SOME REMARKS ON A MINIMIZATION PROBLEM ASSOCIATED TO A FOURTH ORDER NONLINEAR SCHRÖDINGER EQUATION

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Abstract. Let $\gamma > 0$, $\beta > 0$, $\alpha > 0$ and $0 < \sigma N < 4$. In the present paper, we study, for $c > 0$ given, the constrained minimization problem

$$m(c) := \inf_{u \in S(c)} E(u),$$

where

$$E(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\, dx - \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\alpha}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx,$$

and

$$S(c) := \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\, dx = c \right\}.$$

The aim of our study is twofold. On one hand, this minimization problem is related to the existence and orbital stability of standing waves for the mixed dispersion nonlinear biharmonic Schrödinger equation

$$i\partial_t \psi - \gamma \Delta^2 \psi - \beta \Delta \psi + \alpha |\psi|^{2\sigma}\psi = 0, \quad \psi(0,x) = \psi_0(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

On the other hand, in most of the applications of the Concentration-Compactness principle of P.-L. Lions, the difficult part is to deal with the possible dichotomy of the minimizing sequences. The problem under consideration provides an example for which, to rule out the dichotomy is rather standard while, to rule out the vanishing, here for $c > 0$ small, is challenging. We also provide, in the limit $c \to 0$, a precise description of the behavior of the minima. Finally, some extensions and open problems are proposed.

1. Introduction

In this paper we consider a constrained minimization problem which is motivated by the search of standing waves solutions for the biharmonic NLS (Nonlinear Schrödinger Equation) with mixed dispersion

$$i\partial_t \psi - \gamma \Delta^2 \psi - \beta \Delta \psi + \alpha |\psi|^{2\sigma}\psi = 0, \quad \psi(0,x) = \psi_0(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.1}$$

Here $\gamma > 0$, $\beta \in \mathbb{R}$ and $\alpha > 0$ are given parameters and $0 < \sigma N < 4^*$ where we define $4^* := 4N/(N-4)^+$, namely $4^* = +\infty$ if $N \le 4$ and $4^* = 4N/(N-4)$ if $N \ge 5$.

About twenty years ago, equation (1.1) was introduced for several distinct physical motivations, see in particular [22, 23] and [18]. It has been since then the object of intensive studies, some dealing with dynamical issues such as local or global well-posedness, others dealing with the existence and properties of certain kind of solutions. We refer to the introductions of the papers [5, 6, 10, 28] for a presentation of the more recent results concerning (1.1).

Of particular interest are the so called standing waves solutions, i.e. solutions of the form $\psi(t,x) = e^{i\lambda t}u(x)$ with $\lambda > 0$. The function u then satisfies the elliptic equation

$$\gamma \Delta^2 u + \beta \Delta u + \lambda u = \alpha |u|^{2\sigma} u, \quad u \in H^2(\mathbb{R}^N). \tag{1.2}$$

A possible choice is to consider that $\lambda > 0$ in (1.2) is given and to look for solutions as critical points of the functional

$$\mathbb{E}(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\, dx - \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2\, dx - \frac{\alpha}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx. \tag{1.3}$$

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Then, for physical reasons, one usually focus on the so-called least energy solutions, namely solutions which minimize $\mathbb{E}$ on the set

$$\mathcal{N} := \Big\{u \in H^2(\mathbb{R}^N)\backslash\{0\} : \mathbb{E}'(u) = 0\Big\}. \tag{1.4}$$

This is the approach followed in [5, 9].

Alternatively, one can consider the existence of solutions to (1.2) having a prescribed L^2-norm. Since solutions to (1.1) conserve their *mass* along time, it is natural from a physical point view to search for such solutions. We shall focus on this issue. For $c > 0$ given, we consider the problem of finding solutions to

$$\gamma\Delta^2 u + \beta\Delta u + \lambda u = \alpha|u|^{2\sigma}u, \quad u \in H^2(\mathbb{R}^N) \quad \text{with} \quad \int_{\mathbb{R}^N} |u|^2 dx = c. \tag{P_c}$$

It is standard to show that a critical point of the energy functional

$$E(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\, dx - \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx, \tag{1.5}$$

restricted to

$$S(c) := \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\, dx = c\right\}, \tag{1.6}$$

corresponds to a solution of (P_c). The value of $\lambda \in \mathbb{R}$ in (P_c) is then an unknown of the problem and it corresponds to the associated Lagrange multiplier.

In [5, 6] the authors study this problem assuming that $\beta < 0$. First, in [5] the mass subcritical case $0 < \sigma N < 4$ was considered. In this case, the functional E is bounded from below on $S(c)$ for any $c > 0$. Hence, it is possible to search for a critical point of E restricted to $S(c)$ as a global minimizer. Setting, for $c > 0$ given,

$$m(c) := \inf_{u\in S(c)} E(u), \tag{1.7}$$

the following result was obtained.

Theorem 1.1. [5, Theorem 1.1] *Assume $\gamma > 0$, $\beta \le 0$ and $\alpha > 0$. If $0 < \sigma N < 2$, then $m(c)$ is achieved for every $c > 0$. If $2 \le \sigma N < 4$ then there exists a critical mass $\tilde{c} \in [0, \infty)$ such that*

i) *$m(c)$ is not achieved if $c < \tilde{c}$;*
ii) *$m(c)$ is achieved if $c > \tilde{c}$ and $\sigma = 2/N$;*
iii) *$m(c)$ is achieved if $c \ge \tilde{c}$ and $\sigma \ne 2/N$.*

Moreover if σ is an integer and $m(c)$ is achieved, then there exists at least one radially symmetric minimizer.

Remark 1.1. Let us point out that, for $\beta = 0$, it follows that $\tilde{c} = 0$ while, for $\beta < 0$, it holds that $\tilde{c} > 0$. The appearance of a critical mass when $\beta < 0$ and $2 \le \sigma N < 4$ is linked to the fact that each term of E behaves differently with respect to scaling. Such a phenomenon was first observed in [16] and has now been revealed in several distinct settings, see for instance [13, 17, 21] for related results.

In [6] the authors considered, still assuming $\beta < 0$, the mass critical case $\sigma N = 4$ and the mass supercritical case $4 < \sigma N < 4^*$. In particular, it was shown in [6, Theorem 1.2] that standing waves do not exist if $\sigma N = 4$ and that, assuming $c > 0$ sufficiently small, they do exist when $4 < \sigma N < 4^*$, see [6, Theorem 1.3]. Note that in the mass supercritical case $4 < \sigma N < 4^*$ the functional E is not more bounded from below on $S(c)$. The critical points obtained in [6] are of saddle point type. In [6], some multiplicity results for radial solutions were also derived.

Very recently, the case where $\beta > 0$ started to be considered in [26, 27]. The paper [26] is devoted to the mass subcritical case $0 < \sigma N < 4$ and the mass critical case $\sigma N = 4$ while [27] deals with the mass supercritical case $4 < \sigma N < 4^*$ and the Sobolev critical case $\sigma N = 4^*$. We shall come back later, in some details, to these two papers.

In the present work we also deal with the case $\beta > 0$ and restrict ourselves to the mass subcritical case $0 < \sigma N < 4$, see however Section 7 for a result in the mass critical case $\sigma N = 4$. Our first main result reads as follows

Theorem 1.2. *Assume $\gamma > 0$, $\beta > 0$, $\alpha > 0$ and $0 < \sigma N < 4$. For $m(c)$ defined as in (1.7), there exists $c_0 \in [0, \infty)$ such that:*

i) *If $c > c_0$, any minimizing sequence of $m(c)$ is precompact in $H^2(\mathbb{R}^N)$ up to translations. In particular, $m(c)$ is achieved.*
ii) *If $0 < \sigma < \max\left\{\frac{4}{N+1}, 1\right\}$, then we have that $c_0 = 0$.*

In addition, if $u \in S(c)$ is a minimizer of $m(c)$, the associated Lagrange multiplier $\lambda \in \mathbb{R}$ satisfy $\lambda > \frac{\beta^2}{4\gamma}$.

Remark 1.2.

a) In the case where $\beta < 0$ and $2 \le \sigma N < 4$, see Theorem 1.1, there exists a critical mass $\tilde{c} > 0$ such that $m(c)$ is not achieved if $c < \tilde{c}$. The situation is now different. In the case $\beta > 0$ considered in Theorem 1.2, we may find $\sigma > \frac{2}{N}$ such that, for every $c > 0$, $m(c)$ is achieved.
b) It is known from [31], see also [4], that the Cauchy problem associated to (1.1) is locally well-posed in $H^2(\mathbb{R}^N)$ as soon as $0 < \sigma N < 4^*$. Also, in the mass subcritical case $0 < \sigma N < 4$ that we are considering in Theorem 1.2, the global existence for the Cauchy problem holds, see [18, 31]. Thus, having at hand the precompactness up to translations of any minimizing sequence, it is standard to show the orbital stability of the set of global minima following the strategy laid down in [14].
c) When $\sigma > 0$ is an integer, using a very recent result of L. Bugiera, E. Lenzmann and J. Sok [12], it is possible to obtain symmetry properties for the global minimizer of $m(c)$. In view of Theorem 1.2, we shall benefit from these results when $N = 1, 2$. More details will be given in Section 7.

Let us now provide some elements of the proof of Theorem 1.2. First, assuming that $0 < \sigma N < 4$, it is straightforward to show that the functional E is bounded from below on $S(c)$ and coercive, see Lemma 2.2. In particular, $m(c)$ is well defined for any $c > 0$. Then, using a convenient version of the Concentration Compactness principle of P.-L. Lions, we deduce that, for any $c > 0$, either the vanishing of a minimizing sequence occurs or it is precompact up to translations, see Lemma 2.5. Namely, the ruling out of the vanishing also exclude the possibility of dichotomy for the minimizing sequences. Then, in Lemma 4.1, we show that a necessary and sufficient condition to avoid the vanishing is that

$$m(c) < -\frac{\beta^2}{8\gamma}c. \tag{1.8}$$

This condition is derived through the study of an associated minimization problem, see Section 3, which has also an interest by itself. More precisely, for all $c > 0$, we consider

$$m_I(c) := \inf_{u \in S(c)} I(u), \tag{1.9}$$

where

$$I(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

and we show that $m_I(c) = -\frac{\beta^2}{8\gamma}c$, that the infimum is never achieved and that any minimizing sequence is vanishing. See Lemma 3.1.

When $c > 0$ is large, it is direct to show that (1.8) holds. However, when $c > 0$ is small, the situation is surprisingly much more complex and the treatment of this case is a central part of this paper. Under the assumption $0 < \sigma < \max\{4/(N+1), 1\}$, we manage to check (1.8) for all $c > 0$ through the construction of suitable testing functions. We refer to Section 5 for more details. Our choice of testing functions is inspired by the following result which provides a description of the behavior of the minima, when they exist, as $c \to 0$.

