The inverse scattering method for the Davey-Stewartson equation

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1.ISM was introduced by M.Kruskal and others in 1967 [1] for solving evolutionary equations of type $u_t = K[u]$, where $K[u]$- some non-linear operator acting on $u(x,t)$.
2.ISM appeared as non-linear analogue of Fourier Methods for PDE's.
3.The First soliton was observed in 1834 by John Scott Russell. He described it as "a big solitary wave".
4.Using ISM one can find all soliton solutions of the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx} \tag{1}$$

Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

Fourier method for PDE's

Let us consider linearized KdV

$$u_t + u_{xxx} = 0, x \in \mathbb{R} \tag{2}$$

and following Cauchy problem

$$u(x,0) = u_0(x), \text{where } u_0(x) \in L_2(\mathbb{R})$$

After applying Fourier transformation to (2), we get

$$\tilde{u}_t - i\lambda^3 \tilde{u} = 0 \tag{3}$$

where λ is a variable in Fourier space.
General solution of (3) is

$$\tilde{u}(\lambda, t) = C(\lambda) e^{-i\lambda^3 t}$$

where $C(\lambda)$ have to be founded from the initial conditions

$$C(\lambda) = \tilde{u}(\lambda, 0) = \int\limits_{-\infty}^{+\infty} e^{i\lambda x} u(x,0)\,\mathrm{d}x = \mathcal{F}(u_0)$$

To get the solution of (2) we have to apply the Inverse Fourier Transform, then we get

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - i\lambda^3 t} C(\lambda) \, \mathrm{d}\lambda$$

The whole procedure we can be written as a following diagram:

$$\begin{tikzcd}
u(x,0) \arrow[r, "\mathcal{F}"] \arrow[d] & \tilde{u}(\lambda,0) \arrow[d] \\
\text{u(x,t)} & \tilde{u}(\lambda,t) = \tilde{u}(\lambda,0)e^{-i\lambda^3 t} \arrow[l, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

For KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad \text{where } u = u(x,t)$$

the analogue of FT will be the Scattering transformation represented by Schrödinger equation

$$\frac{d^2\Phi}{dx^2} + (\lambda + u)\Phi = 0$$

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\qquad\mathcal{S}\qquad} & \{a(k,0), b(k,0)\} \\ \downarrow & & \downarrow \\ u(x,t) & \xleftarrow{\mathcal{S}^{-1}} & \{a(k,t) = a(k,0), b(k,t) = b(k,0)e^{8ik^3t}\} \end{array}$$

where $\lambda = k^2$. Remark: we will consider that
$\Phi \in L(\mathbb{R}), \lambda \in \mathbb{R}, \int_{-\infty}^{+\infty}(1+|x|)|u(x)|\,\mathrm{d}x < \infty$

lost functions

The eigenvalue problem for Schrödinger operator has formal solutions with following asymptotic properties.

$$\psi_1(x,k) = e^{-ikx} + o(1), \quad \psi_2(x,k) = e^{ikx} + o(1), \quad \text{when} \quad x \to +\infty$$
$$\phi_1(x,k) = e^{-ikx} + o(1), \quad \phi_2(x,k) = e^{ikx} + o(1), \quad \text{when} \quad x \to -\infty$$

Scattering data

The idea is to look at the transformation matrix $T(k)$ between two basis of solutions of Schrödinger equation $\{\psi_1, \psi_2\}$ and $\{\phi_1, \phi_2\}$. It can be denoted

$$T(k) = \begin{pmatrix} a(k) & b(k) \\ c(k) & d(k) \end{pmatrix}$$

Proposition 1

For all real $k \neq 0$ the transformation matrix $T(k)$ is pseudo-unitary matrix, i.e.

$$T(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}$$

and $DetT(k) = |a(k)|^2 + |b(k)|^2$=1.

Remark. Matrix $T(k)$ is fully determined by coefficients $a(k)$ and $b(k)$. Only coefficients $r(k) = \frac{b(k)}{a(k)}$ and $t(k) = \frac{1}{a(k)}$ have physical meaning they are called the reflection coefficient and the the transmission coefficient respectively.

