Estimation and validation of weak FARIMA models

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Introduction

Some definitions

- Long memory processes.
- Weak FARIMA models.

Asymptotic results of the least-squares estimator (LSE)

- Strong consistency and asymptotic normality.
- Asymptotic variance matrix estimation and some simulations.

Diagnostic checking in weak FARIMA models

- Asymptotic joint distribution of the LSE and the noise empirical autocovariances.
- Asymptotic distribution of the residual autocorrelations.
- Limiting distribution of the test statistics.
- Numerical illustrations.

Defining long memory

Let $X = (X_t)_{t\in\mathbb{Z}}$ be a second order stationary stochastic process, and let $\gamma_X(.)$ be its autocovariance function.

Definition A

X is a long memory process if :

$$\sum_{h=-\infty}^{+\infty} |\gamma_X(h)| = +\infty.$$

Definition B

X is a long memory process if :

$$\gamma_X(h) \sim h^{2d-1}\ell(h), \text{ as } h \to +\infty,$$

where d is the so-called long-memory parameter and $\ell(.)$ is a slowly varying function.

FARIMA processes

Let $X = (X_t)_{t\in\mathbb{Z}}$ be a second order stationary process.

Definition 1

X is called a weak FARIMA(p, d_0, q) process if there exists $0 < d_0 < 1/2$, $a_1, ..., a_p$, $b_1, ..., b_q \in \mathbb{R}$ such that the polynomials $a(z) = 1 + \sum_{i=1}^{p} a_i z^i$ and $b(z) = 1 + \sum_{i=1}^{q} b_i z^i$ have all their roots outside of the unit disk with no common factors, and $(\epsilon_t)_{t\in\mathbb{Z}}$ a sequence of uncorrelated variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with zero mean and common variance $\sigma_\epsilon^2 > 0$ such that, for all $t \in \mathbb{Z}$,

$$a(L)(1 - L)^{d_0} X_t = b(L)\epsilon_t, \tag{1}$$

where L is the back-shift operator.

The fractional difference operator $(1 - L)^{d_0}$ is given by :

$$(1-L)^{d_0} = \sum_{j=0}^{+\infty} \alpha_j(d_0) L^j, \text{ where } \alpha_j(d_0) = \frac{d_0(d_0 - 1)\cdots(d_0 - j + 1)}{j!}(-1)^j.$$

Least-squares estimator (LSE)

Framework : Let $\tilde{\Theta}$ be the compact space

$$\tilde{\Theta} := \{\tilde{\theta} = (\theta_1, \theta_2, \ldots, \theta_{p+q})'; \; a_{\tilde{\theta}}(z) = 1 + \theta_1 z + \cdots + \theta_p z^p$$

and $b_{\tilde{\theta}}(z) = 1 + \theta_{p+1} z + \cdots + \theta_{p+q} z^q$ have all their roots outside the unit disk and have no common zero$\}$.

Denote by Θ the cartesian product $\tilde{\Theta} \times [d_1, d_2]$, where $[d_1, d_2] \subset]0, 1/2[$ and containing d_0.

The parameter $\theta_0 := (a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, d_0)'$ belongs to the parameter space Θ.

For all $\theta = \left(\tilde{\theta}', d\right)' \in \Theta$, let $(\epsilon_t(\theta))_{t \in \mathbb{Z}}$ be the second order stationary process defined as the solution of

$$\epsilon_t(\theta) = \sum_{j \geq 0} \alpha_j(d) X_{t-j} + \sum_{i=1}^{p} \theta_i \sum_{j \geq 0} \alpha_j(d) X_{t-i-j} - \sum_{j=1}^{q} \theta_{p+j} \epsilon_{t-j}(\theta).$$

Least-squares estimator (LSE)

Given a realization of length n, $X_1, X_2, ..., X_n$, $\epsilon_t(\theta)$ can be approximated, for $0 < t \leq n$, by $\tilde{\epsilon}_t(\theta)$ defined recursively by

$$\tilde{\epsilon}_t(\theta) = \sum_{j=0}^{t-1} \alpha_j(d) X_{t-j} + \sum_{i=1}^{p} \theta_i \sum_{j=0}^{t-i-1} \alpha_j(d) X_{t-i-j} - \sum_{j=1}^{q} \theta_{p+j} \tilde{\epsilon}_{t-j}(\theta),$$

with $\tilde{\epsilon}_t(\theta) = X_t = 0$ if $t \leq 0$.

