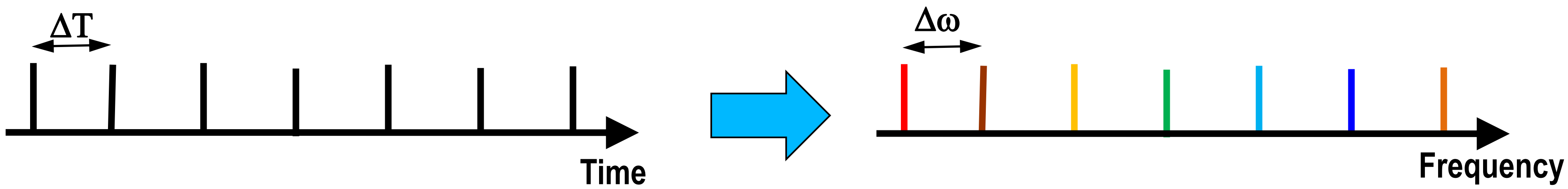
Optical frequency combs

A simple concept…

Optical frequency combs are sets of extremely narrow and equidistant spectral lines in the ultraviolet, visible and infrared spectral ranges



Interestingly the Fourier Transform of a train of regular Dirac spikes is a set of coherent and equidistant Dirac spectral components, that is, a comb.



This is why optical frequency combs are mainly generated using femtosecond-lasers.

Optical frequency combs

... and a Nobel Prize winning technology

Nobel Prize of Physics, 2005

Photo: J.Reed — Roy J. Glauber

Photo: Sears.P.Studio — John L. Hall

Photo: F.M. Schmidt — Theodor W. Hänsch

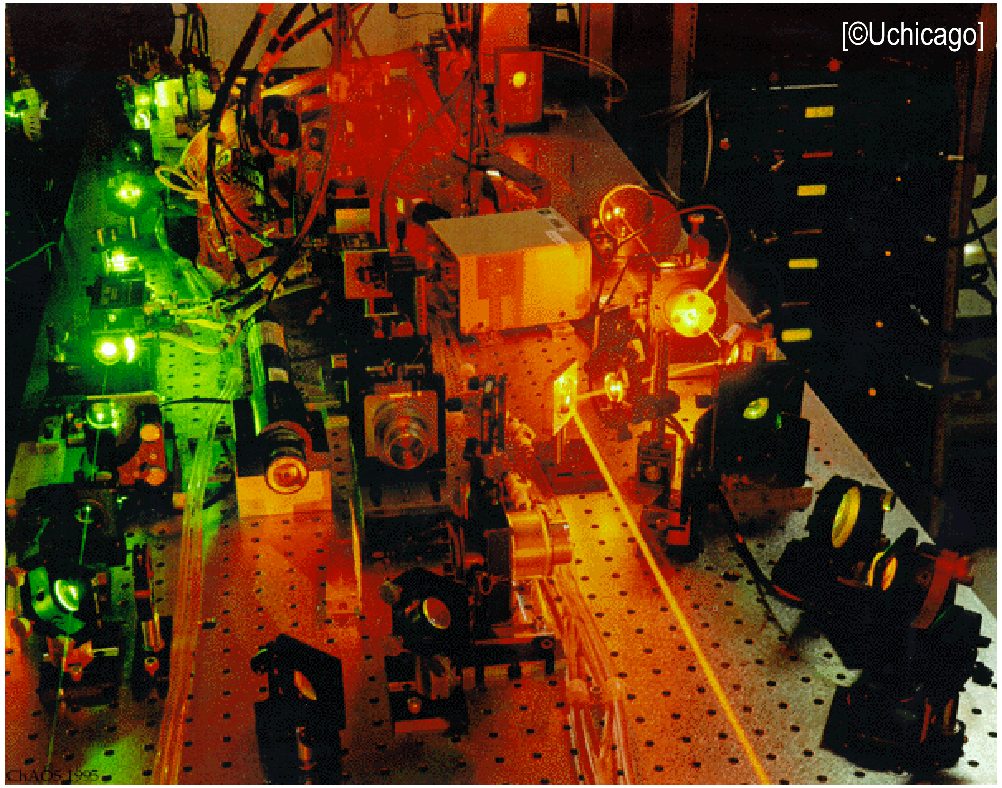
The Nobel Prize in Physics 2005 was divided, one half awarded to Roy J. Glauber *"for his contribution to the quantum theory of optical coherence"*,the other half jointly to John L. Hall and Theodor W. Hänsch *"for their contributions to the development of laser-based precision spectroscopy, including the optical frequency comb technique"*.

Hall and Hänsch have provided key contributions to optical frequency metrology using femtosecond-lasers

Kerr optical frequency combs

What's the point?

Mode-locked Nd: YAG lasers



Our objective is simple: downsize this!

Outline

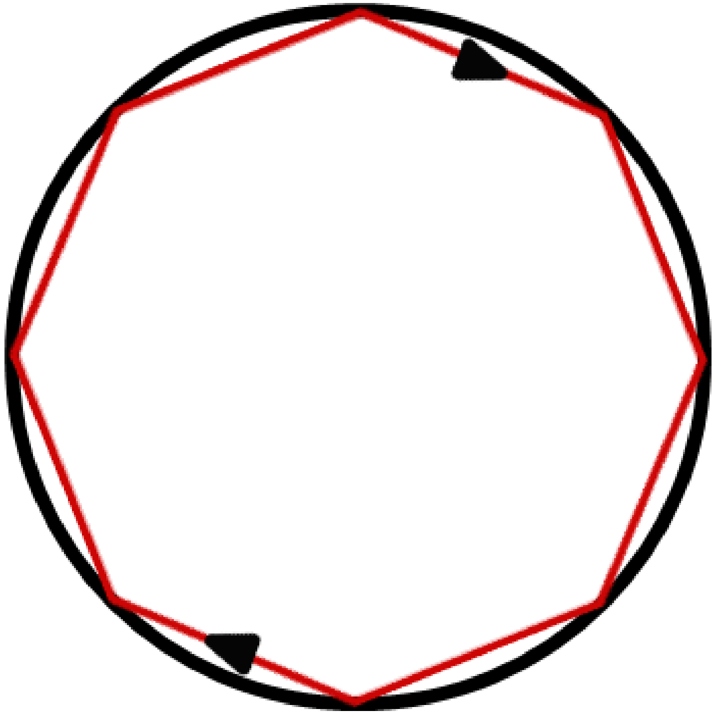
- Whispering-gallery mode resonators
- Kerr comb generation
- Applications of Kerr combs
- Modelling Kerr combs
- LMB/FEMTO-ST collaboration: A new approach

Whispering gallery mode resonators

Whispering gallery modes (WGM)

History of a weird name

Explanation from Lord Rayleigh
[Theory of sound. Vol II, 1878]



Dome of St Paul's
Cathedral in London

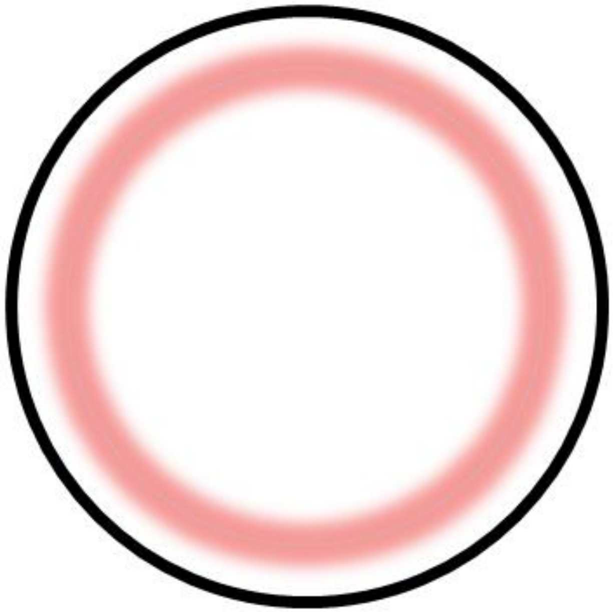
Whispering gallery modes (WGM)

Wave theory: Rayleigh vs Raman & Sutherland

[1001]

CXII. *The Problem of the Whispering Gallery.* *By* Lord Rayleigh, *O.M., F.R.S.**

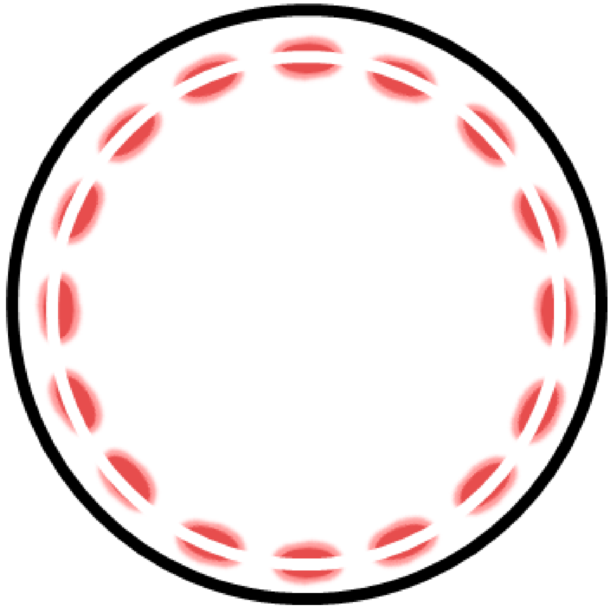
Phil. Mag. S. 6. Vol. 20. No. 120. *Dec.* 1910.



Nature (*London*) **108** 42 (1921)

C V RAMAN
G A SUTHERLAND

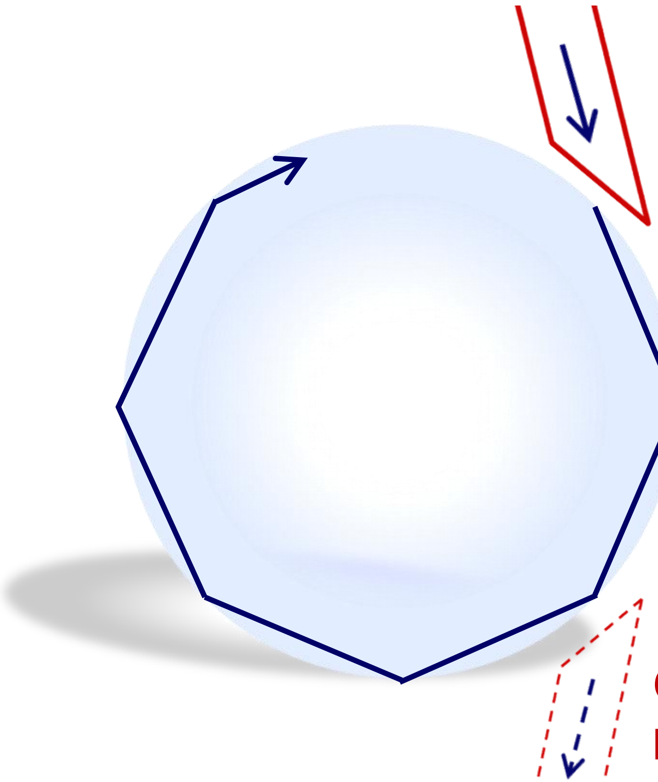
Whispering Gallery phenomena at St. Paul's Cathedral

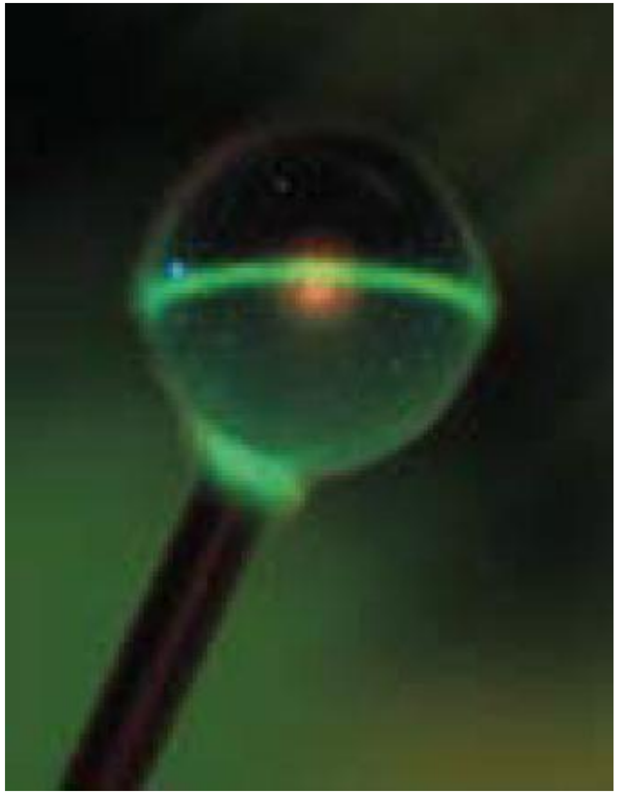


Whispering gallery modes

Trapping photons in an optical resonator

Consider a perfectly spherical optical cavity
(glass, crystal, etc.)

