Ondes non linéaires: bifurcations et dynamique

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$\mu\ \phi$

Séminaires croisés LMB – FEMTO-ST

2 Juillet 2013

Nonlinear Waves on Water

Water wave
[David Sanger Photography]

Solitary wave
Lagoon of Molokai, Hawaii
[photo : R.I. Odom]

Roll wave
Channel in Lions Bay, Canada
[website of N. Balmforth]

Mascaret de St Pardon
Dordogne river

Tsunami in Asia

Rogue wave
Chemical tanker ship Stolt Surf
[photo : K. Petersen]

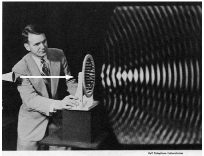
Other Nonlinear Waves

Kelvin-Helmholtz clouds
Mount Duval, Australia
[English Wikipedia : GRAHAMUK]

Fire rainbow
Northern Idaho

Morning Glory cloud
near Burketown, Australia
[author : Mick Petrov]

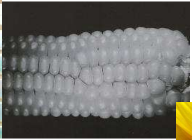
Hurricane



Sound wave
Bell Telephone Laboratories
[book by David C. Knight]

Patterns in Nature

Sand patterns
[photo : R. Niebrugge]



The Mathematics of ... Nonlinear Waves and Patterns

- **observed in nature, experiments, numerical simulations**
- **particular solutions of PDEs or ODEs**
  - *well-defined temporal and spatial structure*
  - *e.g., traveling waves*
- **play a key role in the dynamics of the underlying system**

The Mathematics of ... Nonlinear Waves and Patterns

Questions

- **existence** – *spatial and temporal properties*
- **stability** – *spatial and temporal behavior*
- **interactions**
- ...
- **role in the dynamics of the system**

The Mathematics of ... Nonlinear Waves and Patterns

Methods

- ... many different ...
- ... not enough ...

- numerical
- analytical

First Example

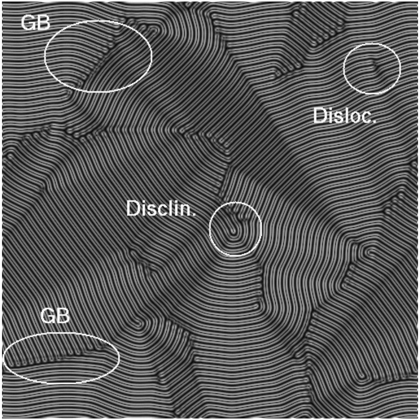
Water Waves

Water Waves

Water-wave Problem

- **gravity-capillary water waves**
  - *three-dimensional inviscid fluid layer*
  - *constant density* ρ
  - *gravity and surface tension*
  - *irrotational flow*

Second Example



[D. Boyer, J. Viñals]

Defects in patterns

- dislocations
- grain boundaries
- disclinations

Defects in Striped Patterns

- **Occur in a wide range of systems**
  - Rayleigh-Bénard convection experiment
  - crystal patterns in material science
  - chemical reactions
  - biology
  - ......

Third Example . . .

Collaboration FEMTO-ST & LMB

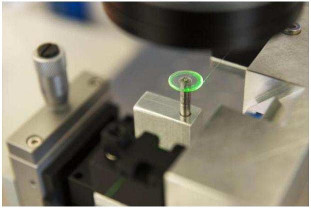
femto-st SCIENCES & TECHNOLOGIES

- Irina Balakireva
- Yanne K. Chembo
- Aurélien Coillet

LmB

- Cyril Godey
- Mariana Haragus

Résonateur optique

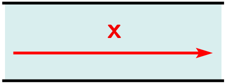


- Permet de générer des horloges de ultra-haute précision.
- Applications en aérospatiale, en télécommunications (radars, GPS...).