The random variable $\widehat{\theta}_n$ is called least-squares estimator if it satisfies, almost surely,

$$\widehat{\theta}_n = \underset{\theta \in \Theta}{\operatorname{argmin}}\, Q_n(\theta), \text{ where } Q_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} \tilde{\epsilon}_t^2(\theta).$$

Strong consistency

Our first two main results concern the strong consistency and the asymptotic normality of the least-squares estimator (LSE) of the weak FARIMA model parameter

$$\theta_0 = (a_1, \ldots, a_p, b_1, \ldots, b_q, d_0)' .$$

The strong consistency of the LSE is proven under the following assumption :

A1. The process $(\epsilon_t)_{t\in\mathbb{Z}}$ is strictly stationary and ergodic.

Theorem I (strong consistency)

Assume that $(\epsilon_t)_{t\in\mathbb{Z}}$ satisfies (1) and belonging to $\mathbb{L}^2$. Let $(\widehat{\theta}_n)_{n\in\mathbb{N}^*}$ be a sequence of least-squares estimators. We have, under Assumption **A1**,

$$\widehat{\theta}_n \xrightarrow[n\to+\infty]{a.s.} \theta_0.$$

Asymptotic normality

Suppose that $(\epsilon_t)_{t\in\mathbb{Z}}$ satisfies the following two additional conditions :

A2. We have $\mathbb{E}|\epsilon_t|^{4+2\nu} < \infty$ and $\sum_{h=0}^{\infty} \{\alpha_\epsilon(h)\}^{\frac{\nu}{2+\nu}} < \infty$ for some $\nu > 0$.

A3. Assume $\sum_{i,j,k\in\mathbb{Z}} |\mathrm{cum}(\epsilon_0, \epsilon_i, \epsilon_j, \epsilon_k)| < \infty$.

Theorem II (asymptotic normality)

Under the hypotheses of Theorem I and Assumptions **A2** and **A3**, the sequence of random variables

$$\left(\sqrt{n}\left(\widehat{\theta}_n - \theta_0\right)\right)_{n\in\mathbb{N}^*}$$

has a limiting centred normal distribution with covariance matrix $\Omega := J^{-1}IJ^{-1}$, where

$$I = \lim_{n\to\infty} \mathrm{Var}\left\{\sqrt{n}\frac{\partial}{\partial\theta}Q_n(\theta_0)\right\} \text{ and } J = \lim_{n\to\infty}\left\{\frac{\partial^2}{\partial\theta\partial\theta'}Q_n(\theta_0)\right\} \text{ a.s.}$$

Asymptotic variance matrix estimation

It would be necessary to estimate the variance matrix $\Omega = J^{-1}IJ^{-1}$ to obtain confidence intervals or to test significance of FARIMA model coefficients.

- The matrix J can easily be estimated empirically by :

$$\widehat{J} = \frac{2}{n}\sum_{t=1}^{n}\left\{\frac{\partial\tilde{\epsilon}_t(\theta)}{\partial\theta}\frac{\partial\tilde{\epsilon}_t(\theta)}{\partial\theta'}\right\}_{\theta=\widehat{\theta}_n}.$$

- The matrix I can be rewritten in the form :

$$I = \lim_{n\to\infty} Var\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\Upsilon_t\right\}, \text{ where } \Upsilon_t = 2\left\{\epsilon_t(\theta)\frac{\partial\epsilon_t(\theta)}{\partial\theta}\right\}_{\theta=\theta_0}.$$

We use the parametric estimation of the spectral density introduced by Berk [1974]. Let $\widehat{\mathbf{\Phi}}_r(z) = I_{p+q+1} + \sum_{i=1}^{r}\widehat{\Phi}_{r,i}z^i$, where $\widehat{\Phi}_{r,1}, \dots, \widehat{\Phi}_{r,r}$ be the least-squares regression coefficients of $\widehat{\Upsilon}_t$ on $\widehat{\Upsilon}_{t-1}, \dots, \widehat{\Upsilon}_{t-r}$ and $\widehat{\Sigma}_{\widehat{u}_r}$ be the empirical variance of these residues.

