MULTIPLICITY RESULTS IN THE NON-COERCIVE CASE FOR AN ELLIPTIC PROBLEM WITH CRITICAL GROWTH IN THE GRADIENT

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Abstract. We consider the boundary value problem

$$(P_\lambda) \qquad -\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x), \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$ is a bounded domain with smooth boundary. It is assumed that c, h belong to $L^p(\Omega)$ for some $p > N$ with $c \gneqq 0$ as well as $\mu \in L^\infty(\Omega)$ and $\mu \geq \mu_1 > 0$ for some $\mu_1 \in \mathbb{R}$. It is known that when $\lambda \leq 0$, problem (P_λ) has at most one solution. In this paper we study, under various assumptions, the structure of the set of solutions of (P_λ) assuming that $\lambda > 0$. Our study unveils the rich structure of this problem. We show, in particular, that what happen for $\lambda = 0$ influences the set of solutions in all the half-space $]0, +\infty[\times(H_0^1(\Omega) \cap L^\infty(\Omega))$. Most of our results are valid without assuming that h has a sign. If we require h to have a sign, we observe that the set of solutions differs completely for $h \gneqq 0$ and $h \lneqq 0$. We also show when h has a sign that solutions not having this sign may exists. Some uniqueness results of signed solutions are also derived. The paper ends with a list of open problems.

1. Introduction

We consider the boundary value problem

$$(P_\lambda) \qquad -\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x), \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

under the assumption

$$(\mathbf{A}) \qquad \begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 3 \text{ is a bounded domain with } \partial\Omega \text{ of class } C^{1,1}, \\ c \text{ and } h \text{ belong to } L^p(\Omega) \text{ for some } p > N \text{ and satisfy } c \gneqq 0, \\ \mu \in L^\infty(\Omega) \text{ satisfies } 0 < \mu_1 \leq \mu(x) \leq \mu_2. \end{cases}$$

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Depending on the parameter $\lambda \in \mathbb{R}$ we study the existence and multiplicity of solutions of (P_λ). By solutions we mean functions $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v \, dx = \lambda \int_\Omega c(x) uv \, dx + \int_\Omega \mu(x) |\nabla u|^2 v \, dx + \int_\Omega h(x) v \, dx \,,$$

for any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

First observe that, by the change of variable $v = -u$, problem (P_λ) reduces to

$$-\Delta v = \lambda c(x) v - \mu(x) |\nabla v|^2 - h(x), \qquad v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Hence, since we make no assumptions on the sign of h, we actually also consider the case where $|\nabla u|^2$ has a negative coefficient.

The study of quasilinear elliptic equations with a gradient dependence up to the critical growth $|\nabla u|^2$ was essentially initiated by Boccardo, Murat and Puel in the 80's and it has been an active field of research until now. Under the condition $\lambda c(x) \leq -\alpha_0 < 0$ a.e. in Ω for some $\alpha_0 > 0$, which is usually referred to as the *coercive case*, the existence of a unique solution of (P_λ) is guaranteed by assumption (A). This is a special case of the results of [8, 9] for the existence and of [6, 7] for the uniqueness. Let us point out that the requirement to deal with bounded solutions in (P_λ) is essential to the uniqueness results. Indeed if one admits more general solutions, the existence of infinitely many solutions is known in several cases, see for example [1, 2].

The limit case where one just require that $\lambda c(x) \leq 0$ a.e. in Ω is more complex. There had been a lot of contributions [2, 13, 19, 21] when $\lambda = 0$ (or equivalently when $c \equiv 0$) but the general case where $\lambda c \leq 0$ may vanish only on some parts of Ω was left open until the paper [4]. It appears in [4] that under assumption (A) the existence of solutions is not guaranteed, additional conditions are necessary. When $\lambda = 0$ this was already observed in [13]. By [4], the uniqueness itself holds as soon as $\lambda c(x) \leq 0$ a.e. in Ω. See also [5] for a related uniqueness result in a more general frame.

The case $\lambda c \gneqq 0$ remained unexplored until very recently. Following the paper [23] which consider a particular case, Jeanjean and Sirakov [18] study a problem directly connected to (P_λ). In [18, Theorem 2], assuming that μ is a positive constant and h is small (in an appropriate sense) but without sign condition, a $\lambda_0 > 0$ is given under which (P_λ) has two solutions whenever $\lambda \in]0, \lambda_0[$. This result have been complemented in [17] where two solutions are obtained, allowing the function c to change sign but assuming that $h \geq 0$ and that $\max\{0, \lambda c\} \gneqq 0$. The restriction that μ is a constant was subsequently removed in [4] under the price of the assumption $h \geq 0$.

If multiplicity results can be observed in case $\lambda c \gneqq 0$, the existence of solution itself may fail. In [4, Lemma 6.1], letting $\gamma_1 > 0$ be the first eigenvalue of

$$(1.1) \qquad - \Delta \varphi_1 = \gamma c(x) \varphi_1, \quad \varphi_1 \in H_0^1(\Omega),$$

it is proved when $h \geq 0$ that problem (P_λ) has no solution when $\lambda = \gamma_1$ and no non-negative solutions when $\lambda > \gamma_1$. This contrasts to what was observed in [3, Theorem 3.3], namely that if $\mu > 0$ is a constant and $h \lneqq 0$, then there exists a negative solution of (P_λ) as soon as $\lambda > 0$. In addition this negative solution is unique [3, Theorem 3.12]. Considered together, the results of [3, 4] show that the sign of h has definitely an influence on the set of solutions of (P_λ) when $\lambda > 0$.