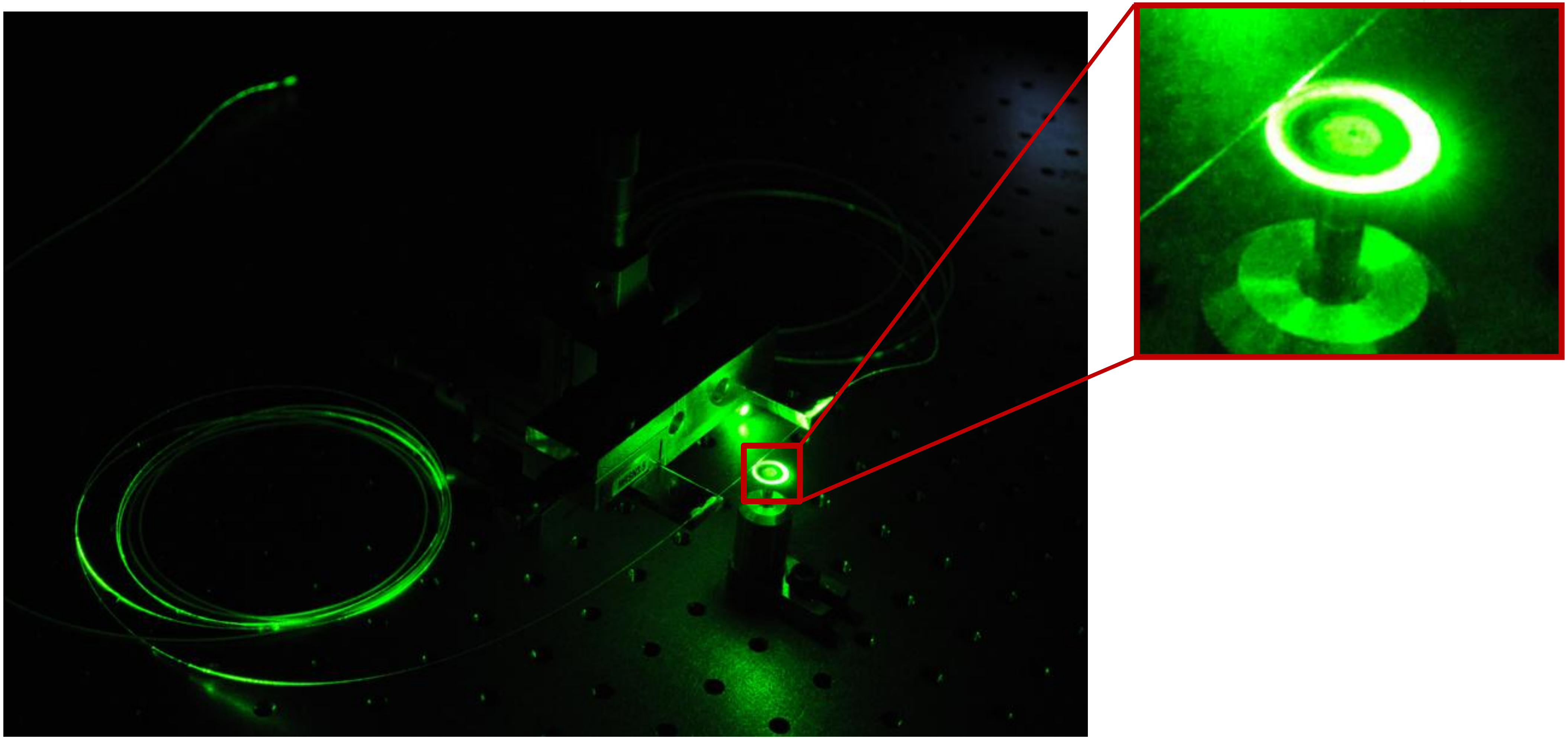




WGM in a microsphere (70 µm). ©Caltech.

Whispering gallery modes

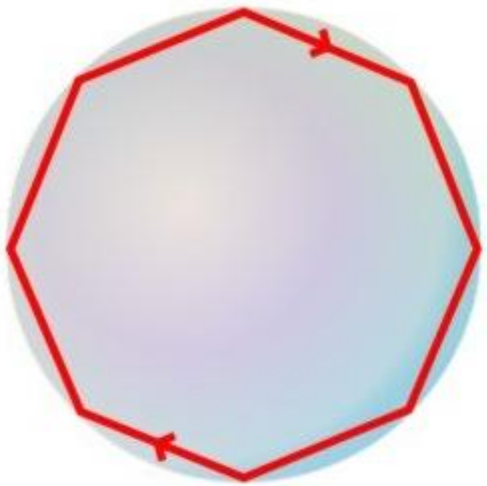
Home-made WGM resonators (Rémi Henriet/ Aurélien Coillet)



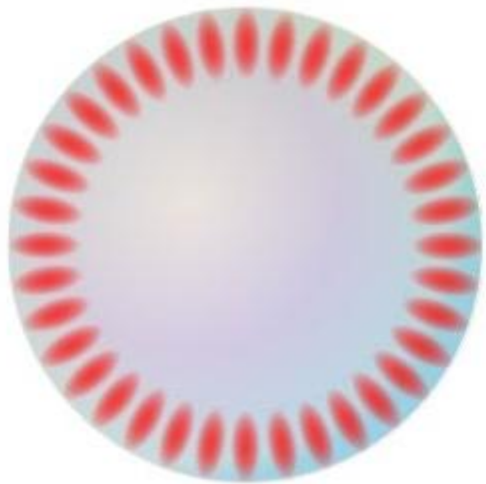
Ray & electromagnetic viewpoints

Toric modes

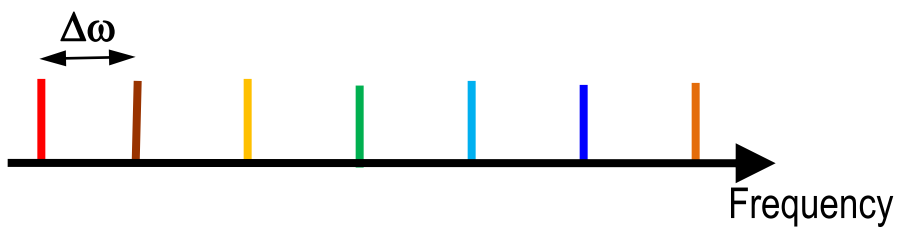
Ray optics interpretation



Electromagnetic wave interpretation



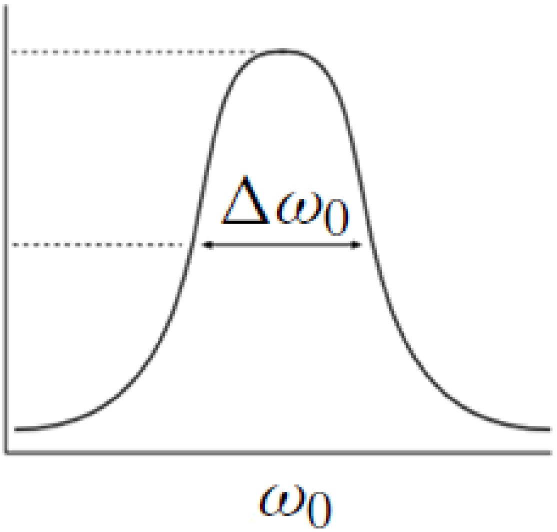
The eigenfrequencies of a WGM resonator are quasi-equidistant



Each toric eigenmode is characterized by a single eigennumber ℓ ,
which is the number of reflections to close a round trip.

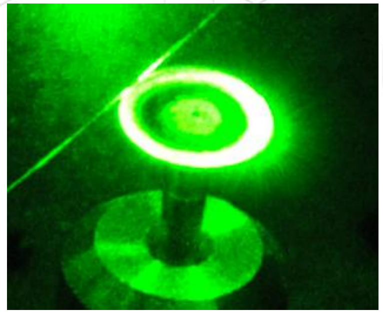
Photon lifetime

How long can you trap a photon in a cavity?



$$\tau_{ph} = \frac{1}{\Delta\omega_0}$$

Narrow linewidths = long photon lifetimes
= low losses



- In the picture, the green photons did undergo ~ 40000 reflections to perform a round-trip.
- They did perform typically ~ 10000 round-trips before annihilation.
- In the picture, the green photons stayed into the cavity for ~ 0.1 µs.
- Record is set at ~ 100 µs (for infrared photons).

Why caring about the photon lifetime?

A (very indirect…) way to win the Nobel Prize!

September 15, 1995 / Vol. 20, No. 18 / OPTICS LETTERS 1835

Splitting of high-Q Mie modes induced by light backscattering in silica microspheres

D. S. Weiss,* V. Sandoghdar, J. Hare, V. Lefèvre-Seguin, J.-M. Raimond, and S. Haroche

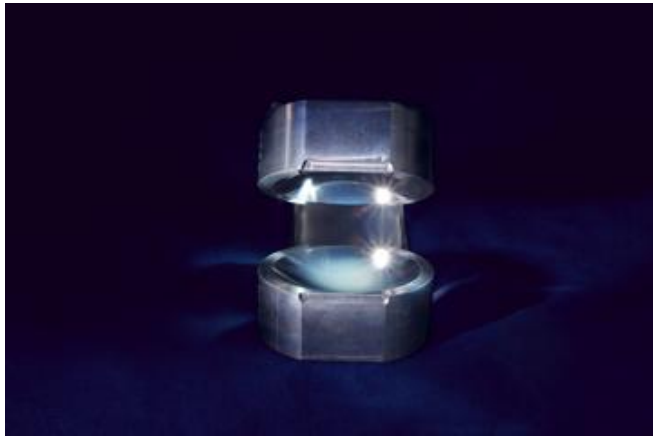
PHYSICAL REVIEW A VOLUME 54, NUMBER 3 SEPTEMBER 1996

Very low threshold whispering-gallery-mode microsphere laser

V. Sandoghdar,* F. Treussart, J. Hare, V. Lefèvre-Seguin, J.-M. Raimond, and S. Haroche

Serge Haroche,
Nobel Prize of Physics 2012
For his experiments on
the non-destructive measurement of photons

The photon box



Photon lifetime ~ 0.1 s

Kerr optical frequency combs

The Kerr effect

The refraction index depends on the field

$$n = n_0 + n_2 I \quad \text{with} \quad I \propto \|\mathbf{E}\|^2$$

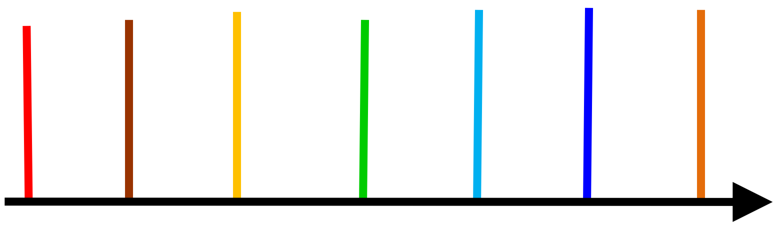
Refractive index

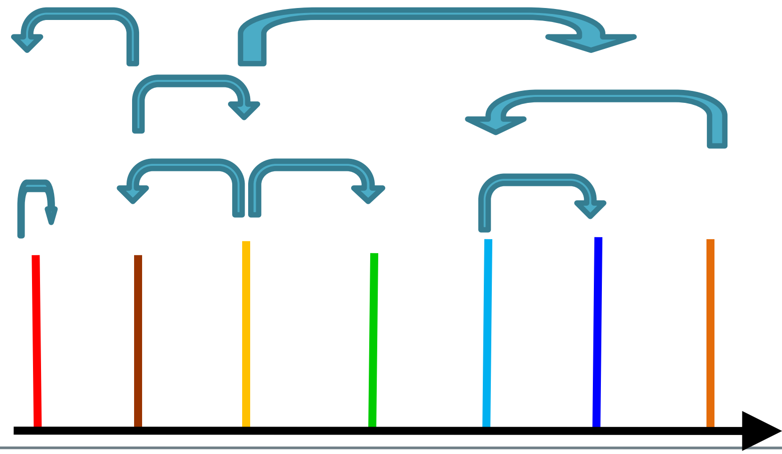
Kerr coefficient

Irradiance

Linear cavity: photons **DO NOT** interact.
Each mode is independent from the others

Nonlinear cavity: photons **DO** interact!
All the modes interact with each other.