Theorem 1.3. *Assume $\gamma > 0$, $\beta > 0$, $\alpha > 0$ and $0 < \sigma N < 4$. Let $\{(u_n, c_n)\} \subset S(c_n) \times \mathbb{R}$ be such that $c_n \to 0$ as $n \to \infty$ and $u_n \in S(c_n)$ be a minimizer of $m(c_n)$ for each $n \in \mathbb{N}$. Then:*

i) *There exists $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$ such that*

$$-\frac{\beta^2}{8\gamma}(1+\varepsilon_n)c_n = (1+\varepsilon_n)m_I(c_n) \le m(c_n) \le m_I(c_n) = -\frac{\beta^2}{8\gamma}c_n, \quad \forall\, n \in \mathbb{N}.$$

ii) *Letting* $\lambda_n \in \mathbb{R}$ *be the Lagrange multiplier associated to* u_n*, it follows that*

$$\lambda_n \to \left(\frac{\beta^2}{4\gamma}\right)^+ \quad as \quad n \to \infty.$$

iii) *It holds that*

$$\frac{\|\Delta u_n\|_2}{\|u_n\|_2} \to \frac{\beta}{2\gamma} \quad and \quad \frac{\|\nabla u_n\|_2}{\|u_n\|_2} \to \sqrt{\frac{\beta}{2\gamma}} \quad as \quad n \to \infty.$$

iv) *Setting* $v_n = \dfrac{u_n}{\|u_n\|_2}$ *we have that* $v_n \to 0$ *in* $L^p(\mathbb{R}^N)$ *for any* $p \in (2, 4^*)$*. Also*

$$\int_{\mathbb{R}^N} \left(|\xi|^2 - \frac{\beta}{2\gamma}\right)^2 |\widehat{v_n}(\xi)|^2 \, d\xi \to 0 \quad as \quad n \to \infty. \tag{1.10}$$

Remark 1.3.

a) From i) we see that the slope of $m(c)$ at the origin is precisely $-\frac{\beta^2}{8\gamma}$. This clearly shows that to check (1.8) for $c > 0$ small is not an easy task.

b) From (1.10) we see that the L^2–norm of $\{\widehat{v_n}\} \subset S(1)$ concentrate near the sphere of radius $\sqrt{\beta/2\gamma}$ centered at the origin when $n \to \infty$. This was one of the keys to find the test functions that allow to show that, under the assumption $0 < \sigma < \max\{4/(N+1), 1\}$, the strict inequality (1.8) holds for every $c > 0$.

Let us now turn back to the works [26, 27] and try to locate our results with respect to the ones presented in these papers. In [26], in the mass subcritical case and mass critical cases, assuming that the non vanishing holds it is shown that any minimizing sequence is precompact, up to translation (and thus that the set of global minima is orbitally stable, see Remark 1.2 b)). Instead of using the approach laid down by P.-L. Lions, the authors rely on a Profile Decomposition of bounded sequences in $H^2(\mathbb{R}^N)$ which was established in [36]. In [26] is derived an explicit lower bound on $c > 0$ above which the non-vanishing holds (this corresponds in [26] to the case $\mu < 0, |\mu|$ small). There are not results in [26] when $c > 0$ is small. The work [27] is devoted to the mass supercritical case $4 < \sigma N < 4^*$ and also the Sobolev critical case $\sigma N = 4^*$. In both cases, assuming that $c > 0$ is small enough, it is shown that E presents a so-called *convex-concave* geometry. Thus, one can hope to find two critical points: one as a local minimizer and another one of mountain pass type. Such geometry and corresponding multiplicity results had been observed recently in a series of papers. In [1, 3], this geometry is created by the presence of an external potential. In [15, 32, 33] it results from the presence of two non linearities having a different behavior under scaling. Such phenomena has also been observed in the case of systems in [19] (see also [30] for results on bounded domains). The set of critical points which are obtained as local minima is then clearly expected to be orbitally stable and that is what is proved in [1, 3, 19, 30, 32, 33]. In [27], concerning the existence of a critical point as local minimizer, the authors seem to have overlook the possibility that the expected local minima may not exists (because of the possible vanishing of any minimizing sequence). This point requires clearly a special care since, to insure the *convex-concave* geometry, it is necessary to assume that $c > 0$ is small.

We end this paper, in Section 7, with some remarks and open problems. First we show, when $\sigma \in \mathbb{N}$, that properties of the minima of $m(c)$ can be obtained exploiting a result from [12]. Also, it should be clear that the condition $0 < \sigma < \max\{4/(N+1), 1\}$ is the consequence of two particular trials of test functions. Nothing guarantees that we have obtained, in Theorem 1.2, the sharpest conditions for the existence of a minimizer for $c > 0$ and deriving necessary and sufficient conditions to insure that it is the case is worth of study. In addition, we indicate how our test functions also prove useful in the cases where the problem is mass critical or mass supercritical. Finally, we note that the existence of a minimizer of $m(c)$ for any $c > 0$ small can be interpreted as a bifurcation result, in the $H^2(\mathbb{R}^N)$ norm, from the bottom of the essential spectrum of the operator $u \mapsto \gamma\Delta^2 u + \beta\Delta u$ defined on $H^2(\mathbb{R}^N)$. It would be interesting to see if such phenomena is also present for the ground states solutions obtained from the functional $\mathbb{E}$ in [5, 9].

We now describe the organization of the paper. In Section 2, we present some preliminary results. In particular, we present the proof of Lemma 2.5 which shows that, if the vanishing do not occurs, then $m(c)$ is reached. In Section 3, we study in details the associated minimization problem (1.9). In Section 4, we derive the sufficient and necessary condition (1.8) which guarantees that the vanishing does not occur. Section 5 is devoted to the construction of our two families of testing functions which permit to rule out the vanishing under the conditions on $\sigma > 0$ given in Theorem 1.2. At this point the proof of Theorem 1.2 is completed. In Section 6, we give the proof of Theorem 1.3, which deals with the behaviour of the minima as $c \to 0$. Finally, in Section 7, we present some additional results and state some open problems.

In the rest of the paper, unless it is stated the contrary, we assume that $N \geq 1$, $\gamma > 0$, $\beta > 0$, $\alpha > 0$ and $0 < \sigma N < 4$.

Notation. *For $1 \leq p < \infty$, we denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue space with norm*

$$\|u\|_p^p := \int_{\mathbb{R}^N} |u|^p \, dx.$$

The Sobolev space $H^2(\mathbb{R}^N)$ is endowed with its standard norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\Delta u|^2 + |\nabla u|^2 + |u|^2 \, dx.$$

We denote by ' $\to$', respectively by ' $\rightharpoonup$', the strong convergence, respectively the weak convergence in corresponding space and denote by $B_R(x)$ the ball in $\mathbb{R}^N$ of center x and radius $R > 0$. We use the notation $o_n(1)$ for any quantity which tends to zero as $n \to \infty$. Finally, we shall denote by $C > 0$ a constant which may vary from line to line but is not essential to the problem.

2. Preliminary results

We shall make use of some inequalities that we now present. First, we recall (see [29, Theorem in Lecture II]) that, for all $0 \leq \sigma < 4/(N-4)^+$, i.e. $0 \leq \sigma$ if $N \leq 4$ and $0 \leq \sigma < 4/(N-4)$ if $N \geq 5$, there exists a constant $B_N(\sigma) > 0$ such that

$$\|u\|_{2\sigma+2}^{2\sigma+2} \leq B_N(\sigma) \|\Delta u\|_2^{\frac{\sigma N}{2}} \|u\|_2^{2+2\sigma-\frac{\sigma N}{2}}, \quad \forall\, u \in H^2(\mathbb{R}^N). \tag{2.1}$$

This is precisely the so-called Gagliardo-Nirenberg inequality. Having at hand (2.1), by interpolation and using the Sobolev inequality, one may infer (see [29, Theorem in Lecture II]) that, for all $0 \leq \sigma < 2/(N-2)^+$, namely $0 \leq \sigma$ if $N \leq 2$ and $0 \leq \sigma < 2/(N-2)$ if $N \geq 3$, there exists a constant $C_N(\sigma) > 0$ such that

$$\|u\|_{2\sigma+2}^{2\sigma+2} \leq C_N(\sigma) \|\nabla u\|_2^{\sigma N} \|u\|_2^{2+\sigma(2-N)}, \quad \forall\, u \in H^2(\mathbb{R}^N). \tag{2.2}$$

We also use the following interpolation inequality that can be easily proved using the Fourier transform

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}, \quad \forall\, u \in H^2(\mathbb{R}^N). \tag{2.3}$$

Lemma 2.1. *For any $\gamma > 0$, $\beta \in \mathbb{R}$ and $u \in H^2(\mathbb{R}^N)$, it follows that*

$$\gamma \|\Delta u\|_2^2 - \beta \|\nabla u\|_2^2 + \frac{\beta^2}{4\gamma} \|u\|_2^2 \geq 0.$$

Thus

$$\inf_{u \in H^2(\mathbb{R}^N)} \left(\frac{\gamma \|\Delta u\|_2^2 - \beta \|\nabla u\|_2^2}{\|u\|_2^2} \right) \geq -\frac{\beta^2}{4\gamma}. \tag{2.4}$$

Proof. It directly follows from (2.3) that

$$\gamma \|\Delta u\|_2^2 - \beta \|\nabla u\|_2^2 + \frac{\beta^2}{4\gamma} \|u\|_2^2 \geq \gamma \|\Delta u\|_2^2 - \beta \|\Delta u\|_2 \|u\|_2 + \frac{\beta^2}{4\gamma} \|u\|_2^2 = \left(\sqrt{\gamma} \|\Delta u\|_2 - \frac{\beta}{2\sqrt{\gamma}} \|u\|_2 \right)^2 \geq 0$$

ending the proof. □

Lemma 2.2. *The functional E is coercive on $S(c)$ and in particular $m(c) > -\infty$ for any $c > 0$.*

Proof. The claim follows directly using (2.1) and arguing as in Lemma 2.1. □

Let us now introduce a scaling that will be useful for the rest of the work. For any $u \in S(c)$ and any $s > 0$, we define

$$u_s(x) := s^{\frac{N}{4}} u(\sqrt{s}x). \tag{2.5}$$

This definition is clearly motivated by the fact that $\|u_s\|_2 = \|u\|_2$ for all $s > 0$. One then easily obtain that

$$E(u_s) = \frac{\gamma s^2}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \frac{\beta s}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\alpha s^{\sigma N/2}}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx, \quad \forall\ s > 0. \tag{2.6}$$

Having at hand this suitable rescaling, we prove several properties of $m(c)$ that will be needed to rule out the possibility of dichotomy for the minimizing sequences.