Despite the works [3, 4, 17, 18], having a clear picture of the set of solutions of (P_λ) in the half-space $]0, +\infty[\times(H_0^1(\Omega) \cap L^\infty(\Omega))$ is still widely open. The present paper aims to be a contribution in that direction. Note that both in [3] and [4], the main results (under this assumption) are obtained assuming that h has a sign, positive in [4], negative in [3], and then these papers look for solutions having the same sign as h. In our paper, we remove in particular the assumption that h has a sign. Also we show that even when h has a sign, solutions not having this sign may exist.

We point out that with respect to [3, 4] we have strengthened our regularity assumptions by requiring c and h in $L^p(\Omega)$ for some $p > N$ while in [4], c and h are in $L^p(\Omega)$ for some $p > \frac{N}{2}$ and in [3], the regularity assumptions are even weaker. Under our assumptions all solutions of (P_λ) lies in $W_0^{2,p}(\Omega) \subset C_0^1(\overline{\Omega})$ (see Theorem 2.2). This permits to use lower and upper solutions arguments together with degree theory. Now, for future reference, we recall,

Definition 1.1. Let $u, v \in C(\overline{\Omega})$. We say that
- $u \leq v$ if, for all $x \in \Omega$, $u(x) \leq v(x)$;
- $u \lneqq v$ if, for all $x \in \Omega$, $u(x) \leq v(x)$ and $u \not\equiv v$;
- $u < v$ if, for all $x \in \Omega$, $u(x) < v(x)$.

Let φ_1 be the first eigenfunction of (1.1). We know that, for all $x \in \Omega$, $\varphi_1(x) > 0$ and, for $x \in \partial\Omega$, $\frac{\partial \varphi_1}{\partial \nu}(x) < 0$ where ν denotes the exterior unit normal.

Definition 1.2. Let $u, v \in C(\overline{\Omega})$. We say that
- $u \ll v$ in case there exists $\varepsilon > 0$ such that, for all $x \in \overline{\Omega}$, $v(x) - u(x) \geq \varepsilon\varphi_1(x)$.

Remark 1.1. Observe that, in case $u, v \in C^1(\overline{\Omega})$, the definition of $u \ll v$ is equivalent to: for all $x \in \Omega$, $u(x) < v(x)$ and, for $x \in \partial\Omega$, either $u(x) < v(x)$ or $u(x) = v(x)$ and $\frac{\partial u}{\partial \nu}(x) > \frac{\partial v}{\partial \nu}(x)$.

Recall that by [4, Theorems 1.2 and 1.3], we have the following result relying on [22, Theorem 3.2].

Theorem 1.1. *Under assumption* (A)*, for* $\lambda \leq 0$ *the problem* (P_λ) *has at most one solution* u_λ*. Denote*

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) \mid (\lambda, u) \text{ solves } (P_\lambda)\}.$$

In case (P_0) *has a solution* u_0*, then there exists a continuum* $\mathcal{C} \subset \Sigma$ *such that* $\mathcal{C} \cap ([0, +\infty[\times C(\overline{\Omega}))$ *is unbounded in* $\mathbb{R} \times C(\overline{\Omega})$ *and* $\mathcal{C} \cap (\{0\} \times C(\overline{\Omega})) = \{u_0\}$*.*

In case $h \gneqq 0$*, this continuum* $\mathcal{C}$ *consists of non-negative functions and its projection* $\mathrm{Proj}_{\mathbb{R}}\mathcal{C}$ *on the* λ*-axis is an interval* $]-\infty, \overline{\lambda}] \subset]-\infty, \gamma_1[$ *containing* $\lambda = 0$ *and* $\mathcal{C}$ *bifurcates from infinity to the right of the axis* $\lambda = 0$*.*

Remark 1.2. From [4, Corollary 3.2], we know that (P_0) has a solution if

$$\inf_{\{u \in H_0^1(\Omega) |\ \|u\|_{H_0^1(\Omega)} = 1\}} \int_\Omega \left(|\nabla u|^2 - \mu_2 h^+(x) u^2 \right) dx > 0, \tag{1.2}$$

where $h^+ = \max\{0, h\}$.

Our first main result gives informations on the behaviour of this continuum without assuming that $h \gneqq 0$.

Theorem 1.2. *Under assumption* (A)*, in case* (P_0) *has a solution, the continuum* $\mathcal{C}$ *of Theorem 1.1 satisfies one of the two cases :*

(i) *it bifurcates from infinity to the right of the axis* $\lambda = 0$ *with the corresponding solutions having a positive part blowing up to infinity as* $\lambda \to 0^+$*;*
(ii) *it is such that its projection* $\mathrm{Proj}_{\mathbb{R}}\mathcal{C}$ *on the* λ*-axis is* $\mathbb{R}$*.*

In Corollary 4.1 below, we show that we are in situation (i) of Theorem 1.2 if (P_0) has a solution and

$$\int_\Omega h(x)\, \varphi_1(x)\, dx \geq 0.$$

In [18, Theorem 2] under conditions insuring that (P_0) has a solution it was proved, assuming that μ is a constant, that (P_λ) has two solutions for $\lambda > 0$ small. Here we remove this restriction on μ.