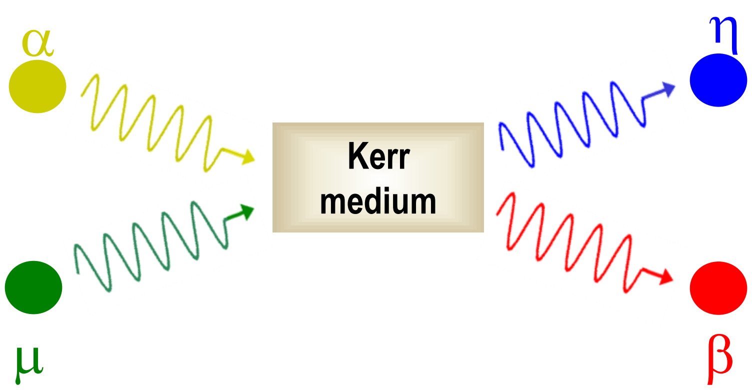
Kerr comb generation

Quantum representation

Four-wave mixing

Two photons are annihilated and two new photons are created with different frequencies:

$$\hbar\omega_\alpha + \hbar\omega_\mu \rightarrow \hbar\omega_\beta + \hbar\omega_\eta$$

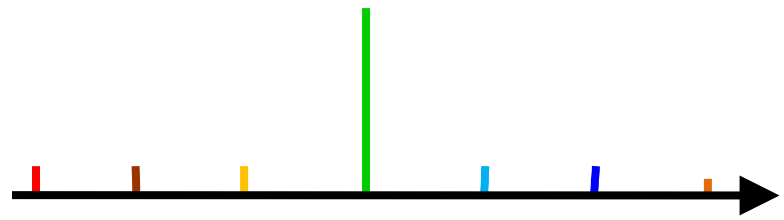


Conservation of energy and angular momentum

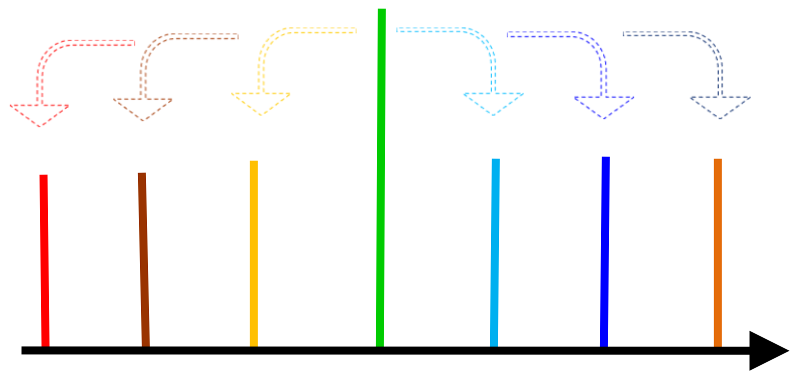
$$\omega_\alpha + \omega_\mu = \omega_\beta + \omega_\eta$$
$$\ell_\alpha + \ell_\mu = \ell_\beta + \ell_\eta$$

Comb generation

Pump under threshold



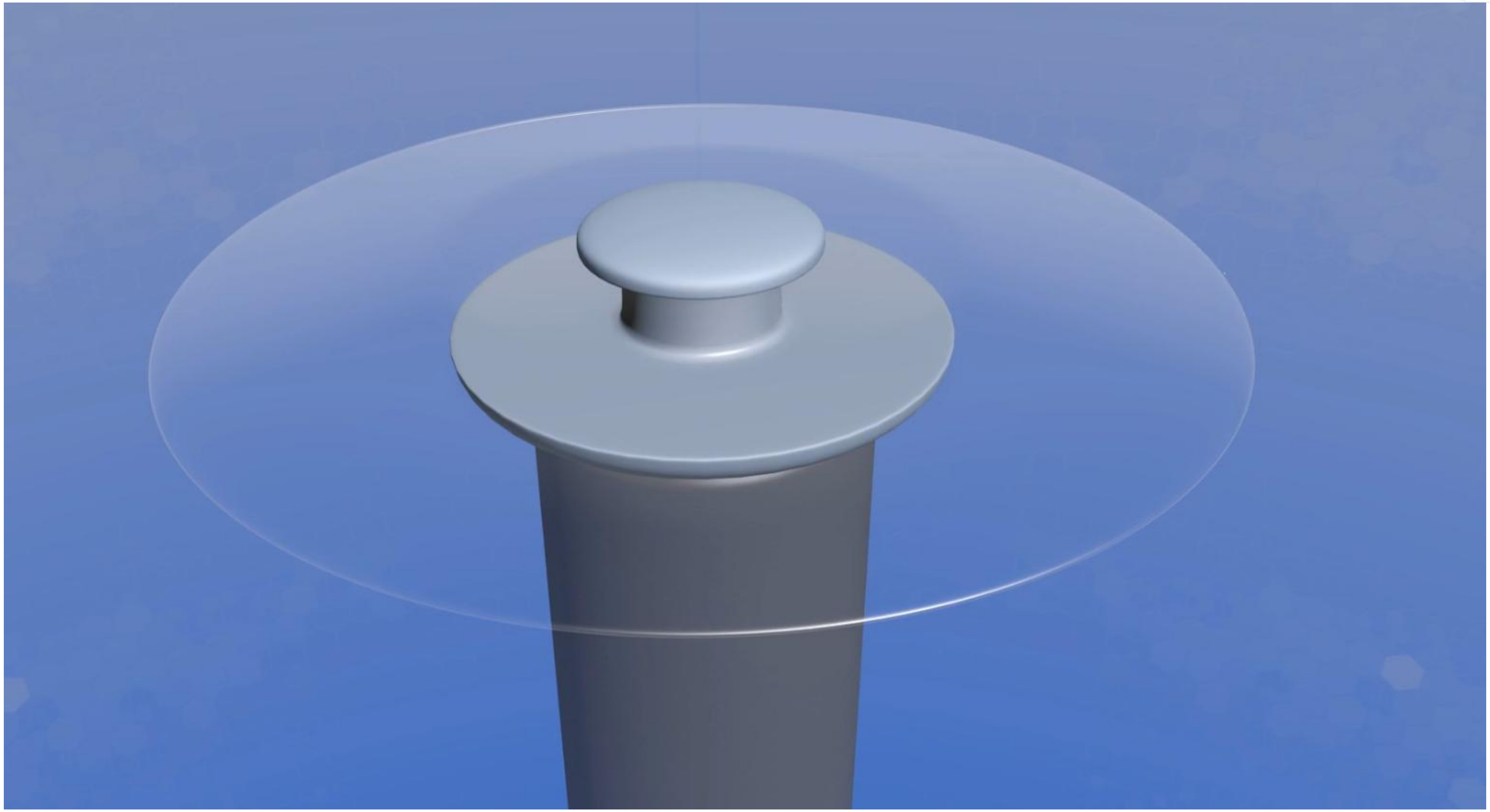
Pump above threshold



$$2\,\hbar\omega_p \rightarrow \hbar[\omega_p + n\Delta\omega] + \hbar[\omega_p - n\Delta\omega]$$

Animated illustration of Kerr combs

An artwork from Rémi Henriet

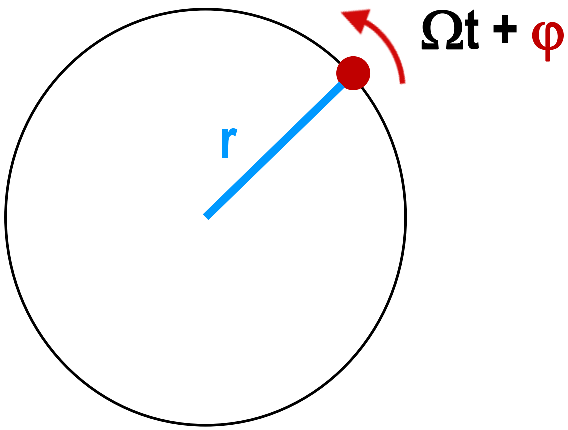


Applications of Kerr combs

Phase noise

Or why normal forms of Hopf bifurcations matter

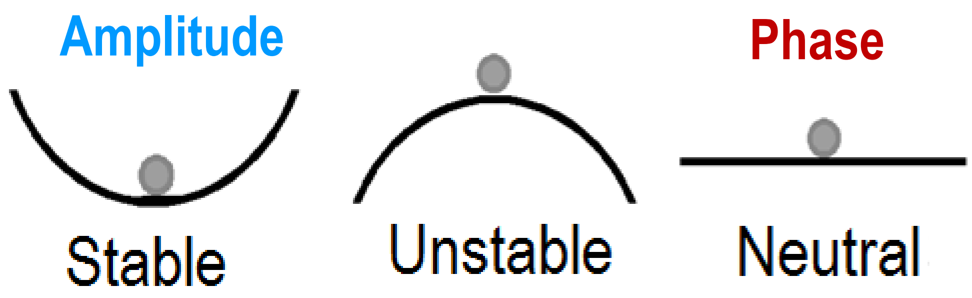
Limit cycle oscillator



x(t)= r cos[Ωt + φ]

$$\dot{z} = \gamma z - |z|^2 z,$$
$$\text{with } z = re^{i\varphi}$$
$$\dot{r} = \gamma r - r^3$$
$$\dot{\varphi} = 0$$

Stability of the equillibria



$$\dot{z} = \gamma z - |z|^2 z + \zeta(t)$$
$$\dot{r} = \gamma r - r^3 + \xi_r(t)$$
$$\dot{\varphi} = \xi_\varphi(t)$$

$$\langle|\varphi|\rangle \propto \sqrt{t}$$

Consequences:

- Phase noise is *more important* than amplitude noise
- Actually, phase noise increases *unboundedly* (divergence)

Ultra-stable microwaves

An aerospace engineering flagship application

Noiseless oscillator

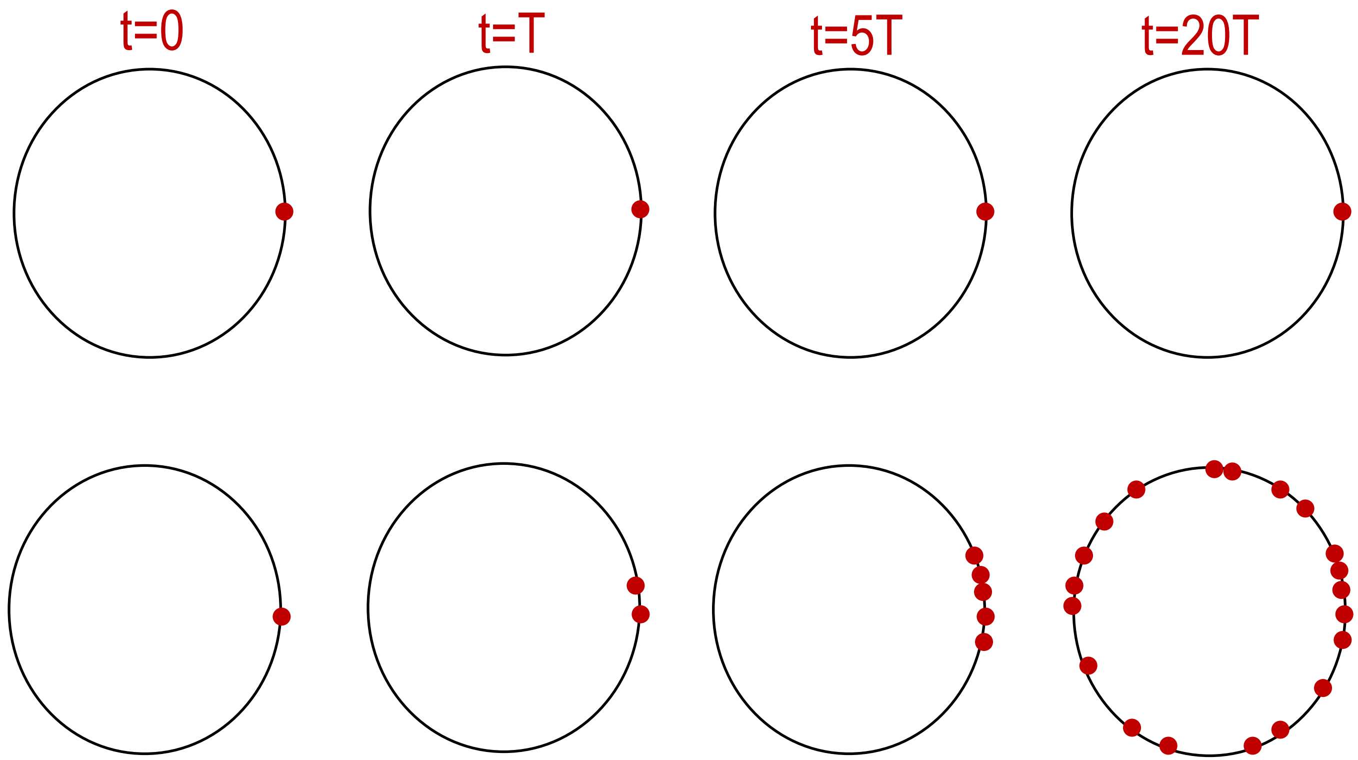
It is a *perfect* clock.