The Existence Problem

... A Dynamical Systems Approach

- **PDEs in unbounded domains**

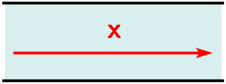


[Kirchgässner, 1982]

  - x timelike coordinate

. . . A Dynamical Systems Approach

- **PDEs in unbounded domains**



[Kirchgässner, 1982]

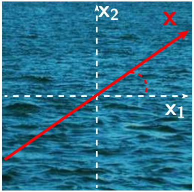
  - **x timelike coordinate**

- **Dynamical system** [Spatial Dynamics]

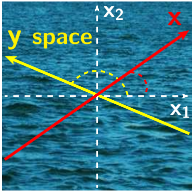
$$\boxed{\frac{d}{dx}U = F(U,\mu), \quad U(x) \in \mathcal{X}}$$

  - **U(x)** *belongs to a Hilbert (Banach) space* $\mathcal{X}$ *of functions depending upon the "space" variables ;*
  - $\mu \in \mathbb{R}^m$ *parameters.*

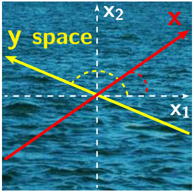
Spatial and Radial Dynamics



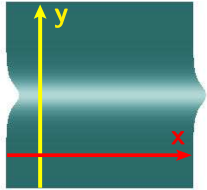
Spatial and Radial Dynamics



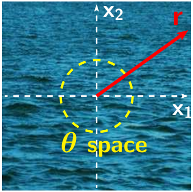
Spatial and Radial Dynamics



[H. & Groves, 2003]



[Kirchgässner, 1994]



[Scheel, 2003]

Spatial Dynamics

- **Dynamical system**

$$\boxed{\frac{d}{dx}U = F(U,\mu), \quad U(x) \in \mathcal{X}}$$

- **Nonlinear waves are found as**

  bounded solutions of the dynamical system

Spatial Dynamics

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$$\boxed{\frac{d}{dx}U = F(U,\mu), \quad U(x) \in \mathcal{X}}$$

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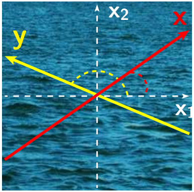
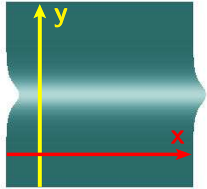
  bounded solutions of the dynamical system

- **What determines the shape of the wave?**

Shape of Solutions . . .

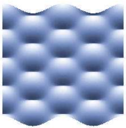
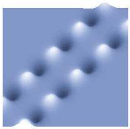
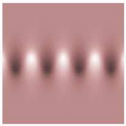
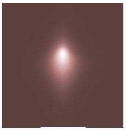




. . . determined by

- boundary conditions in the space variables y
- type of the bounded solution (localized, periodic, . . . in x)

For Instance . . .

Spatial Dynamics Approach

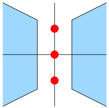
1. **Dynamical system**

$$\boxed{\frac{d}{dx}U = F(U,\mu), \quad U(x) \in \mathcal{X}}$$

1. Dynamical system

$$\frac{d}{dx}U = F(U, \mu), \quad U(x) \in \mathcal{X}$$

2. **Bifurcation points : critical parameter values μ_***
   - *start with a particular solution* U_* *(often* $U_* = 0$*);*
   - *determine the spectrum of* $D_U F(U_*, \mu)$
   - *bifurcation point* μ_* *: if the spectrum of* $D_U F(U_*, \mu_*)$ *contains purely imaginary values*



Reduction

1. Dynamical system

$$\frac{d}{dx} U = F(U, \mu), \quad U(x) \in \mathcal{X}$$

2. Bifurcation points : critical parameter values μ_*
3. **Center manifold reduction**

Reduction

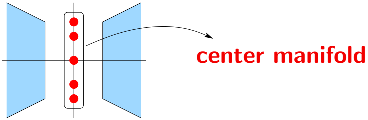
1. Dynamical system

$$\frac{d}{dx} U = F(U, \mu), \quad U(x) \in \mathcal{X}$$

2. Bifurcation points : critical parameter values μ_*

3. **Center manifold reduction**

    - spectrum of $D_U F(U_*, \mu_*)$



- *small bounded orbits lie on a center manifold*
- *finite-dimensional center manifold*
- *study the (reduced) dynamics on the center manifold*

$\longrightarrow$ **reduced ODE**

[Pliss, Kelley, ..., Mielke]