Theorem 1.3. *Under assumption* (A) *and assuming that* (P_0) *has a solution* u_0*, there exists a* $\overline{\lambda} \in]0, +\infty]$ *such that*

(i) *for every* $\lambda \in]0, \overline{\lambda}[$*, the problem* (P_λ) *has at least two solutions with*
  - $u_{\lambda,1} \ll u_{\lambda,2}$*;*
  - $\max\limits_{\overline{\Omega}} u_{\lambda,2} \to +\infty$ *and* $u_{\lambda,1} \to u_0$ *in* $C_0^1(\overline{\Omega})$ *as* $\lambda \to 0$*;*
(ii) *if* $\overline{\lambda} < +\infty$*, the problem* $(P_{\overline{\lambda}})$ *has exactly one solution* u*.*

Next we show that having a sign information on the solution u_0 of (P_0) allows us to give more precise informations on the set of solutions of (P_λ) when $\lambda > 0$.

Theorem 1.4. *Under assumption* (A) *and assuming that* (P_0) *has a solution* $u_0 \geq 0$ *with* $cu_0 \gneqq 0$*, every non- negative solution of* (P_λ) *with* $\lambda > 0$ *satisfies* $u \gg u_0$*. Moreover, there exists* $\overline{\lambda} \in]0, +\infty[$ *such that*

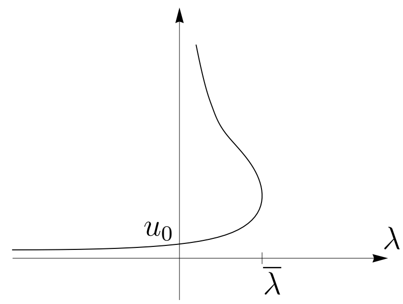

FIGURE 1. Illustration of Theorem 1.4

(i) *for every* $\lambda \in]0, \overline{\lambda}[$, *the problem* (P_λ) *has at least two solutions with*
- $0 \leq u_0 \ll u_{\lambda,1} \ll u_{\lambda,2}$;
- *if* $\lambda_1 < \lambda_2$, *we have* $u_{\lambda_1,1} \ll u_{\lambda_2,1}$;
- $\max_{\overline{\Omega}} u_{\lambda,2} \to +\infty$ *and* $u_{\lambda,1} \to u_0$ *in* $C_0^1(\overline{\Omega})$ *as* $\lambda \to 0$;

(ii) *the problem* $(P_{\overline{\lambda}})$ *has exactly one non-negative solution* u;

(iii) *for every* $\lambda > \overline{\lambda}$, *the problem* (P_λ) *has no non-negative solution.*

Remark 1.3. Since $-\Delta u_0 = \mu(x)|\nabla u_0|^2 + h(x)$, we deduce by the strong maximum principle that, in case $h \gneqq 0$, we have $u_0 \gg 0$ thus $cu_0 \gneqq 0$.

In comparison to Theorem 1.4 we have

Theorem 1.5. *Under assumption* (A) *and assuming that* (P_0) *has a solution* $u_0 \leq 0$ *with* $cu_0 \lneqq 0$, *for every* $\lambda > 0$, *problem* (P_λ) *has two solutions with*

$$u_{\lambda,1} \ll u_{\lambda,2}, \qquad u_{\lambda,1} \ll u_0, \quad \textit{and} \quad \max_{\overline{\Omega}} u_{\lambda,2} > 0.$$

Moreover we have
- *if* $\lambda_1 < \lambda_2$, *then* $u_{\lambda_1,1} \gg u_{\lambda_2,1}$;
- $\max_{\overline{\Omega}} u_{\lambda,2} \to +\infty$ *and* $u_{\lambda,1} \to u_0$ *in* $C_0^1(\overline{\Omega})$ *as* $\lambda \to 0$.

Remark 1.4. Observe that in case (P_0) has a solution u_0 with $cu_0 \equiv 0$, then u_0 is solution for all $\lambda \in \mathbb{R}$.

Remark 1.5. In Proposition 4.3, we prove also that, if (P_0) has a solution $u_0 \leq 0$ with $cu_0 \lneqq 0$, then (P_λ) has at most one solution $u \leq 0$.

Corollary 1.6. *Under assumption* (A) *and assuming that* $h \lneqq 0$, *for every* $\lambda > 0$, *problem* (P_λ) *has two solutions* $u_{\lambda,1}, u_{\lambda,2}$ *satisfying the conclusions of Theorem 1.5.*

Corollary 1.6 should be compared with [3, Theorem 3.3] where the authors prove the existence only of $u_{\lambda,1}$ under however weaker regularity assumptions.

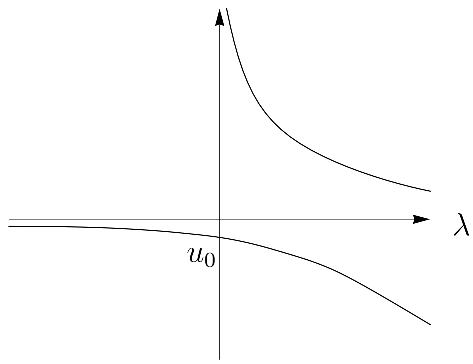

FIGURE 2. Illustration of Theorem 1.5

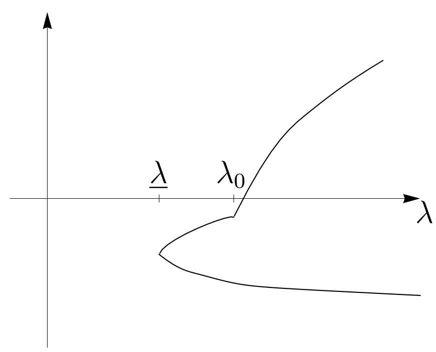

FIGURE 3. Illustration of Theorem 1.7

Our Theorems 1.3 - 1.5 require (P_0) to have a solution and thus we are in a situation where a branch of solutions starts from $(0, u_0)$. In our next results we consider the situation for $\lambda > 0$ "large".