The phase is rigorously constant.

Noisy oscillator

This clock *looses accuracy* with time.

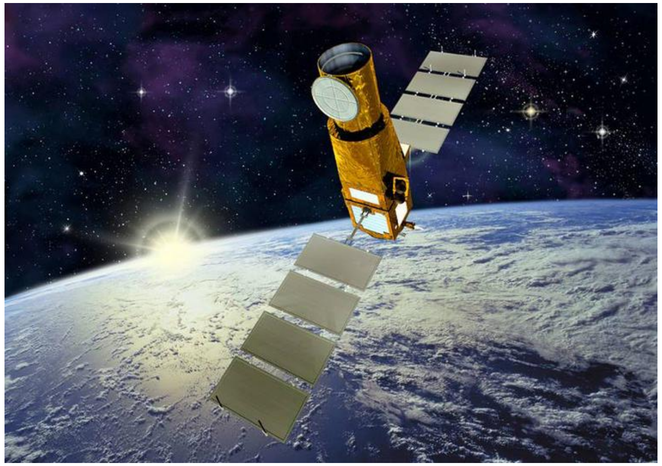
The phase performs a 1D-brownian motion along the circle (stochastic calculus!).



Ultra-stable microwaves

Clocks and radars

Clocks



Radars

Orders of magnitude:

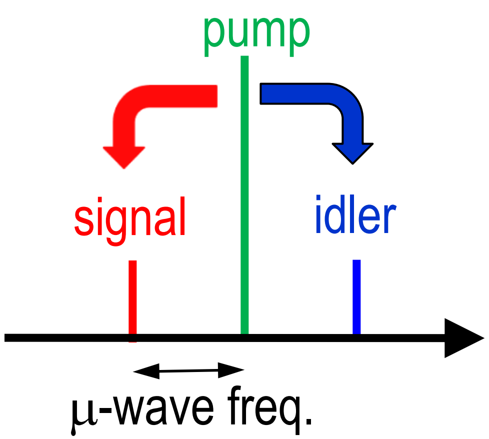
Error of a quartz wristwatch: ~ 1 s /day
Error of a GPS atomic clock: ~ 10^{-8} s /day
Error of a cryogenic clock: ~ 10^{-10} s /day

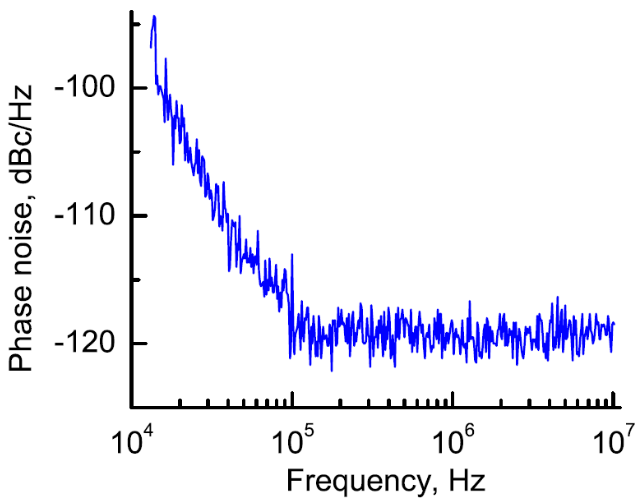
Orders of magniture (~10GHz) :

- Microwave synthetizer: ~ -80 dBc/Hz @10kHz
- Radar probes ~ -140 dBc/Hz @10kHz
- Best microwave sources: ~ -170 dBc/Hz @10kHz

Ultra-stable microwaves

Advantages of Kerr combs

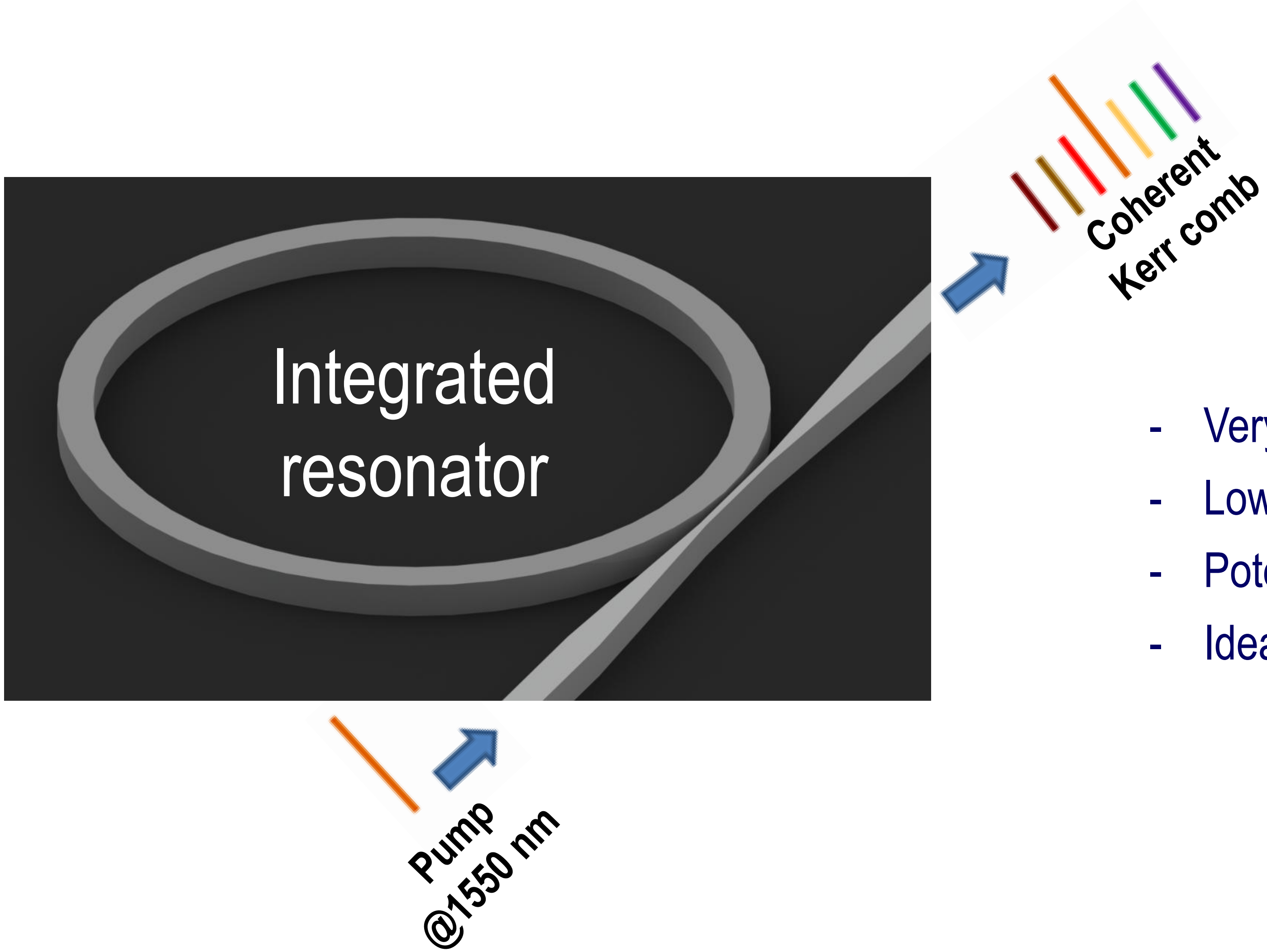




- Compactness
- Small power consumption
- Immunity to interferences
- Frequency versatility
- Compatibility with both microwave and lightwave communication systems
- Ultra-stable signal distribution without penalty over fiber networks (*i.e.*, long distances)

Multi-wavelength coherent source

Interest for fiber telecommunication networks

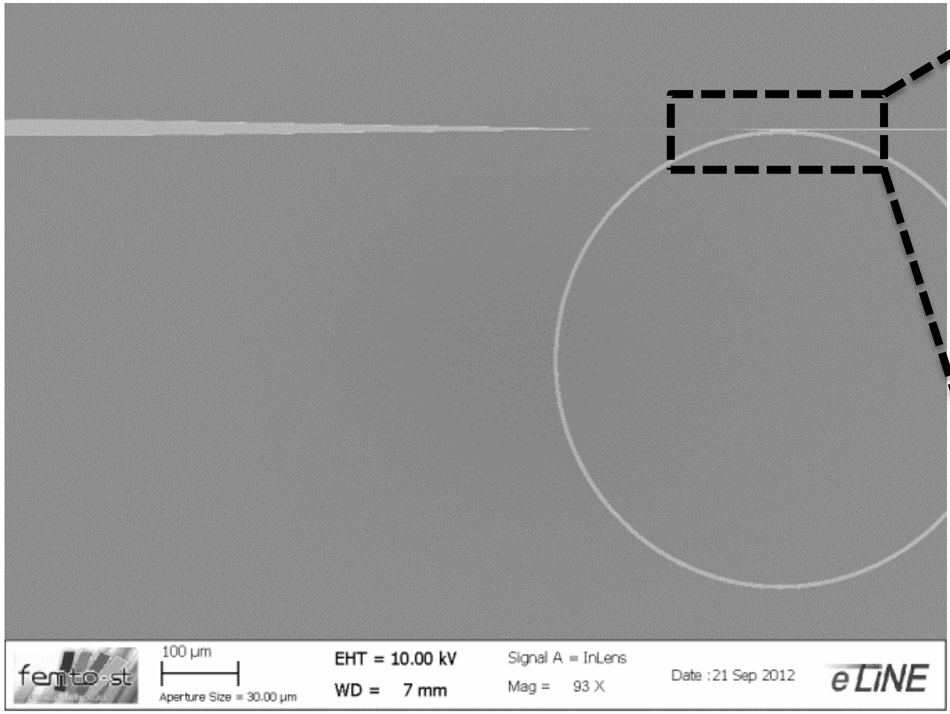


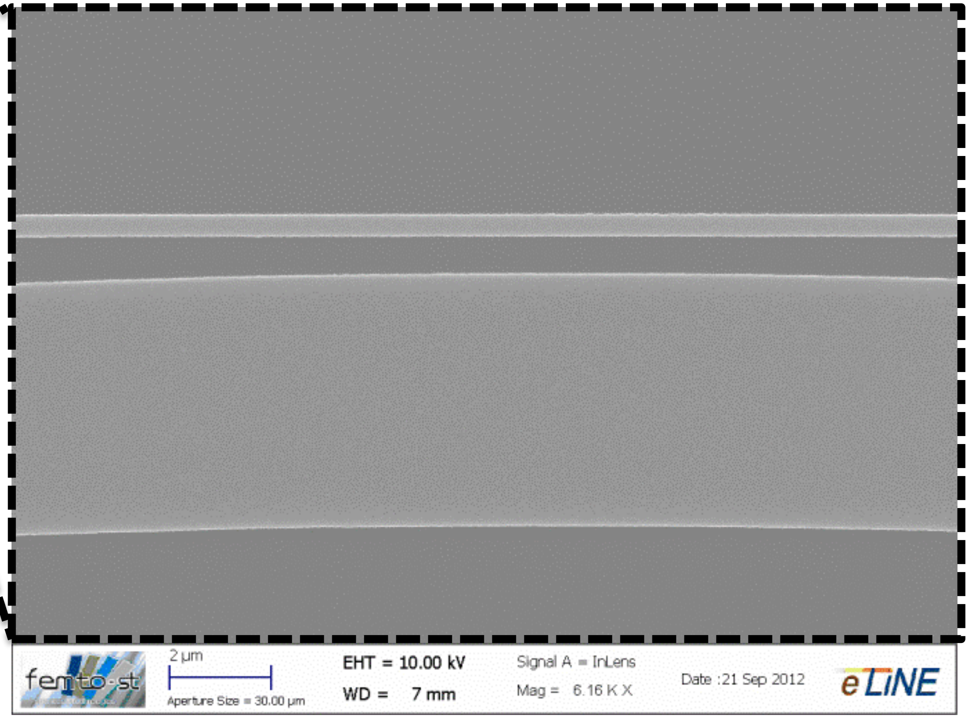
- Very low-cost component
- Low power consumption (~ mW)
- Potentially Multi Tb/s bandwidth
- Ideal for component for FTTH