Reduced System

1. Dynamical system

$$\frac{d}{dx}U = F(U, \mu), \quad U(x) \in \mathcal{X}$$

2. Bifurcation points : critical parameter values μ_*
3. Center manifold reduction : reduced system of ODEs

$$\frac{d}{dx}v = g(v, \mu), \quad v(x) \in \mathbb{R}^d$$

4. **Bounded orbits of the reduced system of ODEs**
   - *e.g., use normal form theory*

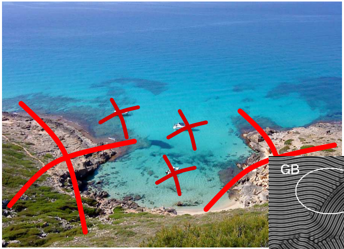
     [Poincaré, Birkhoff, Arnold, Elphick *et al.*, ...]
   - *study a truncated system*
   - *show persistence of the truncated dynamics*

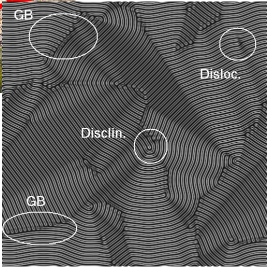
Applications

- **... many different ...**

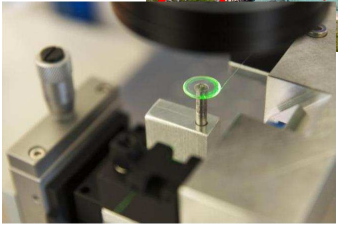
- ... many different ...

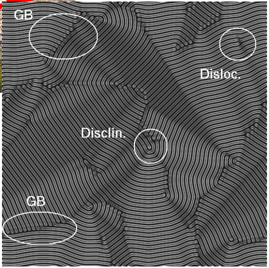
- **. . . many different . . .**





- ... many different ...





First Example

Water Waves

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Water-wave Problem

- **gravity-capillary water waves**
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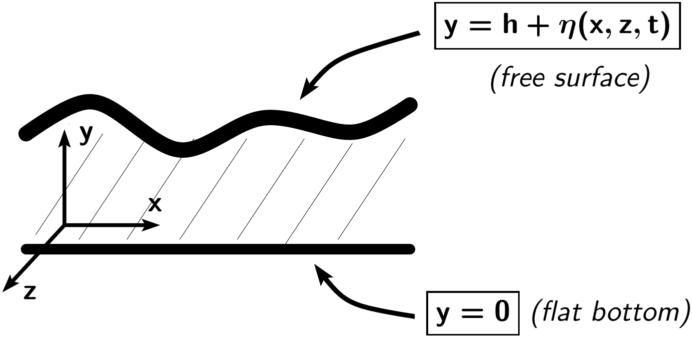


- **Domain**

$$D_\eta = \{(x, y, z) \; : \; x, z \in \mathbb{R}, \; y \in (0, h + \eta(x, z, t))\}$$

  - depth at rest h

Euler Equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for} \quad 0 < y < 1 + \eta$$

$$\phi_y = 0 \quad \text{on} \quad y = 0$$

$$\phi_y = \eta_t + \eta_x + \eta_x \phi_x + \eta_z \phi_z \quad \text{on} \quad y = 1 + \eta$$

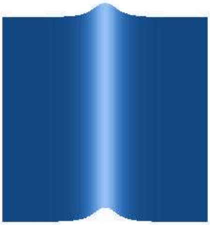
$$\phi_t + \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \alpha\eta - \beta\mathcal{K} = 0 \quad \text{on} \quad y = 1 + \eta$$

- **velocity potential** ϕ; **free surface** $1 + \eta$
- mean curvature $\mathcal{K} = \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x + \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z$
- **parameters**
  - *inverse square of the Froude number* $\boxed{\alpha = \frac{gh}{c^2}}$
  - *Weber number* $\boxed{\beta = \frac{\sigma}{\rho h c^2}}$

Euler Equations

- **very rich dynamics**
- **difficulties**
  - *variable domain (free surface)*
  - *nonlinear boundary conditions*
- **symmetries, Hamiltonian structure**
- **many particular solutions**

The Solitary Wave



John Scott Russell (1808 – 1882)

- Scottish civil engineer
- naval architect
- shipbuilder
- discovery of the solitary wave