Asymptotic result

The third main result is given by :

Theorem III (estimating the asymptotic variance matrix I)

In addition to the assumptions of Theorem II, assume that the process $(\Upsilon_t)_t$ admits an AR(∞) representation of the form $\Phi(L)\Upsilon_t := \Upsilon_t - \sum_{k=1}^{\infty} \Phi_k \Upsilon_{t-k} = u_t$ in which the roots of $\det(\Phi(z)) = 0$ are outside the unit disk, $\|\Phi_k\| = \mathrm{o}(k^{-2})$, and $\Sigma_u = \mathrm{Var}(u_t)$ is non-singular. Moreover we assume that $\mathbb{E}\left[|\epsilon_t|^8\right] < \infty$ and that $\sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_7=-\infty}^{+\infty} |\mathrm{cum}(\epsilon_0, \epsilon_{k_1}, ..., \epsilon_{k_7})| < \infty$. Then, the spectral estimator of I

$$\hat{I}_n^{\mathrm{SP}} := \hat{\Phi}_r^{-1}(1)\hat{\Sigma}_{u_r}\hat{\Phi}_r^{\prime -1}(1) \to I$$

in probability when $r = r(n) \to \infty$ and $r^3/n \to 0$ as $n \to \infty$.

Some simulations

We first study numerically the behavior of the LSE for strong and weak FARIMA models of the form

$$(1-L)^d (X_t + aX_{t-1}) = \epsilon_t + b\epsilon_{t-1}, \tag{2}$$

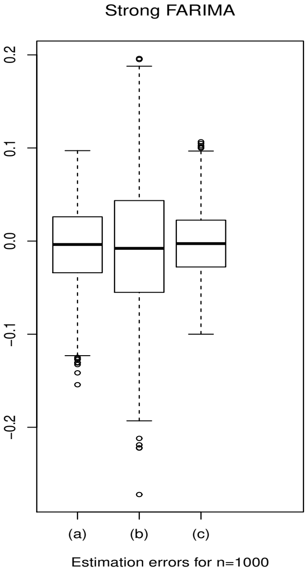
where $(a, b, d) = (0.7, 0.2, 0.4)$.

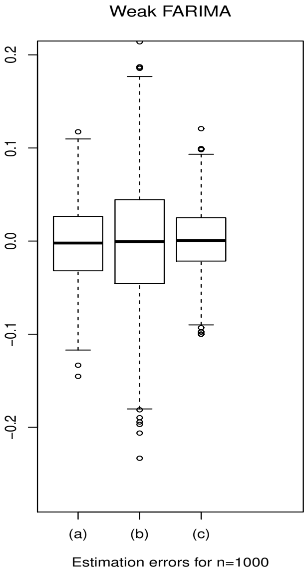
- The process $(\epsilon_t)_t$ is an iid centered Gaussian process with common variance 1 in the strong case.
- In the weak case,

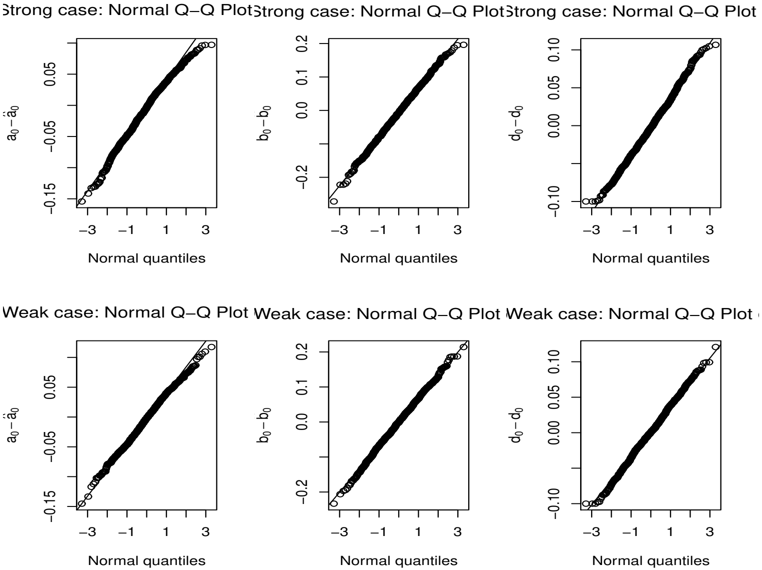
$$\epsilon_t = \frac{\eta_t}{|\eta_{t-1}| + 1}, \quad \text{for all } t \in \mathbb{Z}, \tag{3}$$