Theorem 1.7. *Under assumption (A) and assuming that*

(a) (P_0) *does not have a solution* $u_0 \le 0$*;*
(b) *there exists* $\lambda_0 > 0$ *and* β_0 *an upper solution of* (P_{λ_0}) *with* $\beta_0 \le 0$*.*

Then there exists $0 < \underline{\lambda} \le \lambda_0$ *such that*

(i) *for every* $\lambda \in]\underline{\lambda}, +\infty[$*, the problem* (P_λ) *has at least two solutions with* $u_{\lambda,1} \ll 0$ *and* $u_{\lambda,1} \ll u_{\lambda,2}$*.*
Moreover, if $\lambda_1 < \lambda_2$*, we have* $u_{\lambda_1,1} \gg u_{\lambda_2,1}$*;*
(ii) *the problem* $(P_{\underline{\lambda}})$ *has a unique solution* $u_{\underline{\lambda}} \le 0$*;*
(iii) *for* $\lambda < \underline{\lambda}$*, the problem* $(P_{\underline{\lambda}})$ *has no solution* $u \le 0$*.*

In our last results we change our point of view and consider no more the dependence in λ but in $\|h^+\|$. In proving Theorem 1.8, we shall also obtain, in case $\|h^+\|$ is small enough, the existence of a negative upper solution of (P_{λ_0}) for some $\lambda_0 \geq 0$ as needed in the assumptions of Theorem 1.7.

Theorem 1.8. *Under assumption (A), let $\tilde{h} \in L^p(\Omega)$ and consider $\tilde{h}^+$ and $\tilde{h}^-$ respectively its positive and its negative part. Assume that $\tilde{h}^+ \not\equiv 0$. Let $\nu_1 > 0$ be the first eigenvalue of*

$$-\Delta u + \mu_2 \tilde{h}^-(x) u = \nu_1 c(x) u, \quad u \in H_0^1(\Omega). \tag{1.3}$$

Then, for all $\lambda > \nu_1$, there exists $\overline{k} = \overline{k}(\lambda) \in]0, +\infty[$ such that:

(i) *for all $k \in]0, \overline{k}[$, the problem*

$$-\Delta u = \lambda c(x) u + \mu(x)|\nabla u|^2 + k\tilde{h}^+(x) - \tilde{h}^-(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{$Q_{\lambda,k}$}$$

has at least two solutions $u_{\lambda,1} \ll u_{\lambda,2}$;

(ii) *for all $k > \overline{k}$, the problem $(Q_{\lambda,k})$ has no solution;*

(iii) *for $k = \overline{k}$, the problem $(Q_{\lambda,k})$ has exactly one solution.*

We deduce from Theorems 1.4 and 1.8 the following Corollary that concerns the case $h \gneqq 0$.

Corollary 1.9. *Under assumption (A) and assuming that $h \gneqq 0$, for all $\tilde{\lambda} > \gamma_1$ where $\gamma_1 > 0$ is the first eigenvalue (1.1), there exists $\tilde{k} > 0$ such that, for all $k \in]0, \tilde{k}]$,*

(i) *there exists $\lambda_1 \in]0, \gamma_1[$ such that:*
- *for all $\lambda \in]0, \lambda_1[$, the problem*

$$-\Delta u = \lambda c(x) u + \mu(x)|\nabla u|^2 + k h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{1.4}$$

has at least two positive solutions;
- *for $\lambda = \lambda_1$, the problem (1.4) has exactly one positive solution;*
- *for $\lambda > \lambda_1$, the problem (1.4) has no non-negative solution;*

(ii) *for $\lambda = \gamma_1$ the problem (1.4) has no solution;*

(iii) *there exists $\lambda_2 \in]\gamma_1, \tilde{\lambda}]$ such that:*
- *for $\lambda > \lambda_2$, the problem (1.4) has at least two solutions with $u_{\lambda,1} \ll 0$ and $\min u_{\lambda,2} < 0$;*
- *for $\lambda = \lambda_2$, the problem (1.4) has a unique non-positive solution;*
- *for $\lambda < \lambda_2$, the problem (1.4) has no non-positive solution.*

Remark 1.6. Observe that, as $h \geq 0$, we have $\gamma_1 = \nu_1$, where ν_1 is the first eigenvalue of (1.3) and γ_1 is the first eigenvalue of (1.1).

We conclude this paper considering the case $h \equiv 0$ which can be seen as intermediate between the case $h \gneqq 0$ considered in Corollary 1.9 and the case $h \lneqq 0$ considered in Corollary 1.6. Observe also that if we consider the problem (1.4) with $k \in]-\infty, \tilde{k}]$, then, it is easy to see that the lower of the two solutions tends to 0 and that $\lambda_1 \to \gamma_1$, $\lambda_2 \to \gamma_1$ as $k \to 0$.