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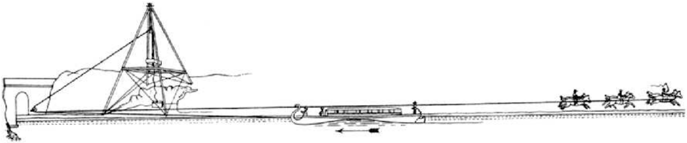
Experimental setup – Union Canal Edinburgh



John Scott Russell (1808 – 1882)

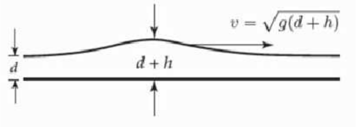
1834 : "The happiest day of my life"

"the boat suddenly stopped – not so the mass of water in the channel which it had put in motion

. . .

a large, solitary, progressive wave"

[Recherches Hydrauliques, par M. H. Darcy et M. H. Bazin, Deuxième Partie, Paris : Imprimerie Impériale, MDCCCLXV, p.9]



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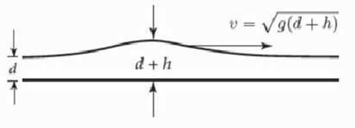


Bridge 11, Hermiston Walk
Heriot Watt University

Airy and Stokes

Sir George Biddell Airy (1801 – 1892)

- mathematician and astronomer
- Airy wave

Sir George Gabriel Stokes (1819 –1903)

- mathematician and physicist
- Stokes wave

Airy and Stokes

Sir George Biddell Airy (1801 – 1892)

Sir George Gabriel Stokes (1819 –1903)

Looking for an Explanation . . .

1895 : Korteweg & de Vries

$$\boxed{u_t + u_x + u_{xxx} + uu_x = 0}$$

1872, 1877 : Boussinesq

1877 : Boussinesq

MÉMOIRES

PRÉSENTÉS PAR DIVERS SAVANTS

À L'ACADÉMIE DES SCIENCES DE L'INSTITUT DE FRANCE.

EXTRAIT DES TOMES XXIII ET XXIV.

ESSAI

SUR

LA THÉORIE DES EAUX COURANTES,

PAR J. BOUSSINESQ.

PARIS.

IMPRIMERIE NATIONALE.

M DCCC LXXVII.

proche en proche les variations de h', c'est-à-dire les changements qu'éprouvera le profil longitudinal de l'onde [1].

Cette détermination faite, il ne restera plus qu'à évaluer la partie non permanente U' de la vitesse. On a pour cela la seconde équation du problème, (236) ou (270 *bis*) [p. 300], dans laquelle $\frac{dh}{dt}$ se réduit sensiblement à $\frac{dh'}{dt}$. Sa comparaison à (283) permet de poser

$$\frac{d}{ds}(hU - h'\omega) = 0,$$

ou bien, en multipliant par ds et intégrant de manière que $hU - h'\omega$ se réduise à HU_0 aux points que les ondes n'ont pas encore atteints,

$$hU - h'\omega = HU_0, \quad \text{c'est-à-dire} \quad (H + h')(U_0 + U') - h'\omega = HU_0.$$

[1] Il aurait été préférable d'obtenir cette équation par l'intégration directe de (281) ou avant de parler des vitesses de propagation ω. A cet effet, on aurait appelé ψ_1, par exemple, l'expression

$$\psi_1 = \frac{dh'}{dt} + \omega_0 \frac{dh'}{ds} + \frac{\omega_0(\omega_0 - U_0)}{2\omega_0 - (1 + \alpha'')U_0} \frac{d}{ds}\left(\frac{2+k}{2}\frac{h'^2}{H} + \frac{k'H^2}{3}\frac{d^2h'}{ds^2}\right),$$

et l'on aurait reconnu, au moyen de (264), que l'équation (281) revient sensiblement à

$$\frac{d\psi_1}{} - [\omega_0 - (1 + \ldots$$ [illegible]

L'intégrale de celle-ci, en raisonnant comme on l'a fait pour celle de (280), serait devenue $\psi_1 = 0$ ou

$$\text{(283 bis)} \qquad \frac{dh'}{dt} + \omega_0 \frac{d}{ds}\left[h' + \frac{k''}{2}\left(\frac{2+k}{2}\frac{h'^2}{H} + \frac{k'H^2}{3}\frac{d^2h'}{ds^2}\right)\right] = 0,$$

ce qui est l'équation cherchée. Ensuite les deux formules (282 *bis*), qu'on peut

$$\omega = \frac{1}{h'}\int_{\pm\infty}^{s}\left(-\frac{dh'}{dt}\right)ds,$$

auraient donné immédiatement, par la substitution à $-\frac{dh'}{dt}$ de sa valeur tirée de (283 *bis*), l'expression (289) de ω.