Lemma 2.3.

i) *$m(c) < 0$, $\forall\ c > 0$;*
ii) *$m(\tau c) \leq \tau m(c)$, $\forall\ \tau > 1$, $\forall\ c > 0$;*
iii) *Assume that there exists a global minimizer $u \in S(c)$ of $m(c)$ for some $c > 0$. Then $m(\tau c) < \tau m(c)$ $\forall\ \tau > 1$;*
iv) *$m(c_1 + c_2) \leq m(c_1) + m(c_2)$, $\forall\ c_1, c_2 > 0$;*
v) *Assume that there exists a global minimizer $u \in S(c_1)$ with respect to $m(c_1)$ for some $c_1 > 0$ and let $c_2 > 0$. Then $m(c_1 + c_2) < m(c_1) + m(c_2)$.*

Proof. i) Taking an arbitrary $u \in S(c)$ and considering u_s as defined in (2.5), we see from (2.6) that $E(u_s) \to 0^-$ as $s \to 0$ and i) follows.

ii) For any $\varepsilon > 0$, there exists $u \in S(c)$ such that $E(u) \leq m(c) + \varepsilon$. Defining $\widetilde{u}(x) = \tau^{\frac{1}{2}} u(x)$ we observe that

$$\|\widetilde{u}\|_2^2 = \tau\|u\|_2^2 = \tau c; \quad \|\Delta\widetilde{u}\|_2^2 = \tau\|\Delta u\|_2^2; \quad \|\nabla\widetilde{u}\|_2^2 = \tau\|\nabla u\|_2^2 \quad \text{and} \quad \|\widetilde{u}\|_{2\sigma+2}^{2\sigma+2} = \tau^{\sigma+1}\|u\|_{2\sigma+2}^{2\sigma+2}.$$

Hence, we have that

$$\begin{aligned} m(\tau c) \leq E(\widetilde{u}) &= \tau\left[\frac{\gamma}{2}\|\Delta u\|_2^2 - \frac{\beta}{2}\|\nabla u\|_2^2 - \frac{\alpha\tau^\sigma}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2}\right] \\ &< \tau\left[\frac{\gamma}{2}\|\Delta u\|_2^2 - \frac{\beta}{2}\|\nabla u\|_2^2 - \frac{\alpha}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2}\right] \\ &= \tau E(u) \leq \tau(m(c)+\varepsilon). \end{aligned} \tag{2.7}$$

Since $\varepsilon > 0$ is arbitrary, we see that ii) holds.

iii) If $m(c)$ is achieved, for example, at some $u \in S(c)$, then we can let $\varepsilon = 0$ in (2.7) and thus the strict inequality follows.

iv) Assume first that $0 < c_2 \leq c_1$. Then, by ii), we have that

$$\begin{aligned} m(c_1+c_2) &\leq \frac{c_1+c_2}{c_1} m(c_1) = m(c_1) + \frac{c_2}{c_1} m(c_1) = m(c_1) + \frac{c_2}{c_1} m(\frac{c_1}{c_2} c_2) \\ &\leq m(c_1) + \frac{c_2}{c_1}\frac{c_1}{c_2} m(c_2) = m(c_1) + m(c_2). \end{aligned}$$

The case $0 < c_1 < c_2$ can be treated reversing the role of c_1 and c_2.

v) Assume first that $0 < c_2 \leq c_1$. Then, using iii), observe that, if $m(c_1)$ is reached, we can write

$$\begin{aligned} m(c_1+c_2) &< \frac{c_1+c_2}{c_1} m(c_1) = m(c_1) + \frac{c_2}{c_1} m(c_1) = m(c_1) + \frac{c_2}{c_1} m(\frac{c_1}{c_2} c_2) \\ &\leq m(c_1) + \frac{c_2}{c_1}\frac{c_1}{c_2} m(c_2) = m(c_1) + m(c_2). \end{aligned}$$

As in iv), the case $0 < c_1 < c_2$ can be treated reversing the role of c_1 and c_2. □

Lemma 2.4. *Let $\{u_n\} \subset H^2(\mathbb{R}^N)$ be a bounded sequence such that $\|u_n\|_2^2 \to c > 0$ and let $\widetilde{u}_n = c^{\frac{1}{2}} \frac{u_n}{\|u_n\|_2}$. Then*

i) *$\widetilde{u}_n \in S(c)$, $\forall n \in \mathbb{N}$.*
ii) *$\lim_{n\to\infty} |E(u_n) - E(\widetilde{u}_n)| = 0$.*

Proof. Point i) is obvious. Hence, we concentrate on point ii). We directly observe that

$$\|\Delta\widetilde{u}_n\|_2^2 = \frac{c}{\|u_n\|_2^2}\|\Delta u_n\|_2^2; \quad \|\nabla\widetilde{u}_n\|_2^2 = \frac{c}{\|u_n\|_2^2}\|\nabla u_n\|_2^2 \quad \text{and} \quad \|\widetilde{u}_n\|_{2\sigma+2}^{2\sigma+2} = \left(\frac{c}{\|u_n\|_2^2}\right)^{\sigma+1}\|u_n\|_{2\sigma+2}^{2\sigma+2}.$$

Hence, we have that

$$\begin{aligned}|E(u_n)-E(\widetilde{u}_n)| &= \left|\frac{\gamma}{2}\left(1-\frac{c}{\|u_n\|_2^2}\right)\|\Delta u_n\|_2^2 - \frac{\beta}{2}\left(1-\frac{c}{\|u_n\|_2^2}\right)\|\nabla u_n\|_2^2 - \frac{1}{2\sigma+2}\left(1-\left(\frac{c}{\|u_n\|_2^2}\right)^{\sigma+1}\right)\|u_n\|_{2\sigma+2}^{2\sigma+2}\right| \\ &\le \frac{\gamma}{2}\left|1-\frac{c}{\|u_n\|_2^2}\right|\|\Delta u_n\|_2^2 + \frac{\beta}{2}\left|1-\frac{c}{\|u_n\|_2^2}\right|\|\nabla u_n\|_2^2 + \frac{1}{2\sigma+2}\left|1-\left(\frac{c}{\|u_n\|_2^2}\right)^{\sigma+1}\right|\|u_n\|_{2\sigma+2}^{2\sigma+2}.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{c}{\|u_n\|_2^2} = 1$ and $\{u_n\} \subset H^2(\mathbb{R}^N)$ is bounded, the result follows. □

Our next result shows that either a minimizing sequence is vanishing or it is precompact up to translations. In other words, the non-vanishing rules out the possibility of dichotomy.

Lemma 2.5. *Let $c > 0$. If $\{u_n\} \subset S(c)$ is a minimizing sequence with respect to $m(c)$ then one of the following alternative holds:*

i) *For all $R > 0$,*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_n|^2dx = 0.$$

ii) *Taking a subsequence if necessary, there exists $u \in S(c)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot - y_n) \to u$ in $H^2(\mathbb{R}^N)$. In particular u is a global minimizer.*

Proof. Suppose that $\{u_n\} \subset S(c)$ is a minimizing sequence with respect to $m(c)$ that do not satisfy i). Then, there exists $R_0 > 0$ such that

$$0 < \lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_{R_0}(y)}|u_n|^2dx \le c,$$

and so, up to a subsequence, there exists a family $\{y_n\} \subset \mathbb{R}^N$ such that

$$0 < \lim_{n\to\infty}\int_{B_{R_0}(y_n)}|u_n(x-y_n)|^2dx \le c. \tag{2.8}$$

Since $\{u_n\}$ is a minimizing sequence, by Lemma 2.2, we deduce that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$ and so, up to a subsequence, that there exists $u \in H^2(\mathbb{R}^N)$ such that

$$u_n(\cdot - y_n) \rightharpoonup u \text{ in } H^2(\mathbb{R}^N) \quad \text{and} \quad u_n(\cdot - y_n) \to u \text{ in } L^p_{loc}(\mathbb{R}^N), \quad \text{for } 1 \le p < \frac{2N}{(N-4)^+}. \tag{2.9}$$

Observe that (2.8) implies that $u \not\equiv 0$. Now, we define $v_n := u_n(\cdot - y_n) - u$ and, by (2.9), we have that $v_n \rightharpoonup 0$ in $H^2(\mathbb{R}^N)$ and so, that

$$\|\Delta u_n\|_2^2 = \|\Delta(u+v_n)\|_2^2 = \|\Delta u\|_2^2 + \|\Delta v_n\|_2^2 + o_n(1),$$
$$\|\nabla u_n\|_2^2 = \|\nabla(u+v_n)\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o_n(1),$$

and

$$\|u_n\|_2^2 = \|u+v_n\|_2^2 = \|u\|_2^2 + \|v_n\|_2^2 + o_n(1). \tag{2.10}$$

On the other hand, by the Brezis-Lieb lemma [11, Theorem 1],

$$\|u_n\|_{2\sigma+2}^{2\sigma+2} = \|u+v_n\|_{2\sigma+2}^{2\sigma+2} = \|u\|_{2\sigma+2}^{2\sigma+2} + \|v_n\|_{2\sigma+2}^{2\sigma+2} + o_n(1).$$

Hence, we have that

$$E(u_n) = E(u_n(\cdot - y_n)) = E(u+v_n) = E(u) + E(v_n) + o_n(1). \tag{2.11}$$

Claim: $\|v_n\|_2^2 \to 0$ *as* $n \to \infty$.

In order to prove this, let us denote $c_1 = \|u\|_2^2 > 0$. By (2.10), if we show that $c_1 = c$, the claim follows. We assume by contradiction that $c_1 < c$ and we define

$$\widetilde{v}_n = \frac{(c-c_1)^{\frac{1}{2}}}{\|v_n\|_2} v_n.$$

By Lemma 2.4 and (2.11), it follows that

$$E(u_n) = E(u) + E(v_n) + o_n(1) = E(u) + E(\widetilde{v}_n) + o_n(1) \geq E(u) + m(c-c_1) + o_n(1).$$

Hence, by Lemma 2.3, iv), we have that

$$m(c) \geq E(u) + m(c-c_1) \geq m(c_1) + m(c-c_1) \geq m(c), \tag{2.12}$$

and so, $E(u) = m(c_1)$, namely u is global minimizer with respect to c_1. Thus, by Lemma 2.3, v), we have that

$$m(c) < m(c_1) + m(c-c_1),$$

which contradicts (2.12). Hence, the claim follows and $\|u\|_2^2 = c$.