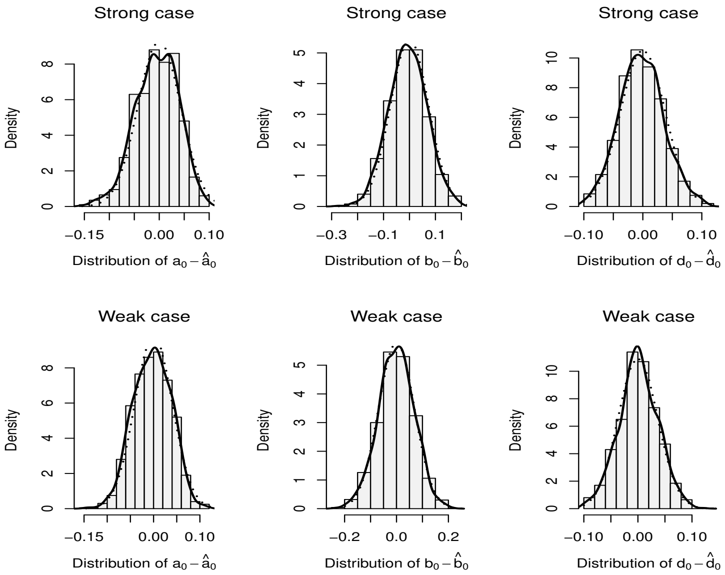
  with $(\eta_t)_t$ is an iid centered Gaussian process with variance 1.

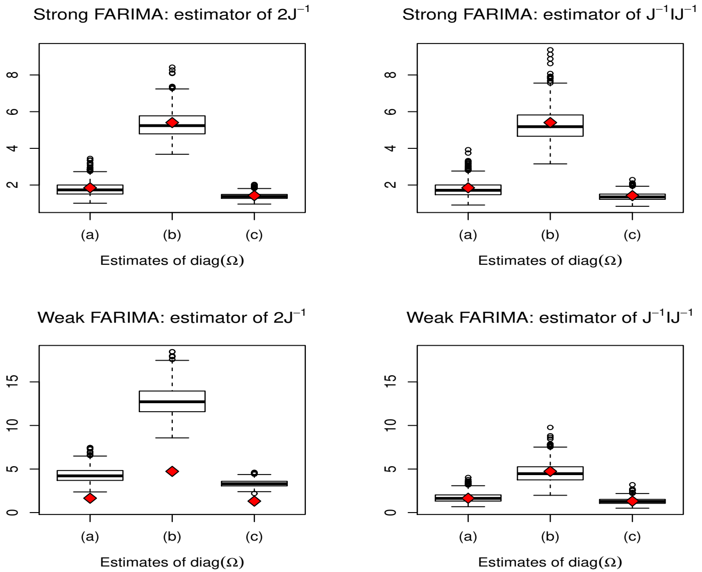
We simulated $N = 1,000$ independent trajectories of size $n = 1,000$ of Model (2), first with the strong Gaussian noise, second with the weak noise (3).


Strong FARIMA
0.2
0.1
0.0
−0.1
−0.2
(a)
(b)
(c)
Estimation errors for n=1000


Weak FARIMA
0.2
0.1
0.0
−0.1
−0.2
(a)
(b)
(c)
Estimation errors for n=1000


Strong case: Normal Q–Q Plot
Strong case: Normal Q–Q Plot
Strong case: Normal Q–Q Plot
Weak case: Normal Q–Q Plot
Weak case: Normal Q–Q Plot
Weak case: Normal Q–Q Plot
$a_0 - \hat{a}_0$
$b_0 - \hat{b}_0$
$d_0 - \hat{d}_0$
Normal quantiles
−3 −1 1 3
−0.15 −0.05 0.05
−0.2 0.0 0.1 0.2
−0.10 0.00 0.05 0.10


Strong case
Density
Distribution of $a_0 - \hat{a}_0$
Strong case
Density
Distribution of $b_0 - \hat{b}_0$
Strong case
Density
Distribution of $d_0 - \hat{d}_0$
Weak case
Density
Distribution of $a_0 - \hat{a}_0$
Weak case
Density
Distribution of $b_0 - \hat{b}_0$
Weak case
Density
Distribution of $d_0 - \hat{d}_0$


Strong FARIMA: estimator of $2J^{-1}$
Strong FARIMA: estimator of $J^{-1}IJ^{-1}$
Weak FARIMA: estimator of $2J^{-1}$
Weak FARIMA: estimator of $J^{-1}IJ^{-1}$
(a) (b) (c)
Estimates of diag(Ω)
2 4 6 8
0 5 10 15