Later . . . The Solitary Wave Becomes a Soliton

1955 : Fermi, Pasta, Ulam recurrence

1963, 1965 : Zabuski, Kruskal *'soliton'*

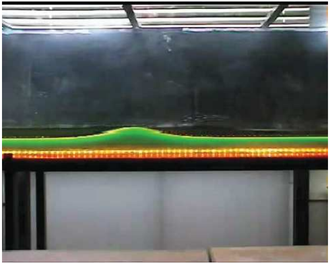
1967 : Gardner, Green, Kruskal, Miura *(inverse scattering transform)*

. . .. . .

- water waves
- nonlinear optics
- nonlinear acoustics
- plasma waves
- . . .

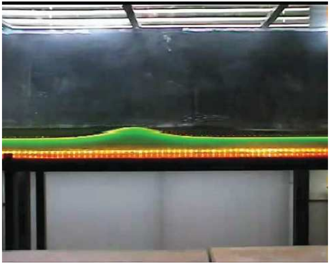
Today ... Many Different Existence Theories

Today . . . Many Different Existence Theories



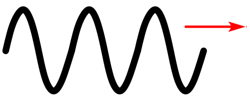
soliton hydrodynamique Search

Today . . . Many Different Existence Theories



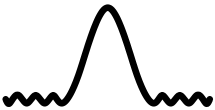
YouTube soliton hydrodynamique Search

Euler Equations : Two-dimensional Traveling Waves

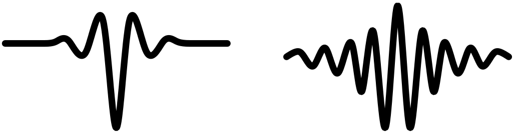


periodic wave

solitary waves



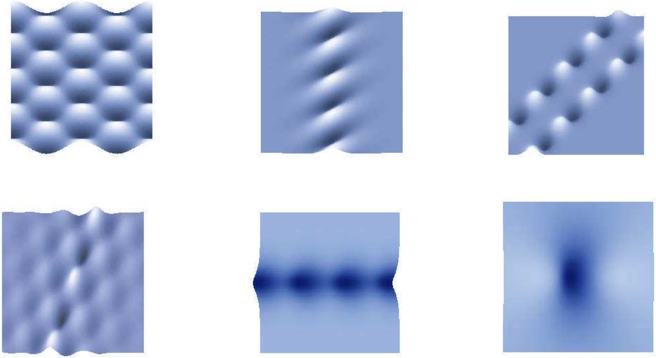
generalized solitary waves



solitary waves

**[Nekrasov, Levi-Civita, Struik, Lavrentiev, Friedrichs & Hyers, . . .
Amick, Kirchgässner, Iooss, Buffoni, Groves, Toland, Lombardi, Sun, . . .]**

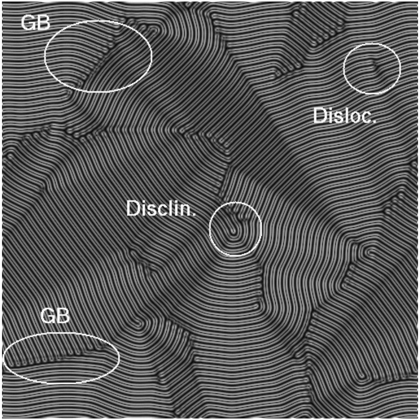
Euler Equations : Three-dimensional Traveling Waves



[Groves, Mielke, Craig, Nicholls, H., Kirchgässner, Deng, Sun, Sandstede, Iooss, Plotnikov, Wahlén, . . .]