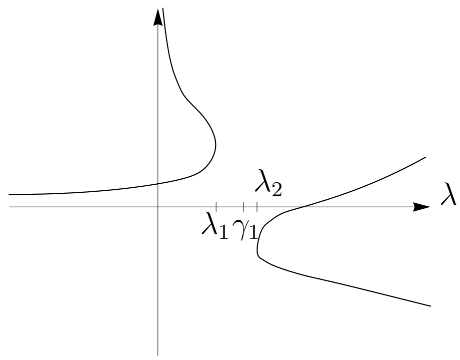

FIGURE 4. Illustration of Corollary 1.9

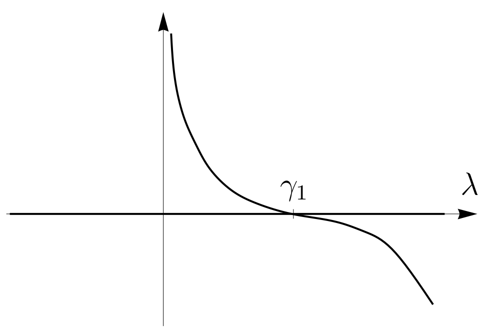

FIGURE 5. Illustration of Theorem 1.10

Theorem 1.10. *Under assumption* (A) *with* $h \equiv 0$ *and recalling that* $\gamma_1 > 0$ *denotes the first eigenvalue* (1.1)*, we have*

(i) *for all* $\lambda \in]0, \gamma_1[$*, the problem*

$$-\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{1.5}$$

has at least two solutions $u_{\lambda,1} \equiv 0$ *and* $u_{\lambda,2} \gneqq 0$ *with* $\max_{\overline{\Omega}} u_{\lambda,2} \to +\infty$ *as* $\lambda \to 0$*;*

(ii) *for* $\lambda = \gamma_1$ *the problem* (1.5) *has only the trivial solution;*

(iii) *for* $\lambda > \gamma_1$*, the problem* (1.5) *has at least two solutions* $u_{\lambda,1} \equiv 0$ *and* $u_{\lambda,2} \ll 0$.

Remark 1.7. Considering the solutions of (P_λ) as stationary solutions for the corresponding parabolic problem, assuming (A) together with $\partial\Omega$ is of class C^2 and c, $h \in L^p(\Omega)$ with $p > N + 2$, and applying [12, Corollary 2.34 and Proposition 2.41], we can prove that, in the above results, the first solution $u_{\lambda,1}$ is $\mathcal{L}$-asymptotically stable from below and $u_{\lambda,2}$ is $\mathcal{L}$-unstable from below. In the particular case of Theorem 1.5, as (P_λ) has a unique negative solution $u_{\lambda,1} \ll u_0$, we have also $u_{\lambda,1}$ is $\mathcal{L}$-asymptotically stable. Fore more informations, see [12].

Our existence results relies on the obtention of a priori bounds on the solutions, see Lemma 3.1 and Theorem 3.3. These results, which are valid for arbitrary solutions, use in a central way the assumption $\mu(x) \geq \mu_1 > 0$ for some $\mu_1 > 0$. Removing this condition seems delicate and in that direction some results are obtained in [24] for non-negative solutions. In [24] it is also shown that some conditions are necessary to obtain a priori bounds for non-negative solutions.

In the case $\mu > 0$ constant it is possible to precise the blow-up rate, as $\lambda \to 0^+$, of our solutions $u_{\lambda,2}$ obtained in Theorems 1.3, 1.4, 1.5 and 1.10. As a by-product, we also obtain that the a priori estimates obtained in Theorem 3.3 are sharp.

The paper is organized as follows. In Section 2 we present some preliminary results. Section 3 is devoted to our a priori bounds results. In Section 4 we prove our main results. Section 5 is devoted to the special case μ constant and in Section 6 the reader can find a list of open problems.

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Notations For $v \in H_0^1(\Omega)$ we set $v^+ = \max\{0, v\}$ and $v^- = \max\{0, -v\}$.

2. Preliminary results

In our proofs we shall need some results on lower and upper solutions that we present here adapted to our setting. We consider the problem

$$\begin{aligned} -\Delta u &= f(x, u, \nabla u), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where f is an L^p-Carathéodory function with $p > N$ and solutions are sought in $W_0^{2,p}(\Omega)$.

Definition 2.1. A *regular lower solution* (respectively a *regular upper solution*) of (2.1) is a function α (resp. β) in $W^{2,p}(\Omega)$ such that

$$\begin{aligned} -\Delta \alpha(x) &\leq f(x, \alpha(x), \nabla \alpha(x)), && \text{for a.e. } x \in \Omega, \\ \alpha(x) &\leq 0, && \text{for all } x \in \partial\Omega, \end{aligned}$$

(respectively

$$\begin{aligned} -\Delta \beta(x) &\geq f(x, \beta(x), \nabla \beta(x)), && \text{for a.e. } x \in \Omega, \\ \beta(x) &\geq 0, && \text{for all } x \in \partial\Omega). \end{aligned}$$

Definition 2.2. We define as a *lower solution* α of (2.1) a function of the form $\alpha := \max\{\alpha_i \mid 1 \le i \le k\}$ where $\alpha_1, \dots, \alpha_k$ are regular lower solutions of (2.1). Similarly, an *upper solution*, β of (2.1) is a function of the form $\beta = \min\{\beta_j \mid 1 \le j \le l\}$ where $\beta_1, \dots, \beta_l$ are regular upper solutions of (2.1).

Remark 2.1. The set of functions w such that $u \ll w \ll v$ is open in $C_0^1(\Omega)$ (the space of the C^1-functions in $\overline{\Omega}$ which vanish on the boundary of Ω).

Problem (2.1) can be transformed into a fixed point problem. The operator

$$\mathcal{L} : W_0^{2,p}(\Omega) \to L^p(\Omega);\ u \mapsto -\Delta u \tag{2.2}$$

is a linear homeomorphism.