Let, for $t \geq 1$, $\hat{e}_t = \tilde{\epsilon}_t(\hat{\theta}_n)$ be the least-squares residuals. Using the expression of $\tilde{\epsilon}_t(.)$ we have $\hat{e}_t = 0$ for $t \leq 0$ and $t > n$. By (1), it holds that

$$\hat{e}_t = \sum_{j=0}^{t-1} \alpha_j(\hat{d}_n)\hat{X}_{t-j} + \sum_{i=1}^{p} \hat{\theta}_{n,i} \sum_{j=0}^{t-i-1} \alpha_j(\hat{d}_n)\hat{X}_{t-i-j} - \sum_{j=p+1}^{p+q} \hat{\theta}_{n,j}\hat{e}_{t-j},$$

for $t = 1, ..., n$, with $\hat{X}_t = 0$ for $t \leq 0$ and $\hat{X}_t = X_t$ for $t \geq 1$.
Let, for $h \geq 0$,

$$\gamma(h) = \frac{1}{n} \sum_{t=h+1}^{n} \epsilon_t \epsilon_{t-h} \quad \text{and} \quad \rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

denote the white noise "empirical" autocovariances and autocorrelations.
The residual autocovariances and autocorrelations are defined by

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=h+1}^{n} \hat{e}_t \hat{e}_{t-h} \quad \text{and} \quad \hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$

For a fixed integer $m \geq 1$, let

$$\gamma_m = (\gamma(1), ..., \gamma(m))^{'} \quad \text{and} \quad \hat{\gamma}_m = (\hat{\gamma}(1), ..., \hat{\gamma}(m))^{'}.$$

Denote also by

$$\hat{\rho}_m = (\hat{\rho}(1), ..., \hat{\rho}(m))^{'}$$

the first m sample autocorrelations.
Based on the residual empirical autocorrelation $\hat{\rho}(h)$, Box-Pierce and Ljung-Box have proposed the following respective statistics for the validation of strong univariate ARMA models :

$$Q_m^{BP} = n \sum_{h=1}^{m} \hat{\rho}^2(h) \quad \text{and} \quad Q_m^{LB} = n(n+2) \sum_{h=1}^{m} \frac{\hat{\rho}^2(h)}{n-h}.$$

Test hypotheses

$\mathbf{(H0)}$: $(X_t)_{t\in\mathbb{Z}}$ satisfies a FARIMA(p, d_0, q) representation ;
against the alternative
$\mathbf{(H1)}$: $(X_t)_{t\in\mathbb{Z}}$ does not admit a FARIMA representation or admits a FARIMA$(p^{'}, d_0, q^{'})$ representation with $p^{'} > p$ or $q^{'} > q$.

Joint distribution of $\hat{\theta}_n$ and the noise empirical autocovariances

Under the assumptions of Theorem II, the random vector

$$\sqrt{n}\left(\left(\hat{\theta}_n - \theta_0\right)^{'}, \gamma_m^{'}\right)^{'}$$

has a limiting centred normal distribution with covariance matrix Ξ, where

$$\Xi = \begin{pmatrix} \Sigma_{\hat{\theta}} & \Sigma_{\hat{\theta},\gamma_m} \\ \Sigma^{'}_{\hat{\theta},\gamma_m} & \Sigma_{\gamma_m} \end{pmatrix} = \sum_{h=-\infty}^{\infty} \mathbb{E}\left[g_t g^{'}_{t-h}\right],$$

with

$$g_t = \begin{pmatrix} g_{1t} \\ g_{2t} \end{pmatrix} = \begin{pmatrix} -2J^{-1}\epsilon_t \frac{\partial}{\partial\theta}\epsilon_t(\theta_0) \\ \left(\epsilon_{t-1}, \dots, \epsilon_{t-m}\right)^{'}\epsilon_t \end{pmatrix}.$$

Asymptotic distribution of the residual autocorrelations

Let Ψ_m be the $m \times (p+q+1)$ matrix defined by

$$\Psi_m = \mathbb{E}\left\{ \left(\epsilon_{t-1}, \dots, \epsilon_{t-m}\right)' \frac{\partial \epsilon_t(\theta_0)}{\partial \theta'} \right\}.$$

The following proposition provides the limit distribution of the residual autocovariances and autocorrelations of weak FARIMA models.