Integrated resonators @FEMTO-ST

LabEx ACTION

Integrated WGM resonators for photonics applications



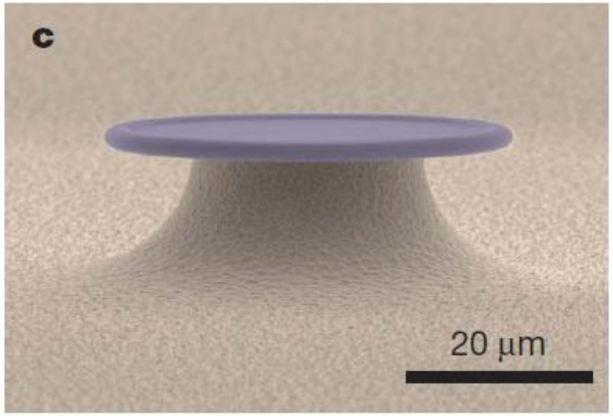


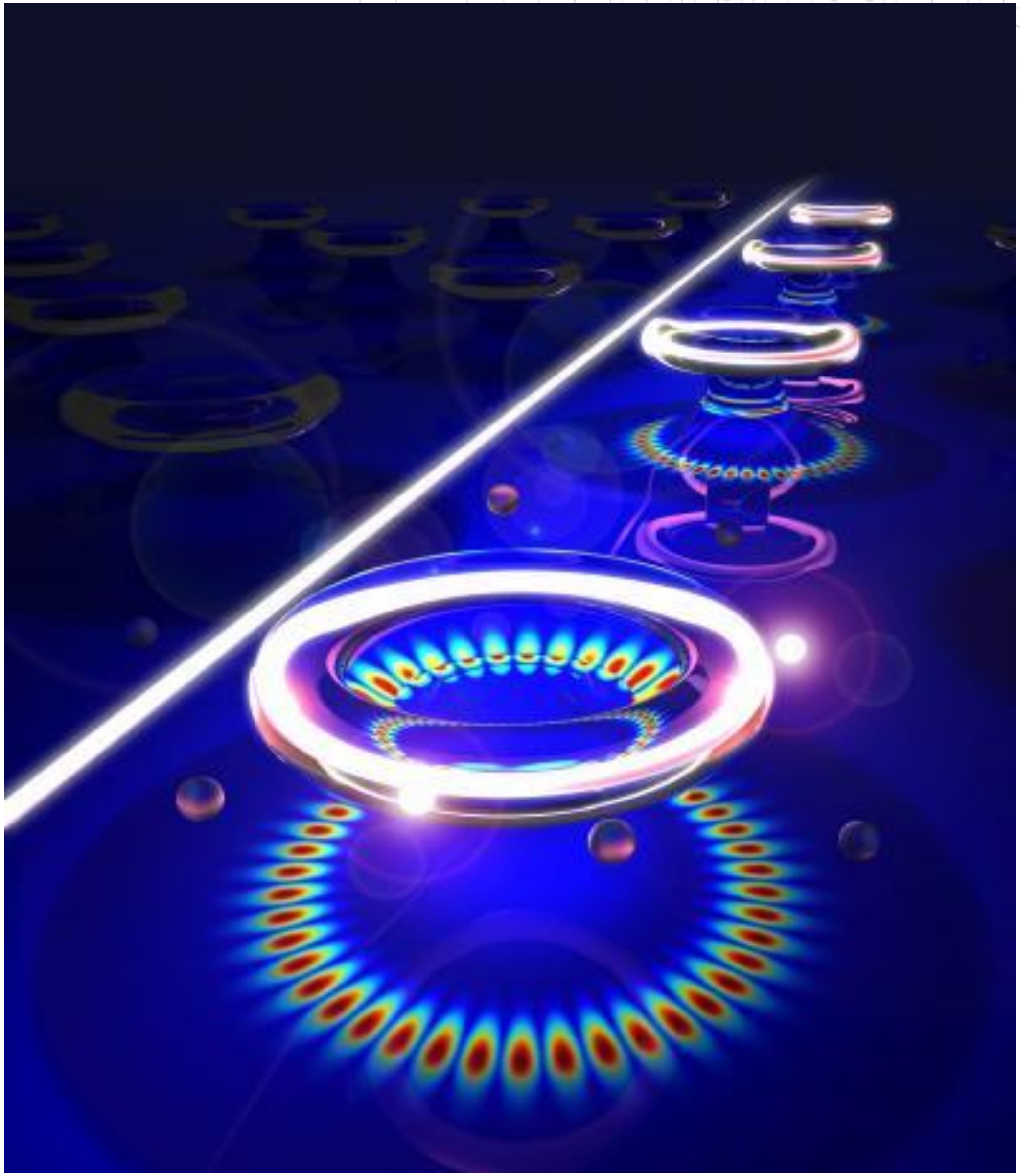
Fabricated by Roland Salut

LabeXction
Integrated smart systems

Photonic chips

Towards full-optical processors?





Full-optical processing:

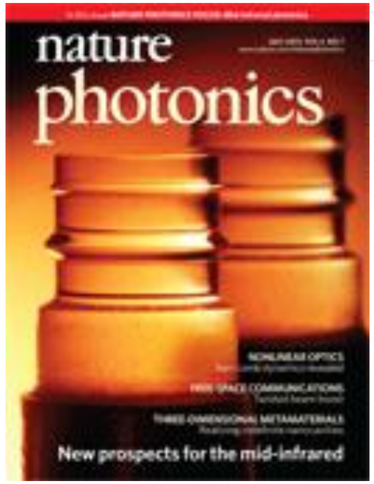
Requires optical components able to perform both nonlinear and linear operations.

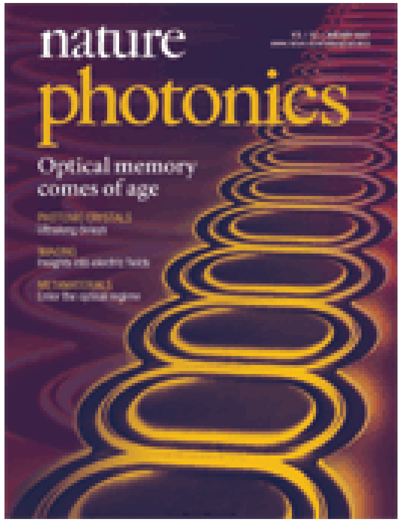
Nonlinear: Kerr comb generation, Brillouin lasing, Raman scattering, two-photon absorption, etc.

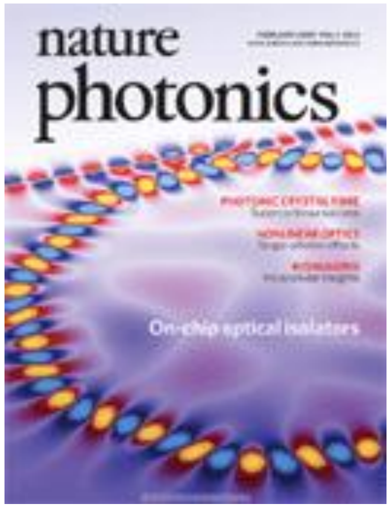
Linear: amplification, filtering, modulation, phase-shifting, etc..

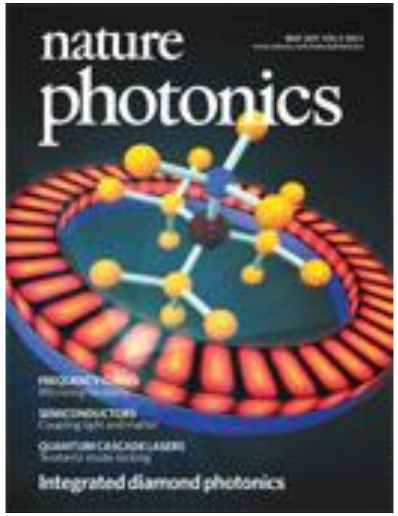
WGMR applications beyond Kerr combs

Some recent ‘*Nature*’ covers on WGM resonators









Modelling Kerr combs

Coupled modes rate equations

One equation per mode (ODEs)

$|\mathcal{A}_\eta|^2$ = number of photons in the mode η

$$\dot{\mathcal{A}}_\eta = -\frac{1}{2}\Delta\omega_\eta \mathcal{A}_\eta + \delta_{\eta\ell_0} \cdot \frac{1}{2}\Delta\omega_\eta \mathcal{F}_\eta e^{i\sigma t}$$
$$-ig_0 \sum_{\alpha,\beta,\mu} \Lambda_\eta^{\alpha\beta\mu} \mathcal{A}_\alpha \mathcal{A}_\beta^* \mathcal{A}_\mu e^{i\varpi_{\alpha\beta\mu\eta} t}$$

- Modal bandwidth: $\Delta\omega_\eta$
- External pumping: $\mathcal{F}_\eta$
- Laser detuning: σ
- Four-wave mixing gain: g_0
- Intermodal coupling tensor: $\Lambda_\eta^{\alpha\beta\mu}$
- Intermodal detuning: $\varpi_{\alpha\beta\mu\eta}$

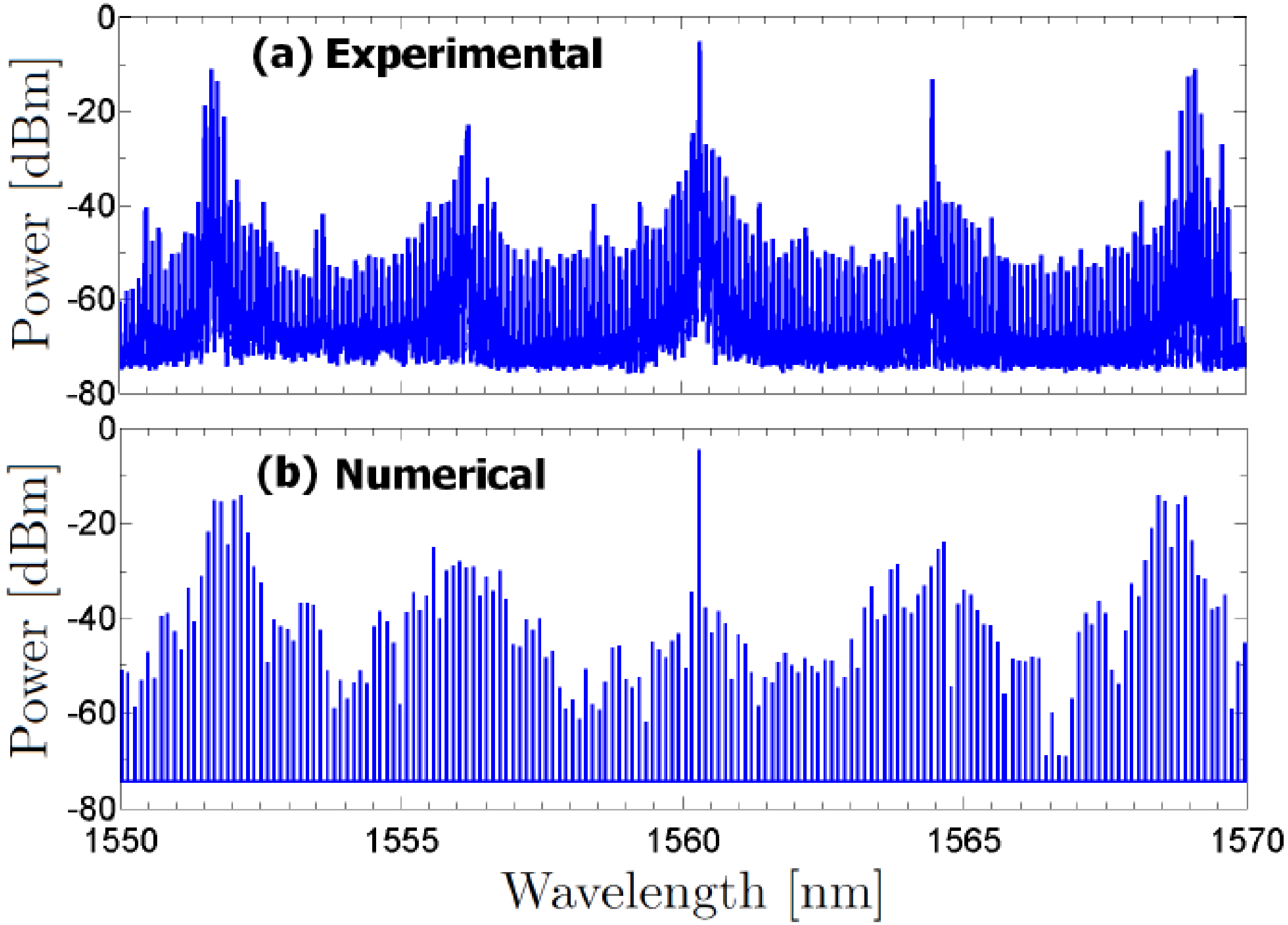
$$g_0 = \frac{n_2 c}{n_0^2}\frac{\hbar\omega_{\eta 0}^2}{V_{\eta 0}}$$

$$\propto \int_V [\boldsymbol{\Upsilon}_\eta^* \cdot \boldsymbol{\Upsilon}_\mu][\boldsymbol{\Upsilon}_\beta^* \cdot \boldsymbol{\Upsilon}_\alpha] dV$$