Properties of the scattering data

Theorem 2

1.The complex valued function $a(k)$ *defined for all real* $k \neq 0$ *admits it's analytic continuation on the upper half plane* $Imk > 0$, *has only simple zeros and asymptotic behavior*

$$a(k) = 1 + \left(\frac{1}{k}\right), \quad \textit{when} \quad |k| \to \infty.$$

2.For $Imk > 0 \quad a(k) = 0$ *if and only if, when* $\lambda = k^2$ *is an eigenvalue of the Shrödinger operator.*

Time evolution of the scattering data

Theorem 3

Evolution of isospectral deformation of Schrödinger equation corresponding to continuous spectrum is described by Gardner-Green-Kruskal-Miura equations:

$$\dot{a}(k,t) = 0, \quad \dot{b}(k,t) = 8ik^3 b(k,t)$$

.

Inverse problem

Theorem 4

Let $K(x,y)$ *is a solution of integral equation*

$$K(x,y)+F(x+y)=\int\limits_x^{+\infty}K(x,z)F(y+z)\,\mathrm{d}z=0,$$

then potential $u(x)$ *can be found by*

$$u(x)=-2\frac{d}{dx}K(x,x)$$

where $y\geq x$ *and* $F(x)=\sum_{n=1}^{N}\frac{b_n e^{-\kappa_n x}}{ia\prime(i\kappa_n)}+\int_{-\infty}^{+\infty}r(k)e^{ikx}\,\mathrm{d}k$

Reflectionless potentials case

Definition 5

We say that potential u of Schrödinger operator called reflectionless if corresponding reflection coefficient $r(k)$ is identically equal to zero.

It means that $b(k)$ identically equal to zero and thus $|a(k)| \equiv 1$ for all $k \in \mathbb{R}$. Let denote eigenvalues again as $i\kappa_n$ where $n = 1, 2, ..., N$ then $a(k) = \sum_{n=1}^{N} \frac{k - i\kappa_n}{k + i\kappa_n}$.then the Kernel in Gelfand-Levitan-Marchenko equation is a sum of exponents.

$$F(x) = \sum_{n=1}^{N} \beta_n e^{-\kappa_n x}, \text{ where } \beta_n = \frac{b_n}{ia'(i\kappa_n)}$$

Solving G-L-M equation we get

$$K(x, y) = \sum_{n=1}^{N} K_n(x) e^{-\kappa_n y},$$

where $K_n(x) = \frac{det A^{(n)}(x)}{det A(x)}$ and matrix $A(x)$ has components $A_{nm}(x) = \delta_{nm} + \frac{\beta_n e^{-(\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m}$

Multisoliton potentials

Finally, we find that

$$K(x,x) = \frac{d}{dx}\ln\det A(x)$$

and using the inverse formula we get explicit expression for the potential

$$u(x) = -2\frac{d^2}{dx^2}\ln\det A(x)$$

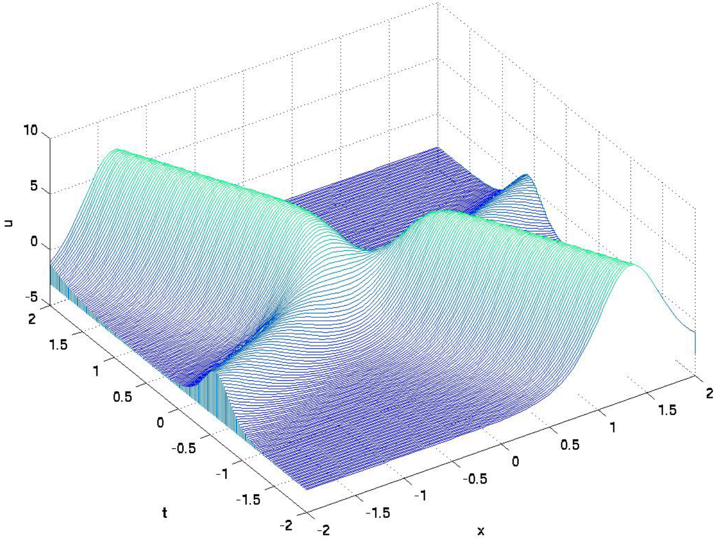
Time evolution for matrix $A(x,t)$ is described by