At this point, since $\{v_n\}$ is a bounded sequence in $H^2(\mathbb{R}^N)$, it follows from (2.1) and (2.3) respectively that $\|v_n\|_{2\sigma+2}^{2\sigma+2} \to 0$ and $\|\nabla v_n\|_2^2 \to 0$ as $n \to \infty$. Thus, we obtain that

$$\liminf_{n\to\infty} E(v_n) = \liminf_{n\to\infty} \frac{\gamma}{2}\|\Delta v_n\|_2^2 \geq 0. \tag{2.13}$$

On the other hand, since $\|u\|_2^2 = c$, we deduce from (2.11) that

$$E(u_n) = E(u) + E(v_n) + o_n(1) \geq m(c) + E(v_n) + o_n(1),$$

and so, that

$$\limsup_{n\to\infty} E(v_n) \leq 0. \tag{2.14}$$

From (2.13) and (2.14) we deduce that $\|\Delta v_n\|_2^2 \to 0$ as $n \to \infty$ and so, that $u_n(\cdot - y_n) \to u$ in $H^2(\mathbb{R}^N)$. □

3. An associated minimization problem

In this section we present a result that we shall use in the proofs of Theorems 1.2 and 1.3 but that we believe is also interesting by itself. Moreover, it enlightens the difficulty of the minimization problem for $m(c)$. Let us introduce

$$I(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\, dx - \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx,$$

and consider the constrained minimization problem

$$m_I(c) := \inf_{u \in S(c)} I(u).$$

Lemma 3.1. *For all* $c > 0$, *it follows that:*

i) $m_I(c) = -\frac{\beta^2}{8\gamma} c$.
ii) *The infimum is never achieved.*
iii) *All minimizing sequences present vanishing.*

Proof. First observe that if, for some $c > 0$, u is a minimizer of $m_I(c)$, then, for any $c_1 > 0$, $\left(\frac{c_1}{c}\right)^{1/2} u$ is a minimizer of $m_I(c_1)$. Hence, if a minimizer exists for some $c_0 > 0$, it exists for any $c > 0$.

Let $c > 0$ arbitrary but fixed, for $u \in S(c)$ arbitrary, let us first minimize I along the ray defined by u_s. From the definition (2.5) we see that the restriction of I is given by

$$I(u_s) = \frac{\gamma s^2}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx - \frac{\beta s}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \tag{3.1}$$

Then, computing the minimum of the function on s, one easily gets that

$$\inf_{s>0} I(u_s) = -\frac{\beta^2}{8\gamma} \frac{\|\nabla u\|_2^4}{\|\Delta u\|_2^2}$$

Thus, we deduce that

$$m_I(c) = \inf_{u \in S(c)} \left[-\frac{\beta^2}{8\gamma} \frac{\|\nabla u\|_2^4}{\|\Delta u\|_2^2} \right], \tag{3.2}$$

or equivalently

$$m_I(c) = \inf_{u \in S(c)} \left[-\frac{\beta^2}{8\gamma} c \frac{\|\nabla u\|_2^4}{\|\Delta u\|_2^2 \|u\|_2^2} \right] = \inf_{u \in H^2(\mathbb{R}^N) \setminus \{0\}} \left[-\frac{\beta^2}{8\gamma} c \frac{\|\nabla u\|_2^4}{\|\Delta u\|_2^2 \|u\|_2^2} \right] = -\frac{\beta^2}{8\gamma} c \sup_{u \in H^2(\mathbb{R}^N) \setminus \{0\}} R(u),$$

where we have set

$$R(u) := \frac{\|\nabla u\|_2^4}{\|\Delta u\|_2^2 \|u\|_2^2} = \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2}{\left(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right) \left(\int_{\mathbb{R}^N} |u|^2 dx \right)}.$$

At this point we deduce that u is a minimizer of $m_I(c)$ if and only if it is a maximizer of

$$\sup_{u \in H^2(\mathbb{R}^N) \setminus \{0\}} R(u).$$

Let us then show that this supremum is equal to 1 and never achieved. Using the Fourier's transform we get

$$R(u) = \frac{\left(\int_{\mathbb{R}^N} |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi \right)^2}{\left(\int_{\mathbb{R}^N} |\xi|^4 |\hat{u}(\xi)|^2 \, dx \right) \left(\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 \, dx \right)}.$$

Because of the Cauchy-Schwartz inequality

$$\left(\int_{\mathbb{R}^N} |fg| \, d\xi \right)^2 \leq \left(\int_{\mathbb{R}^N} |f|^2 \, d\xi \right) \left(\int_{\mathbb{R}^N} |g|^2 \, d\xi \right) \tag{3.3}$$

it follows that $R(u) \leq 1$ (note that this information is precisely (2.4)). Let us now construct a sequence $\{u_n\} \subset H^2(\mathbb{R}^N)$ such that $R(u_n) \to 1$. This will prove that

$$m_I(c) = -\frac{\beta^2}{8\gamma} c.$$

For $\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ we define the sequence $\phi_n(x) = n^{\frac{N}{2}} \phi(n(x-1))$ and we note that $\|\phi_n\|_2^2 = \|\phi\|_2^2$, for all $n \in \mathbb{N}$. Now, we define $\{u_n\}$ as $\hat{u}_n(\xi) = \phi_n(\xi)$, namely

$$u_n(x) = \mathcal{F}^{-1}[\phi_n](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i\xi x} \phi_n(\xi) \, d\xi,$$

and we get that

$$R(u_n) = \frac{\left(\int_{\mathbb{R}^N} |1 + \frac{y}{n}|^2 \phi^2(y) \, dy \right)^2}{\left(\int_{\mathbb{R}^N} |1 + \frac{y}{n}|^4 \phi^2(y) \, dy \right) \|\phi\|_2^2}.$$

It is then clear that $R(u_n) \to 1$ as $n \to \infty$. Hence, $\{u_n\}$ is the desired sequence and i) follows.

Next, let us show that $R(u) = 1$ never holds. If we assume by contradiction that there exists a $u \in H^2(\mathbb{R}^N)$ such that $R(u) = 1$, this corresponds to the equality case in (3.3) and thus

$$|\xi|^4 |\hat{u}(\xi)|^2 = \omega |\hat{u}(\xi)|^2 \text{ a. e. for some } \omega > 0.$$

This may only happen if the support of $\hat{u}$ is contained into a sphere $\{|\xi| = const\}$ but then, we have a contradiction with the fact that $\hat{u} \in L^2(\mathbb{R}^N)\backslash\{0\}$.

At this point, i) and ii) have been established. To conclude the proof it remains to show that any minimizing sequence of $m_I(c)$ is vanishing. Indeed, if for some $c > 0$, we assume that there exists a non-vanishing minimizing sequence, then, following the proof of Lemma 2.5, we get that there exists a $0 < c_1 \leq c$ such that $m_I(c_1)$ is reached. By ii) we know this cannot happen. Hence, the vanishing always holds. □

Remark 3.1.

a) Let us consider the operator $S = \gamma\Delta^2 + \beta\Delta$ defined in $H^2(\mathbb{R}^N)$. As a consequence of Lemma 3.1, see also Lemma 2.1, we deduce that the infimum of the spectrum of S is given by $-\beta^2/4\gamma$ and that it belongs to the essential spectrum.
b) The proof of i) and ii) of the previous lemma relies on the fact that there does not exist a maximizer for the interpolation inequality (2.3). This fact was already observed in [2, Example 2.1].

4. A Necessary and sufficient condition for the existence of a minimizer

The aim of this section is to give a necessary and sufficient condition for the existence of a minimizer of $m(c)$. In particular, this condition will show that a minimizer always exists if $c > 0$ is sufficiently large. Defining

$$c_0 := \inf\left\{c > 0 : m(c) < -\frac{\beta^2}{8\gamma}c\right\}, \tag{4.1}$$

we have the following result.

Lemma 4.1.

i) $c_0 < +\infty$.
ii) *If* $c_0 = 0$, *then* $m(c) < -\frac{\beta^2}{8\gamma}c$ *and it is reached for any* $c > 0$.
iii) *If* $c_0 > 0$, *then:*
   a) $m(c) = -\frac{\beta^2}{8\gamma}c$ *and it is not reached for any* $c \in (0, c_0]$.
   b) $m(c) < -\frac{\beta^2}{8\gamma}c$ *and it is reached for any* $c > c_0$.

Proof. Let us first prove that $c_0 < \infty$. We fix an arbitrary $u \in S(1)$ and we define $u^\tau = \tau^{\frac{1}{2}}u(x)$ with $\tau > 0$. Then, we have that $u^\tau \in S(\tau)$ and

$$E(u^\tau) = \tau\left[\frac{\gamma}{2}\|\Delta u\|_2^2 - \frac{\beta}{2}\|\nabla u\|_2^2\right] - \frac{\alpha\tau^{\sigma+1}}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2}.$$

Since $\sigma + 1 > 1$, we easily deduce that

$$m(\tau) \leq E(u^\tau) < -\frac{\beta^2}{8\gamma}\tau$$

for $\tau > 0$ large enough and so, that $c_0 < \infty$.

Now observe that, if for some $c > 0$ the vanishing occurs for a minimizing sequence $\{u_n\} \subset S(c)$, then, by [25, Lemma I.1], we have that $\|u_n\|_{2\sigma+2}^{2\sigma+2} \to 0$ as $n \to \infty$ and so, that $m(c) \geq m_I(c)$. Hence, the strict inequality $m(c) < m_I(c) = -\frac{\beta^2}{8\gamma}c$ guarantees that the vanishing does not happen. Applying then Lemma 2.5, we deduce that $m(c) < m_I(c)$ implies the existence of a global minimizer.

To end the proof just observe that Lemma 2.3, ii) guarantees that $m(c) < -\frac{\beta^2}{8\gamma}c$ for any $c > c_0$. On the other hand, if $c \in [0, c_0]$, then $m(c) = m_I(c)$ and so, any minimizing sequence of $m(c)$ is a minimizing sequence of $m_I(c)$. Thus, by Lemma 3.1, it must be vanishing and $m(c)$ is not reached. □

5. Some classes of testing functions

This section is devoted to find sufficient conditions on $\sigma > 0$ ensuring that $c_0 = 0$ in Lemma 4.1, i.e. guaranteeing that a minimizer exists for any $c > 0$.

We start with an observation which, although not essential in our proofs, simplifies some computations. To that end, let us introduce the constrained minimization problem

$$m_\Phi(c) := \inf_{u \in S(c)} \Phi(u),$$

where

$$\Phi(u) = \|(\Delta + 1)u\|_2^2 - \frac{\alpha}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2},$$

and $S(c)$ is given in (1.6).