Since f is an L^p-Carathéodory function, the operator

$$\mathcal{N} : C_0^1(\overline{\Omega}) \to L^p(\Omega)\,;\ u \mapsto f(.,u(.),\nabla u(.)) \tag{2.3}$$

is well defined, continuous and maps bounded sets to bounded sets. Since $p > N$, as $W_0^{2,p}(\Omega)$ is compactly embedded in $C_0^1(\overline{\Omega})$, the operator $\mathcal{M} : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ defined by

$$\mathcal{M}(u) = \mathcal{L}^{-1}\mathcal{N}u,$$

where $\mathcal{L}$ and $\mathcal{N}$ are given respectively by (2.2) and (2.3), is completely continuous and the problem (2.1) is equivalent to

$$u = \mathcal{M}u.$$

To be able to associate a degree to a pair of lower and upper solutions we also need to reinforce the definition.

Definition 2.3. A lower solution α of (2.1) is said to be *strict* if every solution u of (2.1) such that $\alpha \le u$ on Ω satisfies $\alpha \ll u$.

In the same way a *strict upper solution* β of (2.1) is an upper solution such that every solution u with $u \le \beta$ is such that $u \ll \beta$.

Our main tool regarding the existence and characterizations of solutions of problem (2.1) by a lower and upper solutions approach is the following theorem. This result, which can be obtained adapting some ideas from [11, 12], will be proved in the Appendix.

Theorem 2.1. *Let Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^{1,1}$ and f be an L^p-Carathéodory function with $p > N$. Assume that there exists a lower solution α and an upper solution β of (2.1) such that $\alpha \le \beta$. Denote $\alpha := \max\{\alpha_i \mid 1 \le i \le k\}$ where $\alpha_1, \dots, \alpha_k$ are regular lower solutions of (2.1) and $\beta = \min\{\beta_j \mid 1 \le j \le l\}$ where $\beta_1, \dots, \beta_l$ are regular upper solutions of (2.1). If there exists $K > 0$ and $h \in L^p(\Omega)$ such that for a.e. $x \in \Omega$, all $u \in [\min\{\alpha_i \mid 1 \le i \le k\}, \max\{\beta_j \mid 1 \le j \le l\}]$ and all $\xi \in \mathbb{R}^N$,*

$$|f(x,u,\xi)| \le h(x) + K|\xi|^2, \tag{2.4}$$

then the problem (2.1) *has at least one solution u satisfying*

$$\alpha \leq u \leq \beta.$$

Moreover, problem (2.1) *has a minimal solution $u_{\min}$ and a maximal solution $u_{\max}$ in the sense that, $u_{\min}$ and $u_{\max}$ are solutions of* (2.1) *with $\alpha \leq u_{\min} \leq u_{\max} \leq \beta$ and every solution u of* (2.1) *with $\alpha \leq u \leq \beta$ satisfies $u_{\min} \leq u \leq u_{\max}$.*

If moreover α and β are strict and satisfy $\alpha \ll \beta$, then, there exists $R > 0$ such that

$$\deg(I - \mathcal{M}, \mathcal{S}) = 1,$$

where

$$\mathcal{S} = \{u \in C_0^1(\overline{\Omega}) \mid \alpha \ll u \ll \beta, \quad \|u\|_{C^1} < R\}.$$

Remark 2.2. If α and β are respectively strict lower and upper solutions of (2.1) with $\alpha \leq \beta$ then $\alpha \ll \beta$. Indeed, from the first part of Theorem 2.1, we deduce the existence of a solution u with $\alpha \leq u \leq \beta$. By definition of strict lower and upper solutions, we obtain $\alpha \ll u \ll \beta$ and hence $\alpha \ll \beta$.

Remark 2.3. We shall apply Theorem 2.1 with $\mathcal{N}(u) = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x)$. Hence, as we are concerned with the λ-dependance, we will denote the fixed point operator $\mathcal{M}_\lambda$ instead of $\mathcal{M}$.

Our assumption (A) implies that the following regularity result applies to problem (P_λ).

Theorem 2.2. *Let Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^{1,1}$, $c \in L^p(\Omega)$, $h \in L^p(\Omega)$ with $p > N$ and $\mu \in L^\infty(\Omega)$. Let u be a solution of*

$$-\Delta u = c(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.5}$$

Then $u \in W_0^{2,p}(\Omega) \subset C_0^1(\overline{\Omega})$.

Remark 2.4. This result is not a simple consequence of classical bootstrap arguments as, for $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\mu|\nabla u|^2 \in L^1(\Omega)$ which does not allow to start a bootstrap process.

Remark 2.5. Observe that any solution $u \in C_0^1(\overline{\Omega})$ of (2.5) belongs to $W_0^{2,p}(\Omega)$.

Proof. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and define the function $g = c\,u + u + h$. Observe that $g \in L^p(\Omega)$ with $p > N$ and u is solution of

$$-\Delta v = -v + \mu(x)|\nabla v|^2 + g(x), \quad v \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.6}$$

From [5, Theorem 1.1] we know that (2.6) admits at most one solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Thus, if we prove that (2.6) has a solution $v \in W_0^{2,p}(\Omega)$, we obtain $u = v \in W_0^{2,p}(\Omega)$. To prove that (2.6) has a solution $v \in W_0^{2,p}(\Omega)$, we shall use Theorem 2.1. Thus, we need to prove that (2.6) has a lower α and an upper solution β with $\alpha \leq \beta$.