$$A_{nm}(x) = \delta_{nm} + \frac{\beta_n e^{-(\kappa_n+\kappa_m)x+8\kappa_n^3 t}}{\kappa_n+\kappa_m}$$

where $\beta_n = \beta_n(0) = const$. Then corresponding solution of the KdV equation can be found by

$$u(x,t) = -2\frac{d^2}{dx^2}\ln\det A(x,t)$$

KdV two-soliton solution



2+1 case: Davey-Stewartson (DS) II system

For the Davey-Stewartson II system

$$\begin{aligned} q_t &= 2iq_{xy} - 4i\phi q, \\ \phi_{xx} + \phi_{yy} &= 4|q|^2_{xy}, \\ q(x,y,0) &= q_0(x,y) \end{aligned}$$

the inverse scattering transform associated with the elliptic system is

$$\frac{\partial \Psi}{\partial \bar{z}} = Q\bar{\Psi}$$

where

$$Q = \begin{pmatrix} 0 & q \\ \pm q & 0 \end{pmatrix}$$

and $\psi(z,k) = e^{i\bar{k}z/2}\mu(z,k)$, also we require that $\lim_{|z|\to\infty} \mu(z,k) = I$

The scattering data for DS II

Then we define scattering data as matrix

$$T(k) = \begin{pmatrix} 0 & T_{12}(k) \\ T_{21}(k) & 0 \end{pmatrix}$$

by formulas

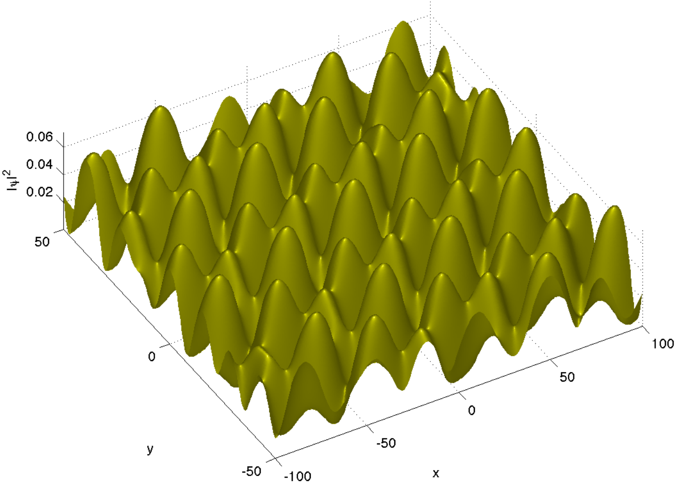
$$T_{12}(k) = \lim_{|z|\to\infty} = \frac{iz}{2}\mu_{12}(z,k)$$

$$T_{21}(k) = \lim_{|z|\to\infty} = \frac{iz}{2}\mu_{21}(z,k)$$

Schematically, these steps can be represented by the following diagram:

$$\begin{array}{ccc} Q(0) & \xrightarrow{\qquad\qquad\mathcal{S}\qquad\qquad} & T(0) \\ \downarrow & & \downarrow \\ Q(t) & \xleftarrow{\mathcal{S}^{-1}} & T(t), \text{ the evolution equation: } \partial_t T = -\frac{k^2-\bar{k^2}}{2}T \end{array}$$

DS II solution



References

1. Method for Solving the Korteweg-deVries Equation. Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura Phys. Rev. Lett. 19, 1095 – Published 6 November 1967
2. Integrable systems, lecture notes. O.Mokhov, S. Smirnov. MSU 2015