Proposition

Under the assumptions of Theorem II, we have

$$\sqrt{n}\hat{\gamma}_m \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}\left(0, \Sigma_{\hat{\gamma}_m}\right) \quad \text{and} \quad \sqrt{n}\hat{\rho}_m \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}\left(0, \Sigma_{\hat{\rho}_m}\right),$$

where

$$\Sigma_{\hat{\gamma}_m} = \Sigma_{\gamma_m} + \Psi_m \Sigma_{\hat{\theta}} \Psi_m^{'} + \Psi_m \Sigma_{\hat{\theta},\gamma_m} + \Sigma_{\hat{\theta},\gamma_m}^{'} \Psi_m^{'} \quad \text{and} \quad \Sigma_{\hat{\rho}_m} = \frac{1}{\sigma_\epsilon^4} \Sigma_{\hat{\gamma}_m}.$$

$\rightarrow$ The matrices $\Sigma_{\hat{\gamma}_m}$ and $\Sigma_{\hat{\rho}_m}$ depend on the unknown matrices Ξ, Ψ_m and the scalar σ_ϵ.

$\rightarrow$ The matrix Ψ_m and the noise variance σ_ϵ^2 can be estimated by its empirical counterpart :

$$\hat{\Psi}_m = \frac{1}{n}\sum_{t=1}^{n}\left\{(\hat{e}_{t-1},...,\hat{e}_{t-m})'\frac{\partial \hat{e}_t}{\partial \theta'}\right\} \quad \text{and} \quad \hat{\sigma}_\epsilon^2 = \frac{1}{n}\sum_{t=1}^{n}\hat{e}_t^2.$$

$\rightarrow$ By interpreting $(2\pi)^{-1}\Xi$ as the spectral density of the stationary process $(g_t)_t$ evaluated at frequency 0, we use Berk's approach to estimate the spectral density of $(g_t)_t$ by fitting a parametric autoregressive model. This estimation technique is based on the following expression

$$\Xi = \Delta^{-1}(1)\Sigma_v\Delta^{'-1}(1)$$

when $(g_t)_t$ satisfies an $\mathrm{AR}(\infty)$ representation of the form

$$\Delta(L)g_t := g_t - \sum_{k\geq 1}\Delta_k g_{t-k} = v_t, \tag{4}$$

where $(v_t)_t$ is a $(p+q+1+m)$-variate weak white noise with variance matrix Σ_v.

Let $\hat{g}_t$ be the vector obtained by replacing ϵ_t by $\hat{e}_t$ in g_t. Let $\hat{\Delta}_r(z) = I_{p+q+1+m} - \sum_{k=1}^{r} \hat{\Delta}_{r,k} z^k$, where $\hat{\Delta}_{r,1}, \ldots, \hat{\Delta}_{r,r}$ denote the coefficients of the least squares regression of $\hat{g}_t$ on $\hat{g}_{t-1}, \ldots, \hat{g}_{t-r}$. Let $\hat{v}_{r,t}$ be the residuals of this regression, and let $\hat{\Sigma}_{\hat{v}_r}$ be the empirical variance of $\hat{v}_{r,1}, \ldots, \hat{v}_{r,n}$.

Theorem IV (estimating the asymptotic variance matrix Ξ)

In addition to the assumptions of Theorem II, assume that the process $(g_t)_t$ admits the AR(∞) representation (4) in which the roots of $\det(\Delta(z)) = 0$ are outside the unit disk, $\|\Delta_k\| = \mathrm{o}(k^{-2})$, and $\Sigma_v = \mathrm{Var}(v_t)$ is non-singular. Moreover we assume that $\mathbb{E}\left[|\epsilon_t|^8\right] < \infty$ and that $\sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_7=-\infty}^{+\infty} |\mathrm{cum}(\epsilon_0, \epsilon_{k_1}, \ldots, \epsilon_{k_7})| < \infty$. Then, the spectral estimator of Ξ

$$\hat{\Xi}_n^{\mathrm{SP}} := \hat{\Delta}_r^{-1}(1) \hat{\Sigma}_{\hat{v}_r} \hat{\Delta}_r'^{-1}(1) \to \Xi$$

in probability when $r = r(n) \to \infty$ and $r^3/n \to 0$ as $n \to \infty$.