Proposition 5.1. *Let $c_0 \in [0, +\infty)$ given in (4.1). We have that $c_0 = 0$ if and only if $m_\Phi(c) < 0$ for all $c > 0$.*

Proof. Let us introduce

$$\widetilde{E}(u) := \|\Delta u\|_2^2 - 2\|\nabla u\|_2^2 - \frac{\alpha}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2} \quad \text{and} \quad \tau = \left(\frac{8\gamma}{\beta^2}\right)^{\frac{1}{\sigma}} \left(\frac{\beta}{2\gamma}\right)^{\frac{N}{2}}. \tag{5.1}$$

As a first step we prove that, for a given $c > 0$, the problem of minimizing E on $S(c)$ is equivalent to minimize $\widetilde{E}$ on $S(\tau c)$. Indeed, letting $v(x) := bu(ax)$ with

$$a = \left(\frac{2\gamma}{\beta}\right)^{\frac{1}{2}} \quad \text{and} \quad b = \left(\frac{8\gamma}{\beta^2}\right)^{\frac{1}{2\sigma}},$$

we obtain that

$$E(u) = b^{-2-2\sigma} a^N \left[\|\Delta v\|_2^2 - 2\|\nabla v\|_2^2 - \frac{\alpha}{2\sigma+2}\|v\|_{2\sigma+2}^{2\sigma+2}\right] = b^{-2-2\sigma} a^N \widetilde{E}(v),$$

and

$$\tau\|u\|_2^2 = b^2 a^{-N}\|u\|_2^2 = \|v\|_2^2.$$

Hence, the mentioned equivalence follows. This being proved, we can assume without loss of generality that $\gamma = 2$ and $\beta = 4$, namely that

$$E(u) = \|\Delta u\|_2^2 - 2\|\nabla u\|_2^2 - \frac{\alpha}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2}.$$

Accordingly, the condition $m(c) < -\frac{\beta^2}{8\gamma}c$ derived in Lemma 4.1 now corresponds to $m(c) < -c$. The result then follows recognizing that $\Phi(u) = E(u) + \|u\|_2^2$. $\square$

Now, we give sufficient conditions on $\sigma > 0$ ensuring that $m_\Phi(c) < 0$ for all $c > 0$. Then, as a consequence of the previous proposition, these conditions guarantee that $c_0 = 0$ in Lemma 4.1.

Proposition 5.2. *Assume that $0 < \sigma < \max\left\{\frac{4}{N+1}, 1\right\}$. Then $m_\Phi(c) < 0$ for all $c > 0$.*

Let us split the proof into two lemmas. The proof of both lemmas consists in finding a suitable test function $u \in S(c)$ such that $\Phi(u) < 0$. As mentioned in Remark 1.3, we will look for functions such that the L^2–norm of their Fourier transform concentrates around the unit sphere. In our first construction, we consider a function whose Fourier transform is a perturbed Gaussian centered at $e_1 = (1, 0, \dots, 0)$.

Lemma 5.3. *Assume that $0 < \sigma < \frac{4}{N+1}$. Then $m_\Phi(c) < 0$ for all $c > 0$.*

Proof. We fix an arbitrary $c > 0$ and, for any $\tau > 0$, we define

$$u_\tau(x) = \pi^{-\frac{N}{4}} c^{\frac{1}{2}} \tau^{\frac{N+1}{2}} e^{ix_1} e^{-\frac{\tau^4 x_1^2 + \tau^2 x_2^2 + \cdots \tau^2 x_N^2}{2}}.$$

It is clear that $u_\tau \in \mathcal{S}(\mathbb{R}^N)$ and an easy computation gives

$$\|u_\tau\|_2^2 = \pi^{-\frac{N}{2}} c \tau^{N+1} \int_{\mathbb{R}^N} e^{-\frac{\tau^4 x_1^2 + \tau^2 x_2^2 + \cdots \tau^2 x_N^2}{2}} dx = \pi^{-\frac{N}{2}} c \int_{\mathbb{R}^N} e^{-|y|^2} dy = c.$$

Also, we have that

$$
(5.2)\qquad \begin{aligned}
\|u_\tau\|_{2\sigma+2}^{2\sigma+2} &= \pi^{-\frac{N(\sigma+1)}{2}} c^{\sigma+1}\tau^{(N+1)(\sigma+1)}\int_{\mathbb{R}^N} e^{-\frac{\tau^4x_1^2+\tau^2x_2^2+\cdots\tau^2x_N^2}{2}}\,dx\\
&= \pi^{-\frac{N(\sigma+1)}{2}}(\sigma+1)^{-\frac{N}{2}} c^{\sigma+1}\tau^{(N+1)\sigma}\int_{\mathbb{R}^N} e^{-|y|^2}\,dy = \pi^{-\frac{\sigma N}{2}}(\sigma+1)^{-\frac{N}{2}}c^{\sigma+1}\tau^{(N+1)\sigma}.
\end{aligned}
$$

It is well-known that the Fourier transform of $f(x) = e^{-\alpha^2|x|^2}$ is given by

$$
\widehat{f}(\xi) = \left(\frac{\pi}{\alpha^2}\right)^{\frac{N}{2}} e^{-\frac{|\xi|^2}{4\alpha^2}}.
$$

Using this fact and the basic properties of the Fourier's transform we get

$$
\widehat{u_\tau}(\xi) = 2^{\frac{N}{2}}\pi^{\frac{N}{4}}c^{\frac12}\tau^{-\frac{N+1}{2}}e^{-\frac{\left(\frac{\xi_1-1}{\tau}\right)^2+\xi_2^2+\cdots+\xi_N^2}{2\tau^2}}.
$$

Hence, by Plancherel's formula, it follows that

$$
\begin{aligned}
\|\Delta u_\tau\|_2^2 - 2\|\nabla u_\tau\|_2^2 + \|u_\tau\|_2^2 &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\Big(|\xi|^4 - 2|\xi|^2+1\Big)|\widehat{u_\tau}(\xi)|^2\,d\xi\\
&= \pi^{-\frac{N}{2}}c\tau^{-(N+1)}\int_{\mathbb{R}^N}(|\xi|^2-1)^2 e^{-\frac{\left(\frac{\xi_1-1}{\tau}\right)^2+\xi_2^2+\cdots+\xi_N^2}{\tau^2}}\,d\xi.
\end{aligned}
$$

Now, using the changes of variables

$$
\xi_1 = 1+\tau^2\eta_1,\qquad \xi_j = \tau\eta_j,\ j=2,\dots,N,
$$

we obtain that

$$
(5.3)\qquad \begin{aligned}
\|\Delta u_\tau\|_2^2 - 2\|\nabla u_\tau\|_2^2 + \|u_\tau\|_2^2 &= \pi^{-\frac{N}{2}}c\int_{\mathbb{R}^N}(\tau^4\eta_1^2+2\tau^2\eta_1+\tau^2\eta_2^2+\cdots+\tau^2\eta_N^2)^2e^{-|\eta|^2}\,d\eta\\
&= \pi^{-\frac{N}{2}}c\tau^4\int_{\mathbb{R}^N}(\tau^2\eta_1^2+2\eta_1+\eta_2^2+\cdots+\eta_N^2)^2e^{-|\eta|^2}\,d\eta\\
&\le A\tau^4, \qquad \forall\ \tau\in(0,1],
\end{aligned}
$$

for some constant $A>0$ (independent of τ). Then, by (5.2) and (5.3), it follows that

$$
\Phi(u_\tau) \le A\tau^4 - \frac{\alpha}{2\sigma+2}\pi^{-\frac{\sigma N}{2}}(\sigma+1)^{-\frac{N}{2}}c^{\sigma+1}\tau^{\sigma(N+1)} =: A\tau^4 - B\tau^{\sigma(N+1)},
$$

with A and B positive constants independent of τ. Since $0<\sigma(N+1)<4$, we may choose $\tau\in(0,1]$ sufficiently small so that $A\tau^4 - B\tau^{(N+1)\sigma}<0$ and then $m_\Phi(c)\le\Phi(u_\tau)\le A\tau^4-B\tau^{(N+1)\sigma}<0$. □

Now, using a different construction based on the fact that

$$
(5.4)\qquad \psi(x) = |x|^{-\frac{N-2}{2}}J_{\frac{N-2}{2}}(|x|),
$$

satisfies

$$
(5.5)\qquad (\Delta+1)\psi = 0, \quad \text{in } \mathbb{R}^N,
$$

we enlarge the range on $\sigma>0$ obtained in Lemma 5.3. Here J_ν denotes the Bessel function of the first kind and order ν and we refer for instance to [20, Appendix B.4] for a proof of (5.5).

Since the asymptotic behavior of ψ will play an important role in our construction, we describe it in the following lemma.

Lemma 5.4. *Assume that $N\ge2$ and let ψ as defined in (5.4). Then:*

i) $\psi(x)\sim\left(\frac12\right)^{\frac{N-2}{2}}\frac{1}{\Gamma(\frac{N}{2})}$ *as* $|x|\to0$.

ii) $\psi(x) \sim |x|^{-\frac{N-1}{2}} \cos\left(|x| - \frac{(N-1)\pi}{4}\right)$ *as* $|x| \to +\infty$.

Proof. The result immediately follows from the asymptotic behavior of $J_\nu(t)$ for $\nu \geq 0$ and $t \geq 0$. We refer for instance to [24, Formula (5.16.1) page 134]. □

Lemma 5.5. *Assume that* $N \geq 4$ *and* $0 < \sigma < 1$. *Then* $m_\Phi(c) < 0$ *for all* $c > 0$.

Remark 5.1.

a) The construction we do here may also be used for $N \leq 3$. Nevertheless, since, for $N \leq 3$, the results we are able to obtain do not improve the ones contained in Lemma 5.3, we focus on $N \geq 4$.
b) Note that, for $N \geq 4$, we cover the full mass subcritical range $0 < \sigma N < 4$.

Proof. First of all observe that, for $N \geq 4$, $\frac{1}{N-1} < \frac{4}{N+1}$. Hence, having at hand Lemma 5.3, we can assume without loss of generality that $\sigma > \frac{1}{N-1}$. Then, for all $m \in \mathbb{N}$, we define

$$\psi_m(x) = \psi(x)\phi\left(\frac{x}{m}\right) \tag{5.6}$$

where ψ is given in (5.4) and $\phi \in C^\infty(\mathbb{R}^N)$ is such that $\phi(x) = 1$ if $|x| \leq 1$, $\phi(x) = 0$ if $|x| \geq 2$ and $0 \leq \phi(x) \leq 1$, for all $x \in \mathbb{R}^N$. Note that, in this proof, for any $\delta > 0$, B_δ denotes the ball $B_\delta(0)$. We now split the rest of the proof into several steps.

Step 1: *There exist* $m_1 \in \mathbb{N}$ *and* $D_1 > 0$ *such that, for all* $m \geq m_1$, *it follows that* $\|\psi_m\|_2^2 \leq D_1 m$.