Second Example

Defects in Striped Patterns



[D. Boyer, J. Viñals]

Existence of defects

- **dislocations**
- **grain boundaries**
- **disclinations**

Defects in Striped Patterns

- **Occur in a wide range of systems**
- **Existence studies**
  - in the frame of modulation equations, e.g., the Newell-Whitehead-Segel equation
    [Boyer, Viñals, Manneville, Pomeau, Newell, Passot, Bowman, Malomed, Nepomnyashchy, Trybelsky, Ercolani, Indik, Lega, . . . see the book of Pismen (2006)]
- **Spatial dynamics : the Swift-Hohenberg equation**

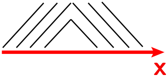
[H. & Scheel, 2012]

The Swift-Hohenberg equation

- **Swift-Hohenberg equation**

$$\boxed{u_t = -(\Delta + 1)^2 u + \mu u - u^3}$$

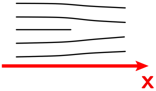
  - **grain boundaries :** *steady solutions*



- anisotropic **Swift-Hohenberg equation**

$$\boxed{u_t = -(\Delta + 1)^2 u + \mu u - u^3 + \beta u_{xx}}$$

  - **dislocations :** *traveling waves*



1. **Dynamical system**

$$\frac{dU}{dx} = \mathcal{A}(\mu, k, c, \beta)U + \mathcal{F}(U)$$

- **rolls** $\longleftrightarrow$ **equilibria**
- **dislocations / grain boundaries** $\longleftrightarrow$ **heteroclinic orbits**

Spatial Dynamics

1. **Dynamical system**

$$\boxed{\frac{dU}{dx} = \mathcal{A}(\mu, k, c, \beta)U + \mathcal{F}(U)}$$

- **rolls** $\longleftrightarrow$ **equilibria**
- **dislocations / grain boundaries** $\longleftrightarrow$ **heteroclinic orbits**

2. **Parameters :**
   - *equation :* μ, β
   - *y-periodic solutions :* wavenumber k
   - *traveling waves :* speed c

- **bifurcation points :** *co-existence of rolls with*

  dislocations : *different wavenumbers*

  grain boundaries : *different orientations*

- **dispersion relation**

Reduced Dynamics

1. Dynamical system

$$\frac{dU}{dx} = \mathcal{A}(\mu, k, c, \beta)U + \mathcal{F}(U)$$

2. Parameters :
3. Center manifold reduction

   ......

4. Reduced system : find a heteroclinic orbit

Reduced Dynamics

1. Dynamical system

$$\boxed{\frac{dU}{dx} = \mathcal{A}(\mu, k, c, \beta)U + \mathcal{F}(U)}$$

2. Parameters :

3. Center manifold reduction

   ......

4. Reduced system : find a heteroclinic orbit

   dislocations : ODE in $\mathbb{R}^4$

   grain boundaries : ODE in $\mathbb{R}^{12}$! !

Existence of Grain Boundaries

- **Reduced system : ODE in $\mathbb{R}^{12}$**

$$
\begin{aligned}
A_0' &= \mathrm{i}A_0 + B_0 - \tfrac{\mathrm{i}}{4}\left(\mu a_0 - a_0(a_0^2 + 6a_+\overline{a_+})\right) + \ldots\ldots \\
B_0' &= \mathrm{i}B_0 - \tfrac{1}{4}\left(\mu a_0 - a_0(a_0^2 + 6a_+\overline{a_+})\right) + \ldots\ldots \\
A_+' &= \mathrm{i}k_x A_+ + B_+ - \tfrac{\mathrm{i}}{4k_x^3}\left(\mu a_+ - 3a_+(a_0^2 + a_+\overline{a_+})\right) + \ldots\ldots \\
B_+' &= \mathrm{i}k_x B_+ - \tfrac{1}{4k_x^2}\left(\mu a_+ - 3a_+(a_0^2 + a_+\overline{a_+})\right) + \ldots\ldots \\
A_-' &= \mathrm{i}k_x A_- + B_- - \tfrac{\mathrm{i}}{4k_x^3}\left(\mu \overline{a_+} - 3\overline{a_+}(a_0^2 + a_+\overline{a_+})\right) + \ldots\ldots \\
B_-' &= \mathrm{i}k_x B_- - \tfrac{1}{4k_x^2}\left(\mu \overline{a_+} - 3\overline{a_+}(a_0^2 + a_+\overline{a_+})\right) + \ldots\ldots
\end{aligned}
$$

$$a_0 = A_0 + \overline{A_0},\ b_0 = B_0 + \overline{B_0},\ a_+ = A_+ + \overline{A_-},\ a_- = A_+ - \overline{A_-},\ b_+ = B_+ + \overline{B_-}$$