We set $\overline{\mu} = ||\mu||_\infty$. Clearly any solution of

$$\begin{aligned} -\Delta u &= -u + \overline{\mu}|\nabla u|^2 + g^+(x), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

is an upper solution of (2.6) and any solution of

$$\begin{aligned} -\Delta u &= -u - \overline{\mu}|\nabla u|^2 - g^-(x), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.8}$$

is a lower solution of (2.6). Now, observe that, if $w \in W_0^{2,p}(\Omega)$ is a solution of

$$\begin{aligned} -\Delta w &= -w + \overline{\mu}|\nabla w|^2 + g^-(x), && \text{in } \Omega, \\ w &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.9}$$

then $u = -w$ satisfies (2.8). Thus, if we find a non-negative solution $u_1 \in W_0^{2,p}(\Omega)$ of (2.7) and a non-negative solution $u_2 \in W_0^{2,p}(\Omega)$ of (2.9) then, setting $\beta = u_1$ and $\alpha = -u_2$, we have the couple of lower and upper solutions required to apply Theorem 2.1.

Let us construct u_1, the construction of u_2 being similar. Our construction makes use of the classical Hopf-Cole change of variable. Let $w_1 \in H_0^1(\Omega)$ be the non-negative solution of

$$\begin{aligned} -\Delta w_1 &= \overline{\mu} g^+(x) w_1 - m(w_1) + g^+(x), && \text{in } \Omega, \\ w_1 &= 0, && \text{on } \partial\Omega \end{aligned}$$

where

$$m(s) = \begin{cases} \frac{1}{\overline{\mu}}(1+\overline{\mu}s)\ln(1+\overline{\mu}s), & \text{if} \quad s \geq 0, \\ -\frac{1}{\overline{\mu}}(1-\overline{\mu}s)\ln(1-\overline{\mu}s), & \text{if} \quad s < 0, \end{cases} \tag{2.10}$$

given by [4, Lemma 3.3]. By [25, Lemma 3.22] and a bootstrap argument, it is easy to prove that $w_1 \in W^{2,p}(\Omega)$. Hence

$$u_1 = \frac{\ln(\overline{\mu} w_1 + 1)}{\overline{\mu}} \in W^{2,p}(\Omega),$$

and one readily shows that $u_1 \geq 0$ is a solution of (2.7). □

Proposition 2.3. *Under assumption (A) if α is a lower solution of (P_0) and β an upper solution of (P_0) then $\alpha \leq \beta$.*

Proof. By Definition 2.2 we have that $\alpha = \max\{\alpha_i \mid 1 \leq i \leq k\}$, $\beta = \min\{\beta_j \mid 1 \leq j \leq l\}$ with α_i and β_j being respectively regular lower and upper solutions of (P_0). Since α_i and β_j are in $W^{2,p}(\Omega)$ they belong to $H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap C(\overline{\Omega})$ and we deduce using [5, Lemma 2.2] that $\alpha_i \leq \beta_j$. By using again the definition of α and β it follows that $\alpha \leq \beta$. □

The following estimates will also be useful.

Lemma 2.4 (Nagumo Lemma). *Let $p > N$, $h \in L^p(\Omega)$, $K > 0$, $R > 0$. Then there exists $C > 0$ such that, for all $u \in W^{2,p}(\Omega)$ satisfying*

$$\begin{array}{ll} |\Delta u| \leq h(x) + K|\nabla u|^2, & \textit{a.e. in } \Omega, \\ u = 0, & \textit{on } \partial\Omega, \end{array}$$

and

$$\|u\|_\infty \leq R,$$

we have

$$\|u\|_{W^{2,p}} \leq C.$$

Proof. see [25, Lemma 5.10]. □

Lemma 2.5. *Assume that $c, h \in L^q(\Omega)$ for some $q > \frac{N}{2}$. Then if $u \in H_0^1(\Omega)$ is solution of*

$$-\Delta u \leq c(x)u + h(x), \qquad (\textit{resp.} \ -\Delta u \geq c(x)u + h(x))$$

in a weak sense, then u is bounded above (resp. below) and

$$\sup_\Omega u^+ \leq C(\|u^+\|_2 + \|h\|_q), \qquad (\textit{resp.} \ \sup_\Omega u^- \leq C(\|u^-\|_2 + \|h\|_q)),$$

where $C > 0$ depends on $N, q, |\Omega|$ and $\|c\|_q$.

Proof. This is a consequence of [26, Theorem 1] combined with Remark 1 on p. 289 in that paper. It can also be obtained by adapting the proof of [15, Theorem 8.15] (which implies the result in the case where $c \in L^\infty(\Omega)$), as remarked at the end of p. 293 of that book. □

We also need the following formulation of the anti-maximum principle. Under slightly more smooth data, this result was established in [16]. Nevertheless, the proof given in [16] directly extend to our regularity assumptions.

Proposition 2.6. *Let $\bar{c}, \bar{h}, \bar{d} \in L^p(\Omega)$ with $p > N$ and assume that $\bar{h} \gneqq 0$. We denote by $\bar{\nu}_1 > 0$ the first eigenvalue of*

$$- \Delta u + \bar{d}(x)u = \bar{\nu}_1 \bar{c}(x)u, \quad u \in H_0^1(\Omega). \tag{2.11}$$

Then there exists $\varepsilon_0 > 0$ such that, for all $\lambda \in]\bar{\nu}_1, \bar{\nu}_1 + \varepsilon_0]$, the solution w of

$$- \Delta w + \bar{d}(x)w = \lambda \bar{c}(x)w + \bar{h}(x), \quad u \in H_0^1(\Omega) \tag{2.12}$$

satisfies $w \ll 0$.

3. A priori bound

This section is devoted to the derivation of some a priori bounds results for the solutions of (P_λ). Most of our results hold under more general assumptions than (A).

First, using ideas of [3], we obtain the following lower bound on the upper solutions of (P_λ).