First observe that, by Lemma 5.4, ii), there exists $R \in \mathbb{N}$ such that, for all $x \in \mathbb{R}^N$ with $|x| \geq R$,

$$|\psi(x)| \leq C\,|x|^{-\frac{N-1}{2}},$$

and so, that, for all $m \geq R$,

$$\|\psi_m\|_2^2 = \int_{\mathbb{R}^N} \left|\psi(x)\phi\left(\frac{x}{m}\right)\right|^2 dx \leq \int_{B_{2m}} |\psi(x)|^2 dx = \int_{B_R} |\psi(x)|^2 dx + \int_{B_{2m}\setminus B_R} |\psi(x)|^2 dx \leq C_1 + C_2(2m - R).$$

Hence, there exist $m_1 \geq R$ and $D_1 > 0$ such that, for all $m \geq m_1$,

$$\|\psi_m\|_2^2 \leq D_1 m.$$

Step 2: *There exists* $m_2 \in \mathbb{N}$ *and* $D_2 > 0$ *such that, for all* $m \geq m_2$, *it follows that* $\|(\Delta + 1)\psi_m\|_2^2 \leq D_2 m^{-1}$.

First of all, using (5.5), one can easily check that

$$(\Delta + 1)\psi_m(x) = \frac{1}{m^2}\Delta\phi\left(\frac{x}{m}\right)\psi(x) + \frac{2}{m}\nabla\phi\left(\frac{x}{m}\right)\cdot\nabla\psi(x),$$

and so, that

$$\begin{aligned}
\|(\Delta+1)\psi_m\|_2^2 &= \frac{1}{m^4}\int_{\mathbb{R}^N}\left|\Delta\phi\left(\frac{x}{m}\right)\psi(x)\right|^2 dx + \frac{4}{m^2}\int_{\mathbb{R}^N}\left|\nabla\phi\left(\frac{x}{m}\right)\cdot\nabla\psi(x)\right|^2 dx \\
&\quad + \frac{4}{m^3}\int_{\mathbb{R}^N}\left|\Delta\phi\left(\frac{x}{m}\right)\psi(x)\right|\left|\nabla\phi\left(\frac{x}{m}\right)\cdot\nabla\psi(x)\right| dx \\
&\leq \frac{1}{m^4}\int_{\mathbb{R}^N}\left|\Delta\phi\left(\frac{x}{m}\right)\psi(x)\right|^2 dx + \frac{4}{m^2}\int_{\mathbb{R}^N}\left|\nabla\phi\left(\frac{x}{m}\right)\cdot\nabla\psi(x)\right|^2 dx \\
&\quad + \frac{4}{m^3}\left(\int_{\mathbb{R}^N}\left|\Delta\phi\left(\frac{x}{m}\right)\psi(x)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\left|\nabla\phi\left(\frac{x}{m}\right)\cdot\nabla\psi(x)\right|^2 dx\right)^{\frac{1}{2}} \\
&=: \frac{1}{m^4}I_1 + \frac{4}{m^2}I_2 + \frac{4}{m^3}(I_1)^{\frac{1}{2}}(I_2)^{\frac{1}{2}}.
\end{aligned} \tag{5.7}$$

Then, arguing as in Step 1, we obtain that, for all $m \geq R$,

$$I_1 \leq \|\Delta\phi\|_\infty^2 \int_{B_{2m}\setminus B_m} |\psi(x)|^2 dx \leq C_1 m. \tag{5.8}$$

Now, since

$$J_\nu'(t) = -J_{\nu+1}(t) + \frac{\nu}{t} J_\nu(t),$$

for all $\nu \geq 0$ and all $t \geq 0$, see for instance [35, (4) page 45], we obtain that

$$\nabla \psi(x) = -\frac{N-2}{2}|x|^{-\frac{N+2}{2}} x J_{\frac{N-2}{2}}(|x|) + |x|^{-\frac{N-2}{2}} \frac{x}{|x|}\left(-J_{\frac{N}{2}}(|x|) + \frac{N-2}{2}|x|^{-1} J_{\frac{N-2}{2}}(|x|)\right), \quad \forall\, x \in \mathbb{R}^N \setminus \{0\},$$

and so, similarly as in Lemma 5.4, there exists $R_2 \in \mathbb{N}$ such that, for all $x \in \mathbb{R}^N$ with $|x| \geq R_2$,

$$|\nabla \psi(x)| \leq C(N-2)|x|^{-\frac{N-1}{2}}.$$

Hence, we deduce that, for all $m \geq R_2$,

$$I_2 \leq \|\nabla \psi\|_\infty^2 \int_{B_{2m}\setminus B_m} |\nabla \psi(x)|^2 dx \leq C(N-2)\|\nabla \psi\|_\infty^2 \int_{B_{2m}\setminus B_m} |x|^{-(N-1)} dx \leq C_2 m. \tag{5.9}$$

Gathering (5.7)-(5.9), we obtain that

$$\|(\Delta+1)\psi_m\|_2^2 \leq C_1 m^{-3} + 4C_2 m^{-1} + 4C_1^{1/2} C_2^{1/2} m^{-2},$$

and so, that there exist $m_2 \geq \max\{R, R_2\}$ and $D_2 > 0$ such that, for all $m \geq m_2$,

$$\|(\Delta+1)\psi_m\|_2^2 \leq D_2 m^{-1}.$$

Step 3: *There exists $D_3 > 0$ such that, for all $m \in \mathbb{N}$, it follows that $\|\psi_m\|_{2\sigma+2}^{2\sigma+2} \geq D_3$.*

Directly observe that, for all $m \in \mathbb{N}$,

$$\|\psi_m\|_{2\sigma+2}^{2\sigma+2} = \int_{\mathbb{R}^N} \left|\psi(x)\phi\left(\frac{x}{m}\right)\right|^{2\sigma+2} dx \geq \int_{B_1} |\psi(x)|^{2\sigma+2} dx.$$

The claim then follows directly from Lemma 5.4, i).

Step 4: *Conclusion.*

We fix an arbitrary $c > 0$ and, for any $m \in \mathbb{N}$, we define $\widetilde{\psi}_m = c^{\frac{1}{2}} \frac{\psi_m}{\|\psi_m\|_2}$. It is clear that $\widetilde{\psi}_m \in S(c)$ for all $m \in \mathbb{N}$. Then, by Steps 1, 2 and 3, we deduce that there exists $m_3 \geq \max\{m_1, m_2\}$ such that, for all $m \geq m_3$,

$$\begin{aligned}
\Phi(\widetilde{\psi}_m) &= \frac{c}{\|\psi_m\|_2^2}\|(\Delta+1)\psi_m\|_2^2 - \frac{\alpha}{2\sigma+2}\frac{c^{\sigma+1}}{\|\psi_m\|_2^{2\sigma+2}}\|\psi_m\|_{2\sigma+2}^{2\sigma+2} \\
&\leq \frac{c}{\|\psi_m\|_2^2} D_2 m^{-1} - \frac{\alpha}{2\sigma+2}\frac{c^{\sigma+1}}{\|\psi_m\|_2^{2\sigma+2}} D_3 \\
&= \frac{1}{\|\psi_m\|_2^2}\left[c D_2 m^{-1} - \frac{\alpha D_3 c^{\sigma+1}}{2\sigma+2}\frac{1}{\|\psi_m\|_2^{2\sigma}}\right] \\
&\leq \frac{1}{\|\psi_m\|_2^2}\left[c D_2 m^{-1} - \frac{\alpha D_3 c^{\sigma+1}}{(2\sigma+2)D_1^\sigma} m^{-\sigma}\right] =: \frac{1}{\|\psi_m\|_2^2}\left[A m^{-1} - B m^{-\sigma}\right],
\end{aligned}$$

with A and B positive constants independent of m. Arguing as in Step 3, one can easily see that $\|\psi_m\|_2^2 \geq D_4 > 0$ for all $m \in \mathbb{N}$. Thus, since $0 < \sigma < 1$, we may choose $m \geq m_3$ sufficiently large so that $Am^{-1} - Bm^{-\sigma} < 0$ and then $m_\Phi(c) \leq \Phi(\widetilde{\psi}_m) \leq Am^{-1} - Bm^{-\sigma} < 0$. □

Proof of Proposition 5.2. It follows directly from Lemmas 5.3 and 5.5. □

At this point we can give the proof of our first main result.

Proof of Theorem 1.2. The existence part of Theorem 1.2 is a direct consequence of Lemma 4.1 and Propositions 5.1 and 5.2. Hence, to conclude we just need to obtain the lower bound on the Lagrange multiplier. We recall that if $u \in S(c)$ is a global minimizer of $m(c)$ (or more generally a constrained critical point), there exists a $\lambda \in \mathbb{R}$ such that $E'(u) = -\lambda u$, namely a $\lambda \in \mathbb{R}$ such that

$$\gamma \Delta^2 u + \beta \Delta u - \alpha |u|^{2\sigma} u = -\lambda u. \tag{5.10}$$

Multiplying (5.10) by u and integrating it follows that

$$-\lambda c = \gamma\|\Delta u\|_2^2 - \beta\|\nabla u\|_2^2 - \alpha\|u\|_{2\sigma+2}^{2\sigma+2}. \tag{5.11}$$

Then, from (5.11) and Lemma 3.1, we deduce that

$$\begin{aligned}-\lambda c &= 2E(u) - \frac{2\alpha\sigma}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2}\\ &= 2m(c) - \frac{2\alpha\sigma}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2} < 2m(c) \le 2m_I(c) = -\frac{\beta^2}{4\gamma}c,\end{aligned}$$

and thus $\lambda > \dfrac{\beta^2}{4\gamma}$. □

6. Behavior of the minimizers as $c \to 0$, proof of Theorem 1.3

In the section we give the proof of Theorem 1.3. Recall that $\{(u_n, c_n)\} \subset S(c_n) \times \mathbb{R}$ is such that $c_n \to 0$ and, for each $n \in \mathbb{N}$, $u_n \in S(c_n)$ is a minimizer of $m(c_n)$ and $\lambda_n \in \mathbb{R}$ is the associated Lagrange multiplier. Hence, without loss of generality we may assume that $c_n \le 1$.

Proof of Theorem 1.3. Let us start with some preliminary observations.

Claim 1: $\left\{\dfrac{\|\Delta u_n\|_2}{\|u_n\|_2}\right\}$ *remains bounded.*

Indeed, using (2.1), (2.3) and Lemma 2.3, i), we have that

$$0 \ge 2E(u_n) \ge \gamma\|\Delta u_n\|_2^2 - \beta\|\Delta u_n\|_2\|u_n\|_2 - \frac{\alpha}{\sigma+1}B_N(\sigma)\|\Delta u_n\|^{\frac{\sigma N}{2}}\|u_n\|_2^{2+2\sigma-\frac{\sigma N}{2}}, \tag{6.1}$$

and so,

$$\|\Delta u_n\|_2^2 \le \frac{\beta}{\gamma}\|\Delta u_n\|_2\|u_n\|_2 + C\|\Delta u_n\|^{\frac{\sigma N}{2}}\|u_n\|_2^{2+2\sigma-\frac{\sigma N}{2}}. \tag{6.2}$$

Dividing (6.2) by $\|u_n\|_2^2$, it follows that

$$\left(\frac{\|\Delta u_n\|_2}{\|u_n\|_2}\right)^2 \le \frac{\beta}{\gamma}\frac{\|\Delta u_n\|_2}{\|u_n\|_2} + C\left(\frac{\|\Delta u_n\|_2}{\|u_n\|_2}\right)^{\frac{\sigma N}{2}}\|u_n\|_2^{2\sigma} \le \frac{\beta}{\gamma}\frac{\|\Delta u_n\|_2}{\|u_n\|_2} + C\left(\frac{\|\Delta u_n\|_2}{\|u_n\|_2}\right)^{\frac{\sigma N}{2}} \tag{6.3}$$

Since $0 < \sigma N < 4$, from (6.3), the boundedness of the left hand side follows and thus the claim is proved.