- **Normal form and scalings ......**

Leading Order System

$$\begin{aligned} C_0'' &= -\tfrac{1}{4}C_0 + \tfrac{3}{4}C_0(|C_0|^2 + 2|C_+|^2 + 2|C_-|^2) \\ C_+'' &= -\frac{1}{4k_x^2}C_+ + \frac{3}{4k_x^2}C_+(2|C_0|^2 + |C_+|^2 + 2|C_-|^2) \\ C_-'' &= -\frac{1}{4k_x^2}C_- + \frac{3}{4k_x^2}C_-(2|C_0|^2 + 2|C_+|^2 + |C_-|^2) \end{aligned}$$

- **existence of a heteroclinic orbit** $(0, C_+^\star, C_-^\star)$

$(C_0^\star, C_+^\star, 0)$?

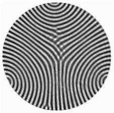
[van den Berg & van der Vorst, 1997]

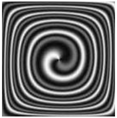
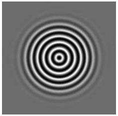
- **persistence of the heteroclinic orbit**
  - *analysis of the linearized operator*
  - *implicit function theorem*

$\ldots\ldots \Longrightarrow$ **existence of grain boundaries**

More Defects

- **Some may be treatable by related methods . . .**



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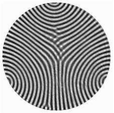
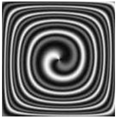
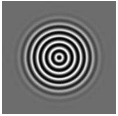


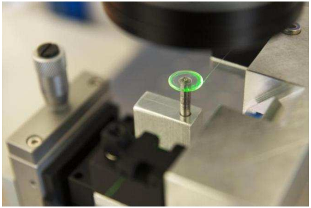



- Some cannot be treated by any of these methods . . .


Third Example

Le problème physique

Résonateur optique



- Permet de générer des horloges de ultra-haute précision.
- Applications en aérospatiale, en télécommunications (radars, GPS...).

Le problème mathématique

Équation de Lugiato-Lefever

$$\frac{\partial \psi}{\partial t} = -(1 + i\alpha)\psi + i\psi|\psi|^2 - i\beta\frac{\partial^2 \psi}{\partial x^2} + F$$

- **But :** *étudier la dynamique des solutions en fonction des paramètres* $\alpha \in \mathbb{R}$ *et* $F > 0$.

L'Équation de Lugiato-Lefever

$$\boxed{\frac{\partial \psi}{\partial t} = -\left(1 + i\alpha\right)\psi + i\psi\left|\psi\right|^2 - i\beta\frac{\partial^2 \psi}{\partial x^2} + F}$$

- Solutions les plus simples : les **équilibres**.

L'Équation de Lugiato-Lefever

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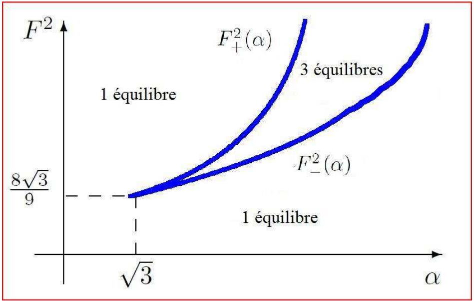

Stabilité des équilibres

- **Stabilité** *en tant que solutions de l'équation différentielle*

$$\boxed{\frac{\partial \psi}{\partial t} = -(1 + i\alpha)\,\psi + i\psi\,|\psi|^2 + F}$$

  - On étudie les valeurs propres de la matrice de l'équation linéarisée.