The exact limiting distribution of Box-Pierce and Ljung-Box statistics is given in the following theorem :

Theorem V (exact asymptotic distribution of the standard portmanteau statistics)

Under the assumptions of Theorem II and **(H0)**, the statistics Q_m^{BP} and Q_m^{LB} converge in distribution, as $n \to \infty$, to

$$Z_m(\xi_m) = \sum_{k=1}^{m} \xi_{k,m} Z_k^2,$$

where $\xi_m = (\xi_{1,m}, ..., \xi_{m,m})'$ is the vector of the eigenvalues of the matrix $\Sigma_{\hat{\rho}_m} = \sigma_\epsilon^{-4} \Sigma_{\hat{\gamma}_m}$ and $Z_1, ..., Z_m$ are independent and identically distributed (i.i.d.) random variables of the same distribution $\mathcal{N}(0, 1)$.

→ Let $\hat{\Sigma}_{\hat{\rho}_m}$ be the matrix obtained by replacing Ξ by $\hat{\Xi}$ and σ_ϵ^2 by $\hat{\sigma}_\epsilon^2$ in $\Sigma_{\hat{\rho}_m}$.

→ Denote by $\hat{\xi}_m = (\hat{\xi}_{1,m}, ..., \hat{\xi}_{m,m})'$ the vector of the eigenvalues of $\hat{\Sigma}_{\hat{\rho}_m}$.

→ At the asymptotic level α, the **LB** test (respectively the **BP** test) consists in rejecting the null hypothesis of the weak FARIMA(p, d_0, q) model (the adequacy of the weak FARIMA(p, d_0, q) model) when

$$Q_m^{LB} > S_m(1-\alpha) \quad \text{(resp.} \quad Q_m^{BP} > S_m(1-\alpha)),$$

where $S_m(1-\alpha)$ is such that $\mathbb{P}(Z_m(\hat{\xi}_m) > S_m(1-\alpha)) = \alpha$.

→ The proposed modified versions of the **BP** and **LB** statistics are more difficult to implement because their critical values have to be computed from the data.

Numerical illustrations

Table – Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of FARIMA$(1, d, 1)$ model with independent noise. The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1,000$. ($\mathrm{ar} = 0.9$ and $\mathrm{ma} = 0.2$)

d_0	Length n	Lag m	LB_{w}	BP_{w}	LB_{s}	BP_{s}
		1	7.4	7.4	n.a.	n.a.
		2	5.5	5.5	n.a.	n.a.
0.20	$n = 1,000$	3	4.7	4.7	n.a.	n.a.
		6	4.4	4.3	6.5	6.4
		12	4.6	4.6	5.1	5.1
		15	5.0	4.7	5.7	5.2
		1	5.6	5.6	n.a.	n.a.
		2	5.1	5.2	n.a.	n.a.
0.20	$n = 5,000$	3	5.2	5.2	n.a.	n.a.
		6	4.9	4.9	6.6	6.6
		12	5.0	5.0	5.7	5.6
		15	5.6	5.5	5.9	5.8

Table – Empirical size (in %) of the modified and standard versions of the LB and BP tests in the case of FARIMA$(1, d, 1)$ with an ARCH(1) noise ($\alpha_1 = 0.45$). The nominal asymptotic level of the tests is $\alpha = 5\%$. The number of replications is $N = 1,000$. ($\mathrm{ar} = 0.9$ and $\mathrm{ma} = 0.2$)

d_0	Length n	Lag m	$\mathrm{LB_w}$	$\mathrm{BP_w}$	$\mathrm{LB_s}$	$\mathrm{BP_s}$
		1	6.4	6.4	n.a.	n.a.
		2	5.2	5.2	n.a.	n.a.
0.20	$n = 1,000$	3	3.9	3.9	n.a.	n.a.
		6	3.8	3.7	10.0	9.8
		12	1.9	1.8	7.3	6.7
		15	1.0	0.9	6.4	6.3
		1	6.3	6.3	n.a.	n.a.
		2	4.3	4.3	n.a.	n.a.
0.20	$n = 5,000$	3	5.5	5.4	n.a.	n.a.
		6	3.8	3.8	11.6	11.6
		12	2.3	2.3	8.3	8.2
		15	1.5	1.5	8.5	8.5

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- **Some perspectives :**
  - Estimation and validation of AR models with fractional noise.
  - Generalization to fractional ARMA models.
  - Comparison with weak FARIMA models.

Thank you for your attention