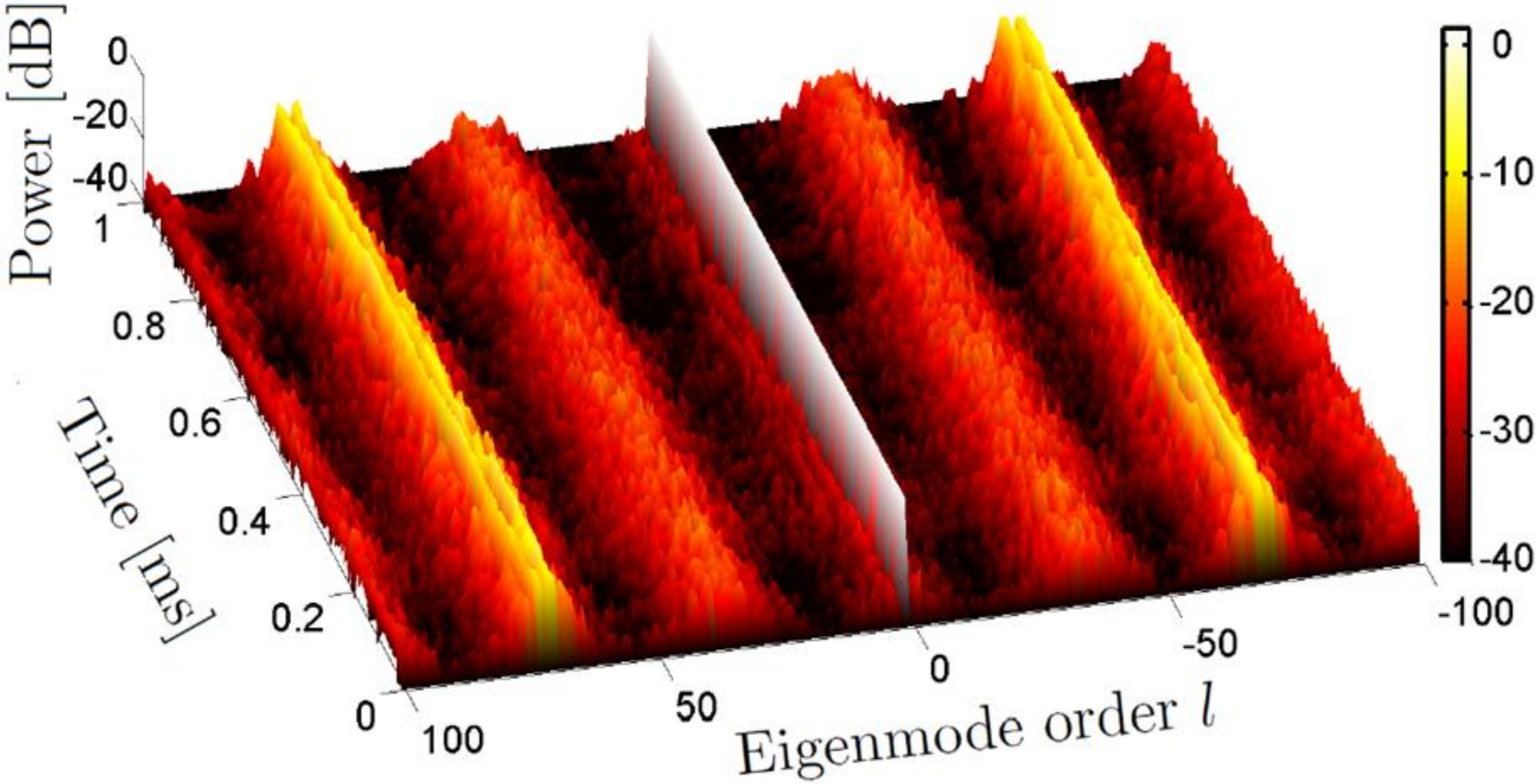
$$\varpi_{\alpha\beta\mu\eta} = \omega_\alpha - \omega_\beta + \omega_\mu - \omega_\eta$$

Numerical and experimental results

An excellent agreement



Spectro-temporal representation



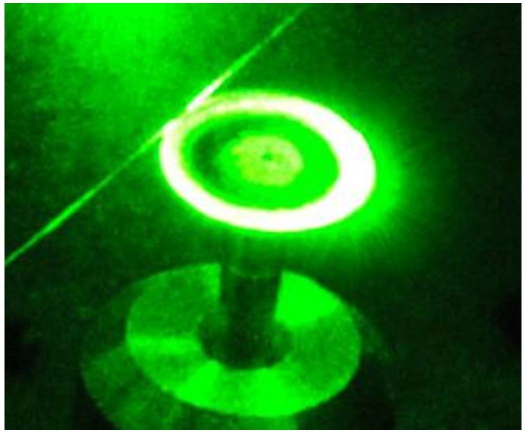
Chaos

Maximal Lyapunov exponent: $\tilde{\lambda} \simeq 8.4 \times 10^4 \text{ s}^{-1}$

A spatiotemporal approach

The modal expansion model

- Provides excellent results (determination of thresholds, explanation of the cascade mechanism, etc.)
- But also has drawbacks, mainly in terms of simulation time of wide-span combs: up to several days!



Light just propagates like in a nonlinear and dispersive optical fiber!

However, the ‘fiber’ is periodic!

<u>Question</u>:

Is it possible to find a kind of nonlinear Schrödinger equation able to model Kerr comb generation?

Equation for the total intra-cavity field

One equation for all modes (PDE)

Total field:

$$\psi(\theta, t) = \sum_{\ell} \mathcal{A}_{\ell}(t) \exp[i(\omega_{\ell} - \omega_0)t - i(\ell - \ell_0)\theta]$$

Dynamical equation for ψ:

$$\frac{\partial \psi}{\partial \tau} = -(1 + i\alpha)\psi + i|\psi|^2\psi - i\frac{\beta}{2}\frac{\partial^2 \psi}{\partial \theta^2} + F$$

Damping — Detuning — Kerr nonlinearity — Dispersion — Laser Pump

This is the Lugiato-Lefever equation

Lugiato-Lefever equation

A paradigm for localized structures in dissipative optical cavities

VOLUME 58, NUMBER 21 PHYSICAL REVIEW LETTERS 25 MAY 1987

Spatial Dissipative Structures in Passive Optical Systems

L. A. Lugiato
Dipartimento di Fisica del Politecnico di Torino, I-10129 Torino, Italy

and

R. Lefever
Service de Chimie-Physique II, University of Brussels, B-1050 Brussels, Belgium

(Received 24 November 1986)

Diffraction

$$\frac{\partial E}{\partial \bar{t}} = -E + E_I + i\eta E(|E|^2 - \theta) + ia\frac{\partial^2 E}{\partial \bar{x}^2}.$$

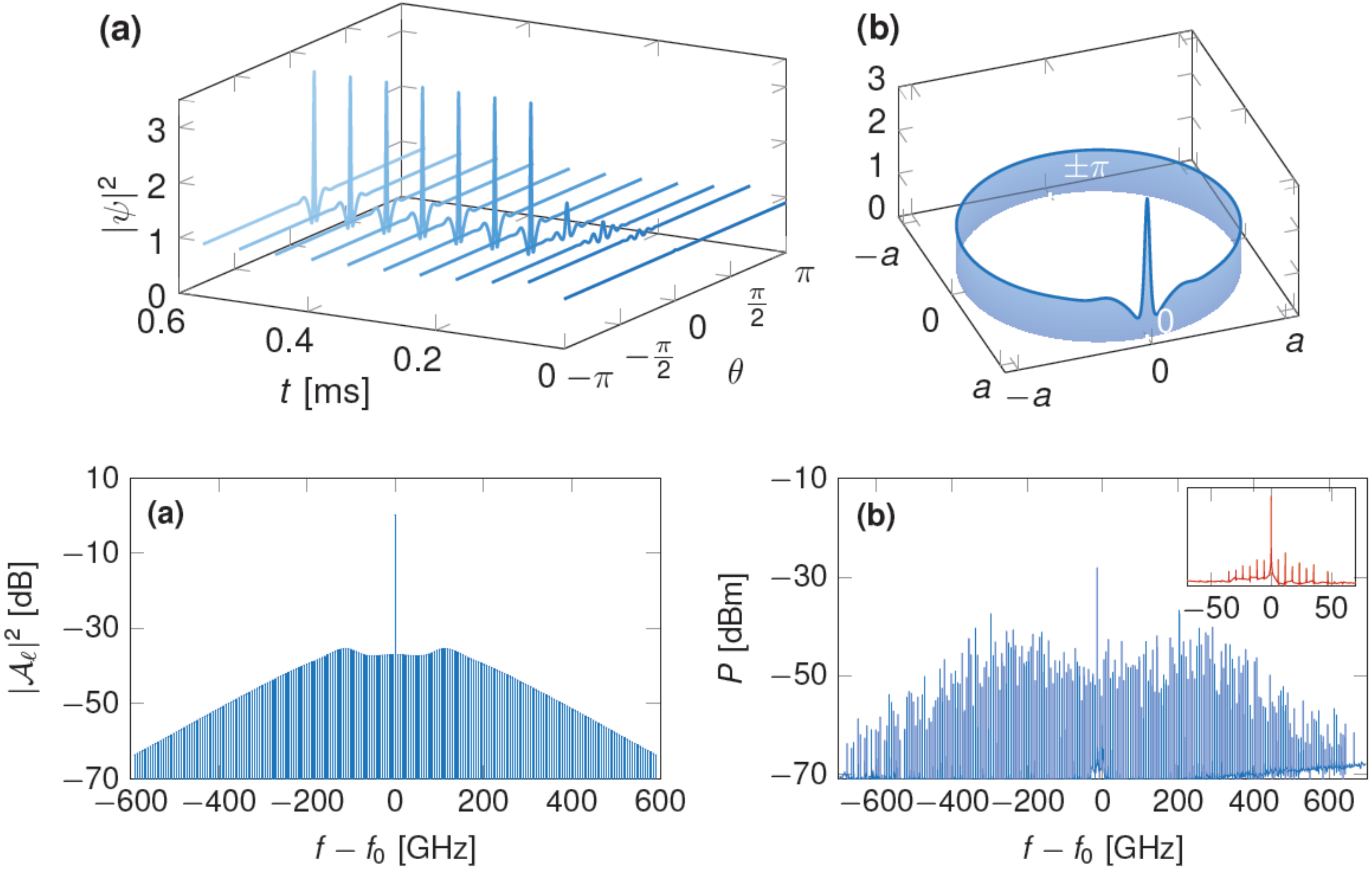
$$\frac{\partial \psi}{\partial \tau} = -(1+i\alpha)\psi + i|\psi|^2\psi - i\frac{\beta}{2}\frac{\partial^2 \psi}{\partial \theta^2} + F$$