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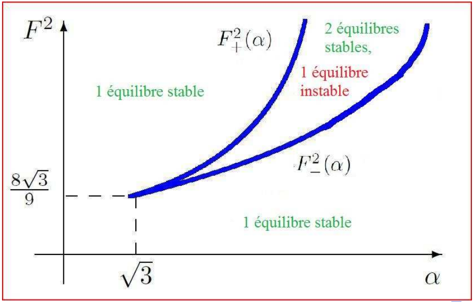

Étude des bifurcations

- **Bifurcations :** *changement qualitatif dans la dynamique des solutions lorsque les paramètres varient.*
  - On se restreint aux solutions stationnaires.
  - On étudie les valeurs propres de la matrice de l'équation linéarisée.

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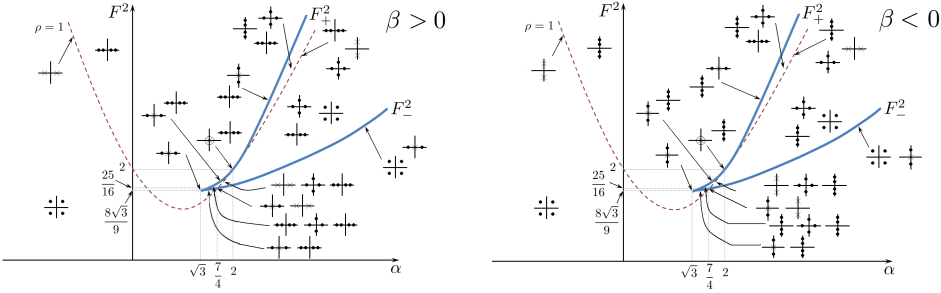
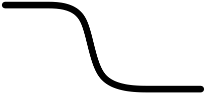


La suite ...

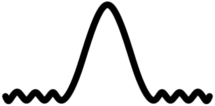
- **Calcul des formes normales : existence des solutions.**

onde périodique

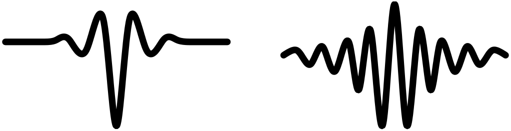
onde solitaire



front



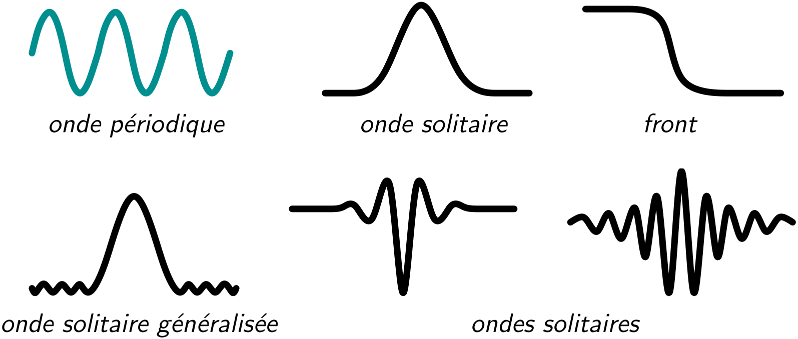
onde solitaire généralisée



ondes solitaires

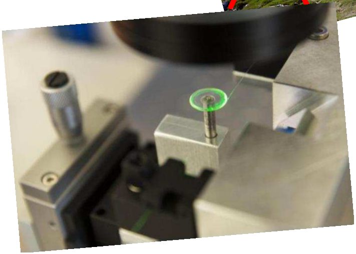
La suite ...

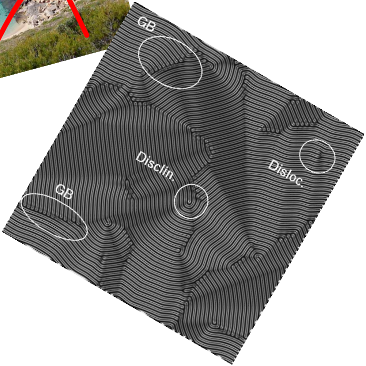
- **Calcul des formes normales : existence des solutions.**



- **Étude de la stabilité des solutions périodiques.**
- **Interprétation physique des résultats mathématiques.**

Nonlinear Waves and Patterns





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