Claim 2: $\dfrac{\|u_n\|_{2\sigma+2}^{2\sigma+2}}{\|u_n\|_2^2} \to 0$ *as* $n \to \infty$.

By (2.1) we know that

$$\|u_n\|_{2\sigma+2}^{2\sigma+2} \le B_N(\sigma)\|\Delta u_n\|_2^{\frac{\sigma N}{2}}\|u_n\|_2^{2+2\sigma-\frac{\sigma N}{2}}.$$

Then, dividing by $\|u_n\|_2^2$ one gets

$$\frac{\|u_n\|_{2\sigma+2}^{2\sigma+2}}{\|u_n\|_2^2} \le B_N(\sigma)\left(\frac{\|\Delta u_n\|_2}{\|u_n\|_2}\right)^{\frac{\sigma N}{2}}\|u_n\|_2^{2\sigma} = B_N(\sigma)\left(\frac{\|\Delta u_n\|_2}{\|u_n\|_2}\right)^{\frac{\sigma N}{2}}c_n^{\sigma}$$

From Claim 1 and the fact that $c_n \to 0$ as $n \to \infty$, the claim follows. Now, we split the rest of the proof into several steps:

Step 1: *Proof of* i).

It just remains to prove the first inequality. The other statements have already been established, see Section 4. Directly observe that

$$m(c_n) = E(u_n) = I(u_n) - \frac{\alpha}{2\sigma+2}\|u_n\|_{2\sigma+2}^{2\sigma+2} \ge m_I(c_n) - \frac{\alpha}{2\sigma+2}\|u_n\|_{2\sigma+2}^{2\sigma+2} = m_I(c_n) - \frac{\alpha}{2\sigma+2}\frac{\|u_n\|_{2\sigma+2}^{2\sigma+2}}{\|u_n\|_2^2}c_n.$$

The desired inequality then follows from Claim 2.

Step 2: *Proof of* ii).

By Theorem 1.2 we know that $\lambda_n > \frac{\beta^2}{4\gamma}$. Now, recording that

$$-\lambda_n c_n = \gamma\|\Delta u_n\|_2^2 - \beta\|\nabla u_n\|_2^2 - \alpha\|u_n\|_{2\sigma+2}^{2\sigma+2}, \tag{6.4}$$

and using Lemma 3.1 and Claim 2, one can write

$$-\lambda_n c_n \geq 2m_I(c_n) - \alpha\|u_n\|_{2\sigma+2}^{2\sigma+2} = -\frac{\beta^2}{4\gamma}c_n - \frac{\|u_n\|_{2\sigma+2}^{2\sigma+2}}{\|u_n\|_2^2}c_n = -\frac{\beta^2}{4\gamma}c_n - \mu_n c_n$$

for a sequence $\mu_n \to 0$ as $n \to \infty$. Thus, we have that

$$\frac{\beta^2}{4\gamma} < \lambda_n \leq \frac{\beta^2}{4\gamma} + \mu_n$$

for a sequence $\mu_n \to 0$ and ii) follows.

Step 3: *Proof of* iii)

By [7, Lemma 2.1], we know that $u_n \in S(c_n)$ satisfies the Pohozaev type identity

$$\gamma\|\Delta u_n\|_2^2 - \frac{\beta}{2}\|\nabla u_n\|_2^2 - \frac{\sigma N}{2(2\sigma+2)}\|u_n\|_{2\sigma+2}^{2\sigma+2} = 0. \tag{6.5}$$

Dividing (6.5) by $\|u_n\|_2^2$ and using Claim 2 we get that

$$\gamma\frac{\|\Delta u_n\|_2^2}{\|u_n\|_2^2} - \frac{\beta}{2}\frac{\|\nabla u_n\|_2^2}{\|u_n\|_2^2} \to 0. \tag{6.6}$$

Also, by i), we know there exists $\mu_n \to 0$ as $n \to \infty$ such that

$$E(u_n) = \frac{\gamma}{2}\|\Delta u_n\|_2^2 - \frac{\beta}{2}\|\nabla u_n\|_2^2 - \frac{\alpha}{2\sigma+2}\|u_n\|_{2\sigma+2}^{2\sigma+2} = -\frac{\beta^2}{8\gamma}\|u_n\|_2^2 + \mu_n\|u_n\|_2^2 - \frac{\alpha}{2\sigma+2}\|u_n\|_{2\sigma+2}^{2\sigma+2}.$$

Then, using again Claim 2, we deduce that

$$\frac{\gamma}{2}\frac{\|\Delta u_n\|_2^2}{\|u_n\|_2^2} - \frac{\beta}{2}\frac{\|\nabla u_n\|_2^2}{\|u_n\|_2^2} \to -\frac{\beta^2}{8\gamma}. \tag{6.7}$$

Having at hand (6.6) and (6.7) one easily deduce iii).

Step 4: *Proof of* iv).

Let us define $v_n = \frac{u_n}{\|u_n\|_2}$ for all $n \in \mathbb{N}$. Since $u_n \in S(c_n)$ satisfies (6.4) and by ii) we know that $\lambda_n \to \frac{\beta^2}{4\gamma}$, as $n \to \infty$, we get

$$\gamma\|\Delta v_n\|_2^2 - \beta\|\nabla v_n\|_2^2 + \frac{\beta^2}{4\gamma}\|v_n\|_2^2 = \alpha\frac{\|u_n\|_{2\sigma+2}^{2\sigma+2}}{\|u_n\|_2^2} + \mu_n\|v_n\|_2^2 \tag{6.8}$$

for some $\mu_n \to 0$. On the other hand, by Plancherel's formula, it follows that

$$\gamma\|\Delta v_n\|_2^2 - \beta\|\nabla v_n\|_2^2 + \frac{\beta^2}{4\gamma}\|v_n\|_2^2 = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\left(\gamma|\xi|^4 - \beta|\xi|^2 + \frac{\beta^2}{4\gamma}\right)|\widehat{v_n}(\xi)|^2\,d\xi. \tag{6.9}$$

Gathering (6.8) and (6.9) we have that,

$$\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\left(\sqrt{\gamma}|\xi|^2 - \frac{\beta}{2\sqrt{\gamma}}\right)^2|\widehat{v_n}(\xi)|^2\,d\xi = \alpha\frac{\|u_n\|_{2\sigma+2}^{2\sigma+2}}{\|u_n\|_2^2} + \mu_n \tag{6.10}$$

where $\mu_n \to 0$ as $n \to \infty$. Using Claim 2 we see that the right hand side goes to 0 as $n \to \infty$. This proves that (1.10) holds. Now, from (1.10) and (6.9), we deduce that

$$I(v_n) = \frac{\gamma}{2}\|\Delta v_n\|_2^2 - \frac{\beta}{2}\|\nabla v_n\|_2^2 \to -\frac{\beta^2}{8\gamma}.$$

In view of Lemma 3.1, 1), we have that $\{v_n\} \subset S(1)$ is a minimizing sequence for $m_I(1)$. Then, by Lemma 3.1, iii), it follows that $\{v_n\}$ is a vanishing sequence. Applying then [25, Lemma I.1], we deduce that $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 4^*)$. This completes the proof of the theorem. □

7. Some extensions and related problems.

In this last section we make some additional remarks and discuss possible extensions of our results.

7.1. **Symmetry of the minimizers in Theorem 1.2.**

As a consequence of the arguments developed in the very recent preprint [12], when $\sigma \in \mathbb{N}$, we can obtain the following description of the minima obtained in Theorem 1.2.

Proposition 7.1. *Let $\sigma \in \mathbb{N}$ and, for any arbitrary $c > 0$ such that $m(c)$ is reached, let Q be a minima for $m(c)$. Then, it follows that*

$$Q(x) = e^{i\tau} Q^\bullet(x + x_0),$$

with some constants $\tau \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$. Here, $Q^\bullet : \mathbb{R}^N \to \mathbb{C}$ is a smooth bounded and positive definitive function in the sense of Bochner. As a consequence, it holds that

$$Q^\bullet(-x) = \overline{Q^\bullet(x)} \quad \text{and} \quad Q^\bullet(0) \geq |Q^\bullet(x)| \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Remark 7.1. Our proof is a essentially a consequence of [12, Theorem 2]. We provide some details for the benefit of the reader in trying to keep the notation introduced by the authors in [12].

Proof. First note that our operator $u \mapsto \gamma\Delta^2 u + \beta\Delta u$ falls within the class of pseudo-differential operators considered in [12]. Indeed, our symbol, which is given by $p(\xi) = \gamma|\xi|^4 - \beta|\xi|^2$, satisfies the **Assumption 2** of [12] with $s = 2$.

Now, let Q be a minima for $m(c)$ and let $\lambda \in \mathbb{R}$ be the associated Lagrange multiplier. By Theorem 1.2, we know that $\lambda > \beta^2/4\gamma$. Hence, by [5, Theorem 3.10], we have that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^N)$ for some $a > 0$. Note that such decay may be also obtained as in [12, Theorem 3]. Now, defining

$$Q^\bullet := \mathcal{F}^1(|\mathcal{F}Q|),$$

we observe, from [12, Lemma 2.1], that

$$\|\Delta Q^\bullet\|_2 = \|\Delta Q\|_2, \quad \|\nabla Q^\bullet\|_2 = \|\nabla Q\|_2, \quad \|Q^\bullet\|_2 = \|Q\|_2 \quad \text{and} \quad \|Q^\bullet\|_{2\sigma+2} \geq \|Q\|_{2\sigma+2}.$$

Thus, we easily deduce that $Q^\bullet$ is also a minima for $m(c)$. Having at hand the suitable exponential decay of Q and the fact that $Q^\bullet$ is also a minima for $m(c)$, the rest of the proof follows repeating almost verbatim the proofs of [12, Lemma 4.1] and the first part of [12, Theorem 2]. □

Remark 7.2. The conclusions of Proposition 7.1 hold for any $c > 0$, if $N = 1$ or $N = 2$ and $\sigma = 1$. In particular, we cover the physical relevant case $N = 2$ for the Kerr nonlinearity.