Dispersion

Angular variable

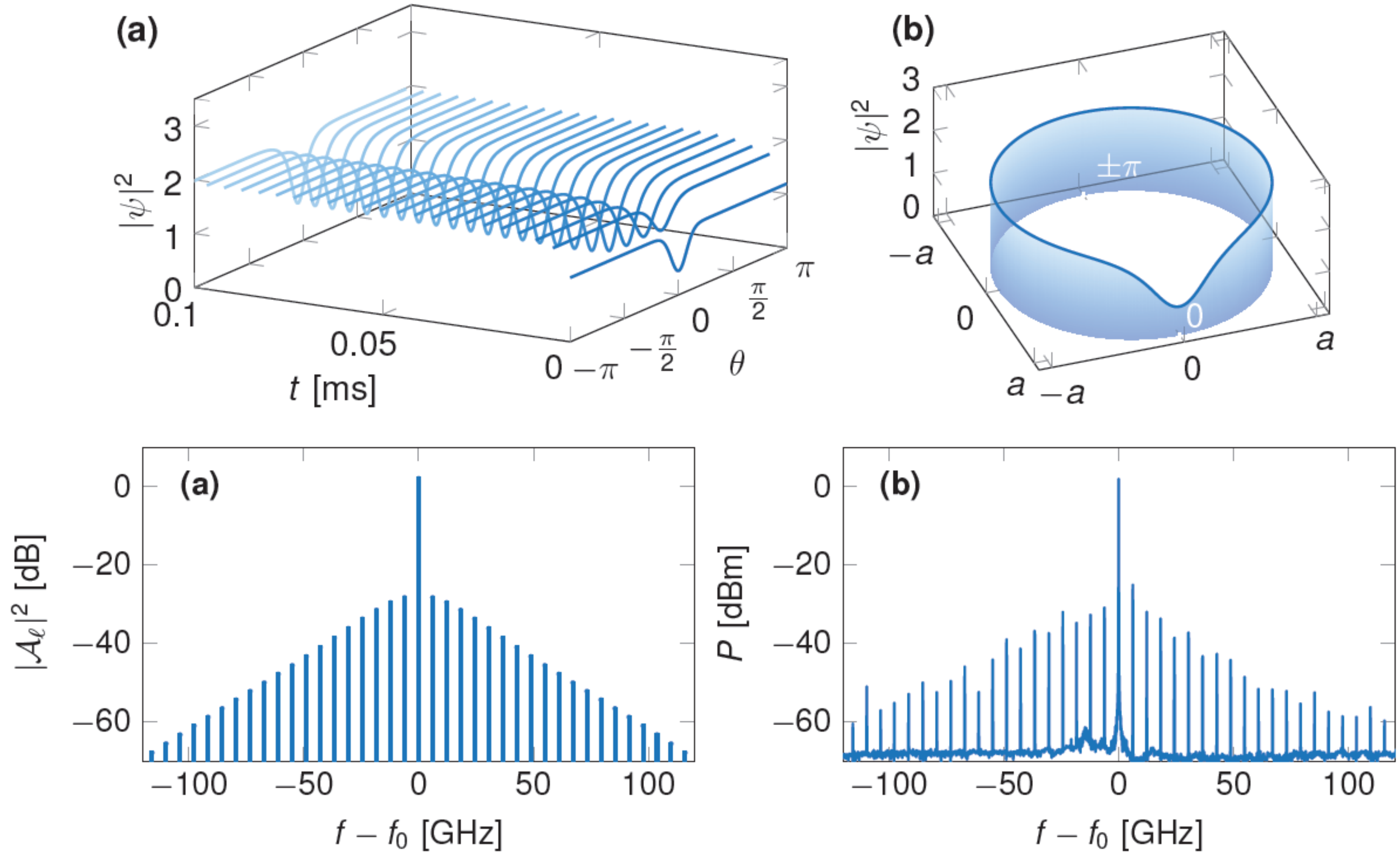
Lugiato-Lefever equation for Kerr combs

Bright azimuthal cavity solitons



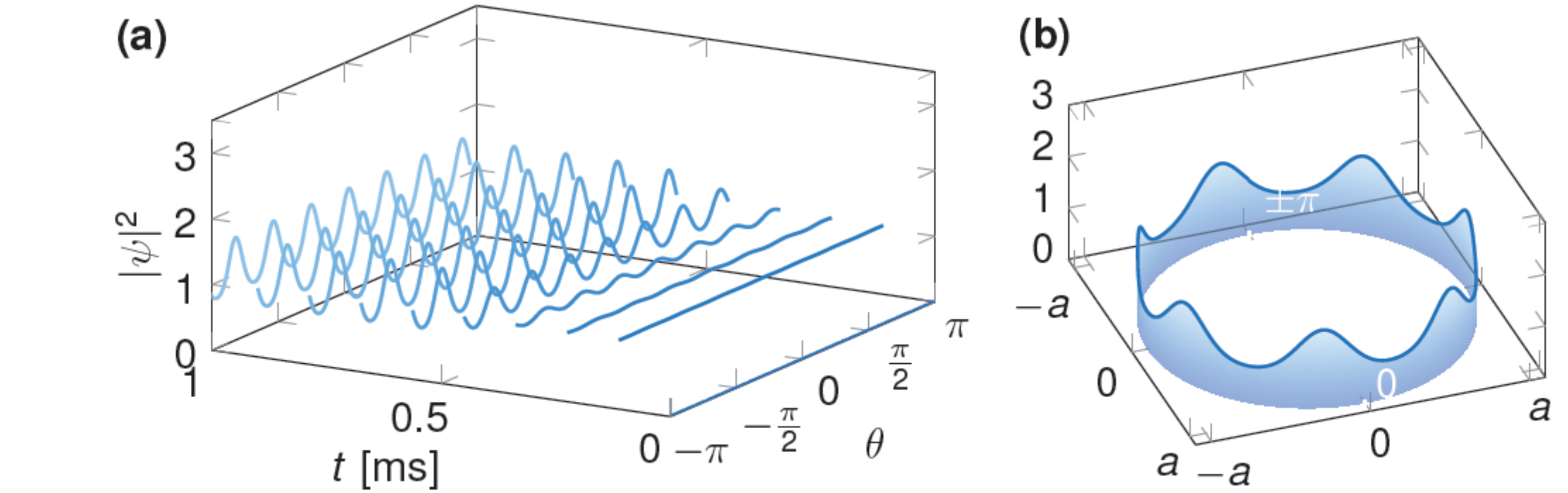
Lugiato-Lefever equation for Kerr combs

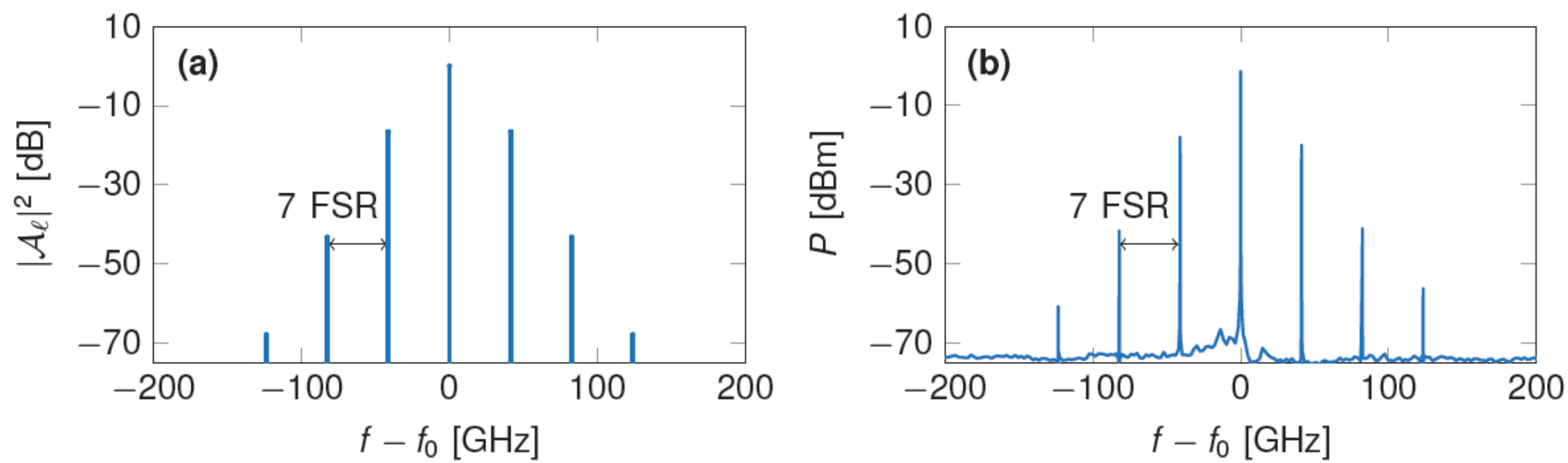
Dark azimuthal cavity solitons



Lugiato-Lefever equation for Kerr combs

Stripe Turing patterns





Lugiato-Lefever equation (LLE)

A theoretical breakthrough for Kerr combs

- The LLE belongs to the NLSE family. It can easily integrate new features like:
  - Nonlinearities,
  - Higher-order dispersion,
  - New phenomena like Brillouin, Raman and Rayleigh scattering.

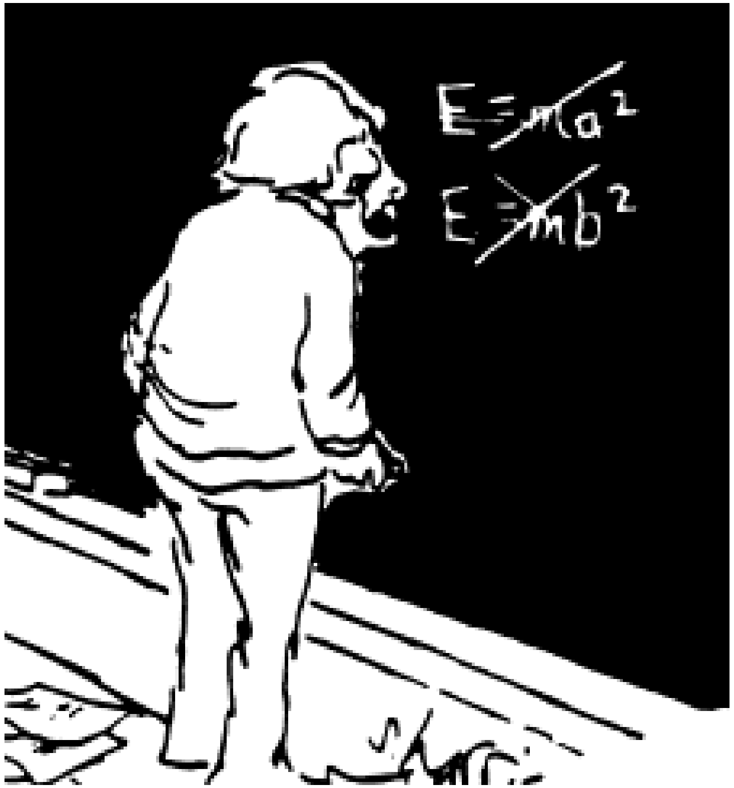
- The LLE enables easily enables to spot phenomena that are not easy to spot with the modal approach (solitons for example)

- Simulating the LLE is (damned) fast!
  Owing the periodic-boundary nature of the FFT algorithm, it is 1000 to 10000 times faster than the modal expansion model.

- The spatiotemporal LLE and the modal approaches provide two twin-models that are complementary and powerful in distinct ways.

A new approach: nonlinear dynamics with PDEs

Collaboration between LMB & FEMTO-ST



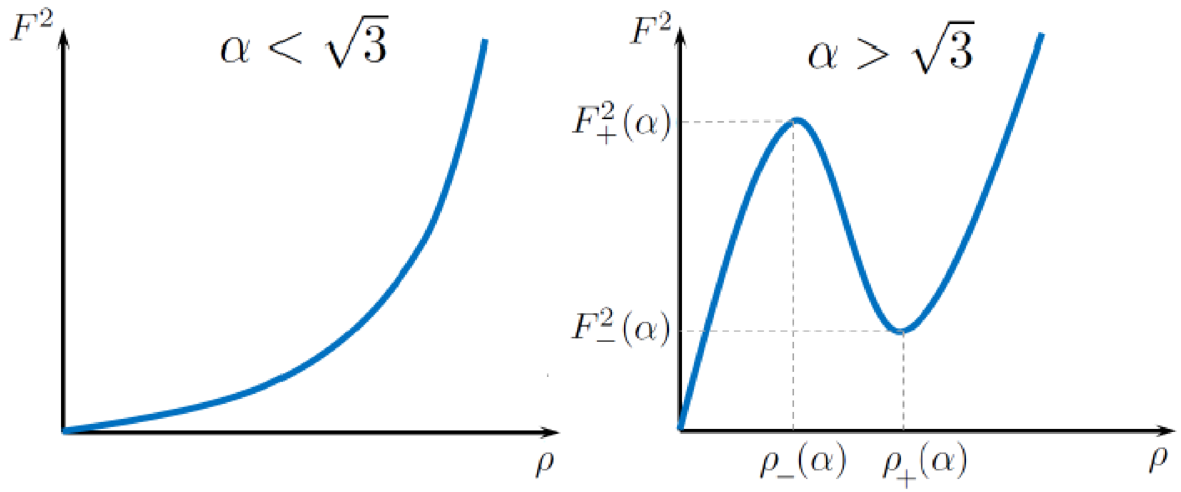
Transverse project:

Periodic waves in nonlinear and dispersive media: Applications Kerr combs

Equilibria and their stability

$$\rho = |\psi_0|^2$$

$$F^2 = [1 + (\rho^2 - \alpha)^2]\rho^2$$

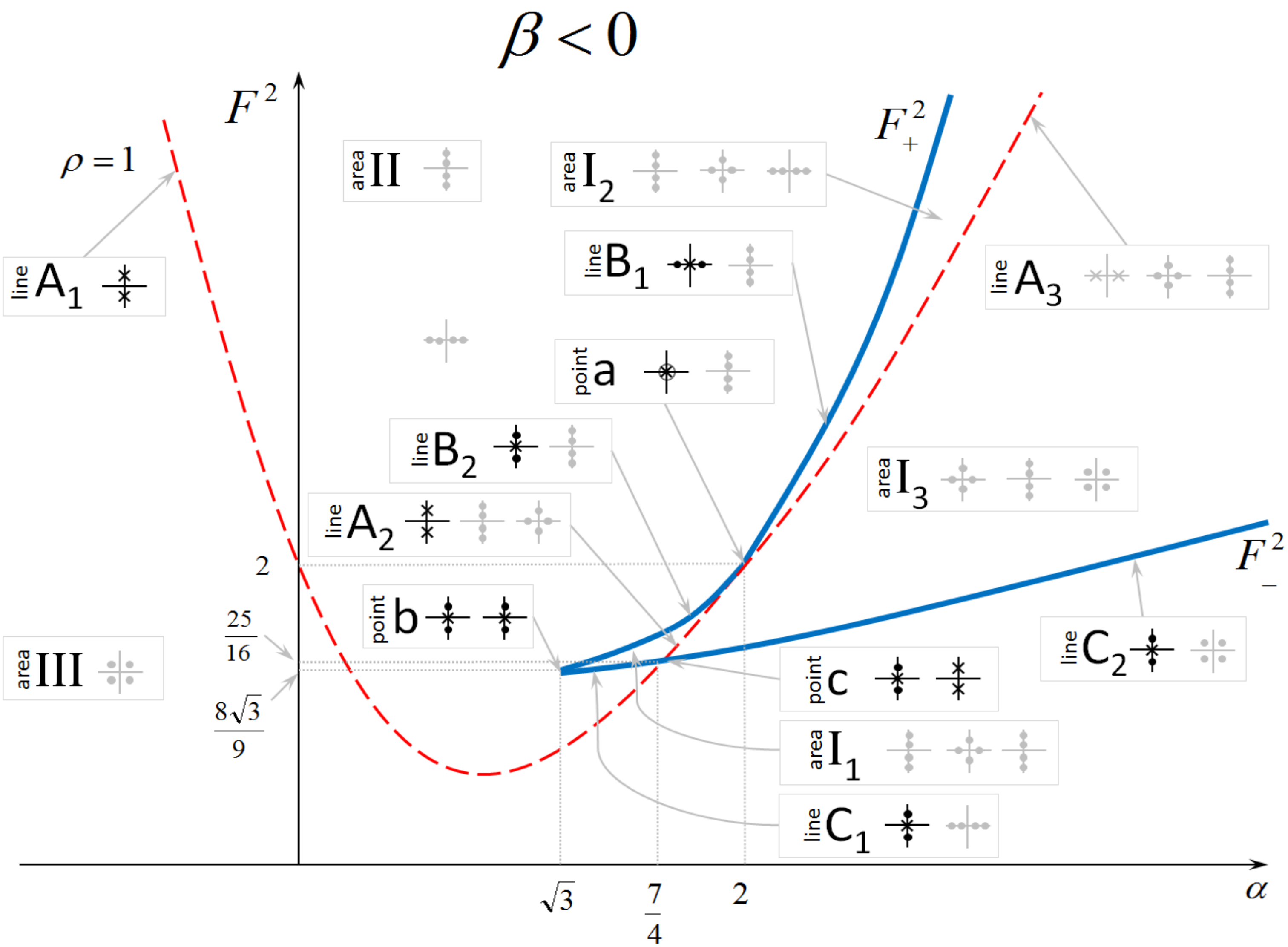


Stability is determined by the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{\beta}(3\psi_{e,r}^2 + \psi_{e,i}^2 - \alpha) & 0 & \frac{2}{\beta}(2\psi_{e,r}\psi_{e,i} - 1) & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2}{\beta}(2\psi_{e,r}\psi_{e,i} + 1) & 0 & \frac{2}{\beta}(\psi_{e,r}^2 + 3\psi_{e,i}^2 - \alpha) & 0 \end{bmatrix}$$

Bifurcation map

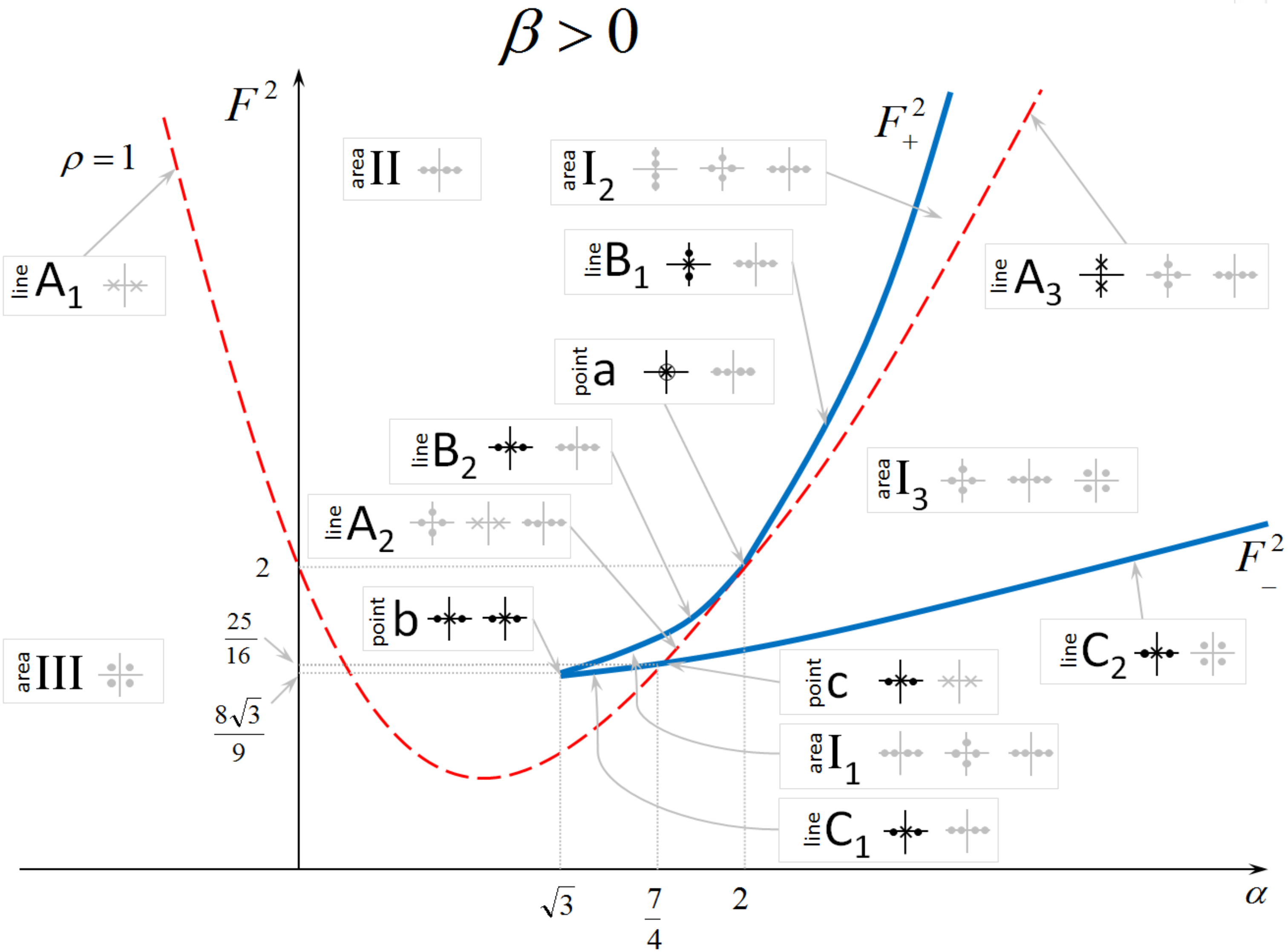
Case of anomalous dispersion



Pictogram	Bifurcation
	O^4
	O^2
	$O^2(i\omega)$
	$(i\omega)^2$

Bifurcation map

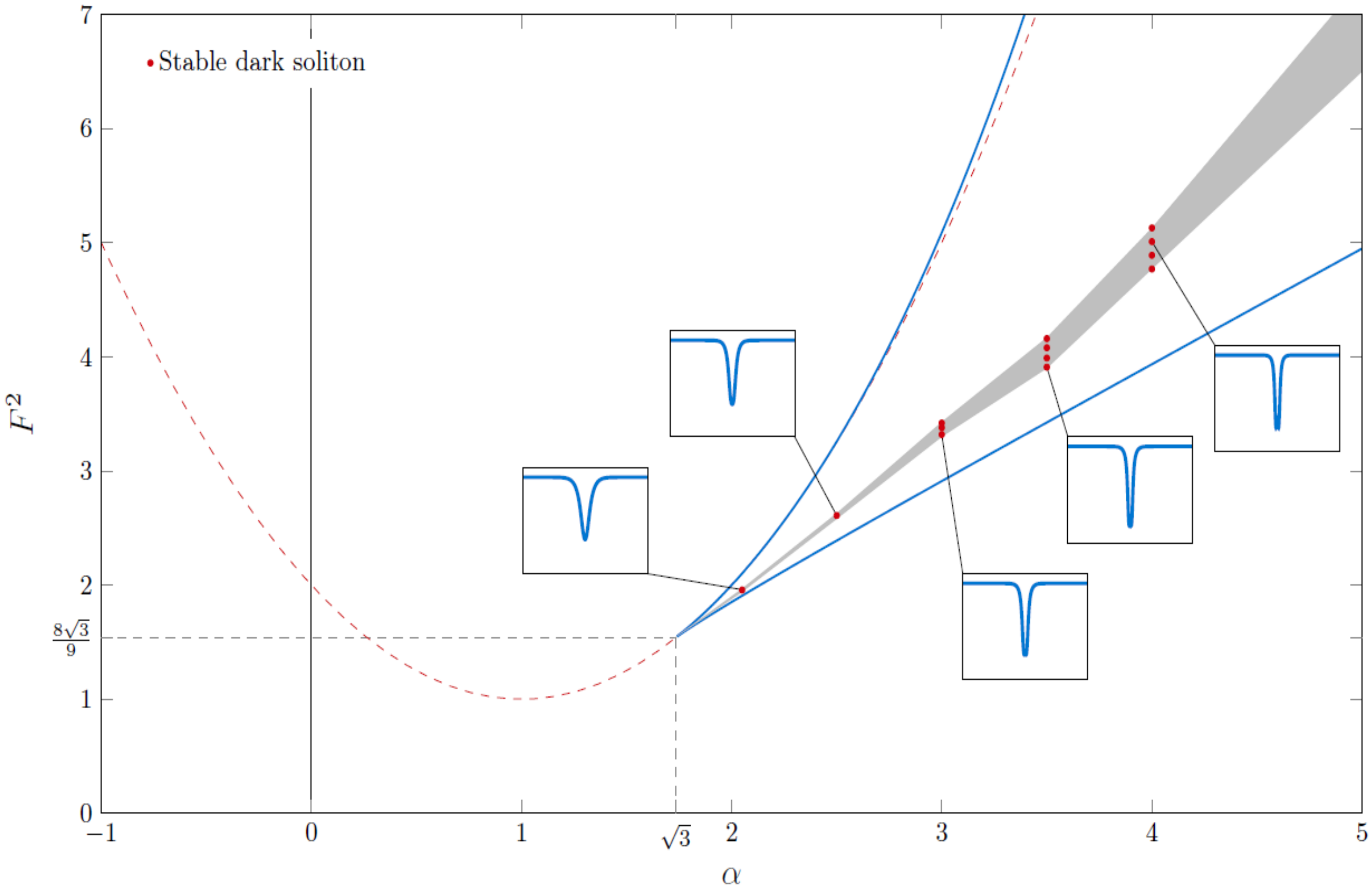
Case of normal dispersion



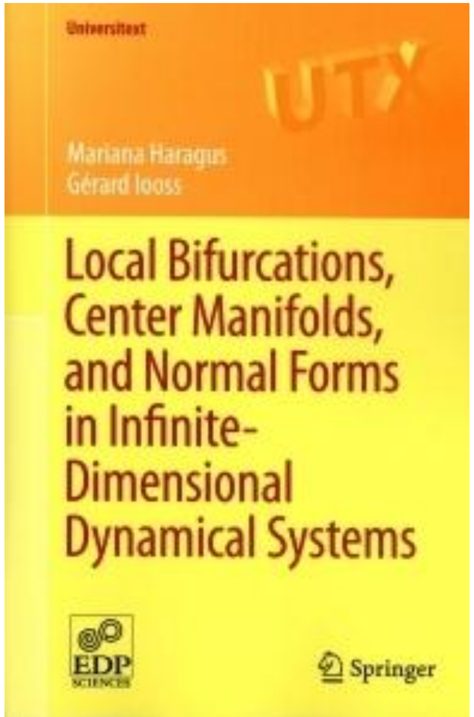
Pictogram	Bifurcation
	O^4
	O^2
	$O^2(i\omega)$
	$(i\omega)^2$

Bifurcation map

Helpful to understand what's going on



Next step:
Normal forms!!!



Outro

- Nonlinear WGM resonators are sources of very interesting phenomena
- They have a great potential for applications in Aero & Com engineering
- Their mathematical analysis is both difficult and exciting
- The spatiotemporal approach requires a sharp knowledge in PDEs

What is next ?

Science: Strong need for answers to several open questions, such as higher order effects, nonlinear and stochastic dynamics

Technology: Performance optimization in terms of stability, versatility, robustness, power consumption, etc.