7.2. **Optimal range of σ.**

The condition $0 < \sigma < \max\{4/(N+1), 1\}$ is the consequence of two particular trials of test functions. It would be nice to put in light an optimal upper bound on $\sigma > 0$, which permits a minimizer to exists for every $c > 0$. See Figure 1.

More generally, the question to consider seems to be the following : Let $0 < \sigma N < 4^*$ and define

$$A(1) := \{u \in S(c) : \|\Delta u\|_2^2 \leq 1\}.$$

Which condition on $\sigma > 0$ guarantees that, for any $c > 0$,

$$\inf_{u \in S(c) \cap A(1)} I(u) < \inf_{u \in S(c) \cap A(1)} E(u)\,?$$

We conjecture that the optimal bound is $\sigma < 2$.

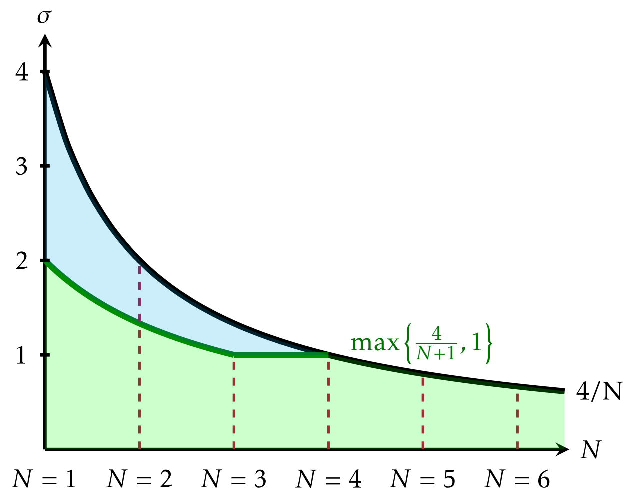

FIGURE 1. The existence of a global minimizer on $S(c)$ for $c > 0$ small is open in the smaller region.

7.3. **The mass critical and mass supercritical cases.**

As already mentioned, the results we have derived in the mass subcritical case are also useful in the mass critical and mass supercritical cases. First, concerning the mass critical case, we note the following

Lemma 7.2. *Assume $\gamma > 0$, $\beta > 0$, $\alpha > 0$, $\sigma N = 4$ and $N \geq 1$. There exists $c_N^* > 0$ such that $m(c) \in (-\infty, 0)$ if $c \in (0, c_N^*)$ and $m(c) = -\infty$ if $c \geq c_N^*$. Actually*

$$c_N^* = \left(\frac{\gamma}{\alpha} C(N)\right)^{\frac{N}{4}} \quad where \quad C(N) := \frac{N+4}{N B_N(\frac{4}{N})}, \tag{7.1}$$

and $B_N(\sigma)$ is the smallest constant satisfying (2.1). In addition, E is coercive on $S(c)$ if $c \in (0, c_N^*)$.

Proof. First note that, when $\sigma N = 4$, one has

$$\frac{\alpha N}{N+4} \|u\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \leq \left(\frac{c}{c_N^*}\right)^{\frac{4}{N}} \gamma \|\Delta u\|_2^2, \quad \forall\ u \in S(c). \tag{7.2}$$

Indeed, (7.2) follows from the Gagliardo-Nirenberg inequality (2.1) using the definition of c_N^* given in (7.1). Now, using (2.3) and (7.2), we have that

$$\begin{aligned} E(u) &\geq \frac{\gamma}{2}\|\Delta u\|_2^2 - \frac{\beta}{2}\|\Delta u\|_2 \|u\|_2 - \frac{\alpha N}{2N+8}\|u\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \\ &\geq \frac{\gamma}{2}\left(1 - \left(\frac{c}{c_N^*}\right)^{\frac{4}{N}}\right)\|\Delta u\|_2^2 - \frac{\beta}{2} c^{\frac{1}{2}} \|\Delta u\|_2, \qquad \forall\ u \in S(c). \end{aligned}$$

Therefore, we deduce that E is coercive if $c < c_N^*$ and then, in particular, that $m(c) > -\infty$. The fact that $m(c) < 0$ when $c \in (0, c_N^*)$ follows directly from (2.6) letting, for an arbitrary $u \in S(c)$, $s \to 0$.

Now, let us prove that $m(c) = -\infty$ for $c \geq c_N^*$. It follows from [10], see also [2], that the best constant $B_N(\frac{4}{N})$ in (2.1) is achieved, i.e. there exists $U \in H^2(\mathbb{R}^N)$ satisfying

$$\|U\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} = B_N\left(\frac{4}{N}\right) \|U\|_2^{\frac{8}{N}} \|\Delta U\|_2^2. \tag{7.3}$$

Choosing

$$w := c^{\frac{1}{2}} \frac{U}{\|U\|_2} \in S(c),$$

and taking (2.6) and (7.3) into account, we get

$$
\begin{aligned}
E(w_s) &= \frac{c}{2\|U\|_2^2}s^2\gamma\|\Delta U\|_2^2 - \frac{c}{2\|U\|_2^2}s\beta\|\nabla U\|_2^2 - \frac{N}{2N+8}\left(\frac{c^{\frac{1}{2}}}{\|U\|_2}\right)^{2+\frac{8}{N}} s^2\alpha\|U\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \\
&= \frac{c}{2\|U\|_2^2}\gamma\left(1-\left(\frac{c}{c_N^*}\right)^{\frac{4}{N}}\right)s^2\|\Delta U\|_2^2 - \frac{c}{2\|U\|_2^2}s\beta\|\nabla U\|_2^2 \\
&\leq -\frac{c}{2\|U\|_2^2}s\beta\|\nabla U\|_2^2
\end{aligned} \tag{7.4}
$$

which implies that $E(w_s) \to -\infty$ as $s \to \infty$ for any $c \geq c_N^*$. □

In view of Lemma 7.2 we obtain the following result on the line of Theorem 1.2.

Theorem 7.3. *Assume $\gamma > 0$, $\beta > 0$, $\alpha > 0$, $\sigma N = 4$ and $N \geq 5$. Then for any $c \in (0, c_N^*)$, any minimizing sequence of $m(c)$ is precompact in $H^2(\mathbb{R}^N)$ up to translations. In particular, $m(c)$ is achieved. In addition, if $u \in S(c)$ is a minimizer of $m(c)$, the associated Lagrange multiplier $\lambda \in \mathbb{R}$ satisfy $\lambda > \frac{\beta^2}{4\gamma}$.*

Proof. We know, from Lemma 7.2, that E is coercive on $S(c)$ for any $c \in (0, c_N^*)$. Thus, the arguments developed in the proof of Theorem 1.2 remain valid. Note also that, when $N \geq 5$, we have $\sigma < 1$ and it guarantees that the minimizing sequences do not vanish. □

Remark 7.3. It should be possible, when $N \leq 4$, to derive a lower bound $\tilde{c}_N > 0$ such that, for any $c \in (\tilde{c}_N, c_N^*)$, the conclusions of Theorem 7.3 holds. We refer to [26] for elements in that direction.

Turning now to the mass supercritical case $\sigma N > 4$, one see directly by considering (2.6), for an arbitrary $u \in S(c)$, and letting $s \to +\infty$, that $m(c) = -\infty$ for any $c > 0$. However, by taking $c > 0$ sufficiently small, it is possible to explicit a local minima structure, More precisely, setting

$$A(R) := \{u \in S(c) : \|\Delta u\|_2^2 \leq R\}$$

one can prove, for a $R > 0$ suitably chosen, that

$$\inf_{u \in S(c)\cap A(R)} E(u) < \inf_{u \in S(c)\cap \partial A(R)} E(u).$$

See [27] in that direction. The presence of such geometry opens the possibility to search for a critical point as a local minima. Following the arguments developed in the proof of Theorem 1.2 it should be the case if $\sigma \in (\frac{4}{N}, 1)$.

7.4. **Bifurcation from the infimum of the essential spectrum.**

First observe that, by Theorem 1.3, iii), we know that, when the minimizers for $m(c)$ exist, they converge to 0 in the $H^2(\mathbb{R}^N)$ norm as $c \to 0$. Also, by Theorem 1.3, ii), we know that the associated Lagrange multipliers converge to $\beta^2/4\gamma$. Hence, in view of Lemma 3.1 and Remark 3.1, we can speak of a bifurcation phenomenon from the bottom of the essential spectrum of the operator $u \mapsto \gamma\Delta^2 u + \beta\Delta u$.

In [5, Theorem 1.2], see also [9], the authors show that when $0 < \sigma N < 4^*$ the equation

$$\gamma\Delta^2 u + \beta\Delta u + \lambda u = |u|^{2\sigma}u, \tag{7.5}$$

admits a ground state solution $u_\lambda \in H^2(\mathbb{R}^N)$ whenever $\beta < 2\sqrt{\gamma\lambda}$, namely if $\lambda > \frac{\beta^2}{4\gamma}$. By a ground state it is intend here a least energy solution for the free functional

$$\mathbb{E}(u) = \frac{\gamma}{2}\|\Delta u\|_2^2 - \frac{\beta}{2}\|\nabla u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2}.$$

Note also that, when $\lambda < \frac{\beta^2}{4\gamma}$, it is expected that (7.5) has no solutions in $H^2(\mathbb{R}^N)$, see [8] in that direction. Worth of interest, in our opinion, would be to investigate if, and under which conditions on $\sigma \in (0, 4^*/N)$, the ground states solutions u_λ to (7.5) satisfy $u_\lambda \to 0$ in $H^2(\mathbb{R}^N)$ as $\lambda \to \beta^2/4\gamma$ from the right, thus presenting a bifurcation phenomenon.

This kind of questions were first addressed in the 80's by C. A. Stuart [34] for equations whose model is given by

$$-\Delta u - \lambda u = |u|^{2\sigma} u, \quad u \in H^1(\mathbb{R}^N). \tag{7.6}$$

Here the bottom of the essential spectrum is $\lambda = 0$. For (7.6), the existence of a sequence of solutions $(u_n, \lambda_n) \subset H^1(\mathbb{R}^N) \times (0, +\infty)$ with $u_n \to 0$ in $H^1(\mathbb{R}^N)$ and $\lambda_n \to 0$ was established under the condition that $0 < \sigma N < 2$. Note that this condition is somehow optimal since, for (7.6), it corresponds to the range for which the associated functional is coercive and no local minima structure is present if $\sigma N > 2$. C. A. Stuart analysis relies on the control of the ground state level by the use of suitable testing functions. We conjecture that such bifurcation phenomenon will take place for (7.5) not only when $\sigma N < 4$, under the the conditions that guarantee the existence of a global minimizer for $m(c)$ when $c > 0$ is small, but also when $\sigma N > 4$ at least when $\sigma \in (\frac{4}{N}, 1)$. In this second case our conjecture is supported by what has been observed in [1].

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