Lemma 3.1. *Under conditions (A), for any $\Lambda_2 > 0$, there exists a constant $M := M(\Lambda_2, \mu_1, \|c\|_{N/2}, \|h^-\|_{N/2}) > 0$ such that, for any $\lambda \in [0, \Lambda_2]$, any function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ verifying $u \geq 0$ on $\partial\Omega$ and such that, for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω,*

$$\int_\Omega \nabla u \nabla v \, dx \geq \int_\Omega [\lambda c(x) u + \mu(x)|\nabla u|^2 + h(x)] v \, dx \tag{3.1}$$

satisfies

$$\min_\Omega u > -M.$$

Remark 3.1. This result is valid under less regularity conditions than (A) and without sign conditions on c and h. More precisely, it holds under the conditions: $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with $\partial\Omega$ of class $C^{1,1}$, c and h belong to $L^p(\Omega)$ for some $p > N/2$, $\mu \in L^\infty(\Omega)$ satisfies $0 < \mu_1 \leq \mu(x) \leq \mu_2$.

Moreover, the lower bound does not depend on h^+ and depends only on an upper bound on $\lambda \geq 0$.

Proof. Let us take $v = u^-$ as test function in (3.1). We obtain

$$\begin{aligned} -\int_\Omega |\nabla u^-|^2 \, dx &\geq -\lambda \int_\Omega c^+ (u^-)^2 \, dx + \mu_1 \int_\Omega |\nabla u^-|^2 u^- \, dx - \int_\Omega h^- u^- \, dx \\ &\geq -\Lambda_2 \int_\Omega c^+ (u^-)^2 \, dx + \mu_1 \frac{4}{9} \int_\Omega |\nabla (u^-)^{3/2}|^2 \, dx - \int_\Omega h^- u^- \, dx \end{aligned}$$

and hence

$$\mu_1 \frac{4}{9} \int_\Omega |\nabla (u^-)^{3/2}|^2 \, dx + \int_\Omega |\nabla u^-|^2 \, dx \leq \int_\Omega h^- u^- \, dx + \Lambda_2 \int_\Omega c^+ (u^-)^2 \, dx.$$

For every $\varepsilon > 0$ we have

$$\begin{aligned} \Lambda_2 \int_\Omega c^+ (u^-)^2 \, dx &= \int_\Omega (\Lambda_2 c^+)^{1/2} (u^-)^{1/2} (\Lambda_2 c^+)^{1/2} (u^-)^{3/2} \, dx \\ &\leq \frac{1}{2\varepsilon} \Lambda_2 \int_\Omega c^+ u^- \, dx + \frac{\varepsilon}{2} \Lambda_2 \int_\Omega c^+ \left((u^-)^{3/2} \right)^2 \, dx. \end{aligned}$$

Also, for some constant C_N, by Sobolev's embedding, we get

$$\int_\Omega c^+ \left((u^-)^{3/2} \right)^2 \, dx \leq \|c^+\|_{N/2} \|(u^-)^{3/2}\|_{2^*}^2 \leq \frac{1}{C_N} \|c^+\|_{N/2} \|\nabla (u^-)^{3/2}\|_2^2.$$

We then obtain

$$\begin{aligned}\mu_1\frac{4}{9}\int_\Omega |\nabla(u^-)^{3/2}|^2\,dx &+ \int_\Omega |\nabla u^-|^2\,dx \\ &\le \int_\Omega h^-u^-\,dx + \frac{1}{2\varepsilon}\Lambda_2\int_\Omega c^+u^-\,dx + \frac{\varepsilon}{2}\frac{\Lambda_2}{C_N}\|c^+\|_{N/2}\|\nabla(u^-)^{3/2}\|_2^2.\end{aligned}$$

Hence, by choosing $\varepsilon = \dfrac{C_N}{\Lambda_2\|c^+\|_{N/2}}\mu_1\dfrac{4}{9}$, it comes

$$\begin{aligned}\mu_1\frac{2}{9}\int_\Omega |\nabla(u^-)^{3/2}|^2\,dx &+ \int_\Omega |\nabla u^-|^2\,dx \\ &\le \int_\Omega h^-u^-\,dx + \frac{9\Lambda_2^2\|c^+\|_{N/2}}{8\mu_1 C_N}\int_\Omega c^+u^-\,dx \\ &\le C\Big(\|h^-\|_{N/2}\|\nabla u^-\|_2 + \frac{\Lambda_2^2}{\mu_1}\|c^+\|_{N/2}^2\|\nabla u^-\|_2\Big)\end{aligned}$$

from which we deduce that

$$\|u^-\|_{H_0^1} \le C(\|h^-\|_{N/2} + \frac{\Lambda_2^2}{\mu_1}\|c^+\|_{N/2}^2).$$

By Lemma 2.5 we obtain that

$$u \ge -M := -M(\Lambda_2,\mu_1,\|c\|_{N/2},\|h^-\|_{N/2}),$$

which allows to conclude. □

As a simple corollary, we have the following result.

Corollary 3.2. *Under conditions* (A), *for any* $\Lambda_2 > 0$, *there exists a constant* $M := M(\Lambda_2,\mu_1,\|c\|_{N/2},\|h^-\|_{N/2}) > 0$ *such that, for any* $\lambda \in [0,\Lambda_2]$, *any upper solution* β *of* (P_λ) *satisfies*

$$\min_\Omega \beta > -M.$$

Proof. As $\beta = \min\{\beta_j \mid 1 \le j \le l\}$ where β_j are regular upper solutions, they belong to $H^1(\Omega)\cap L^\infty(\Omega)$ and satisfy (3.1). We conclude by Lemma 3.1. □

Let $\tilde\nu_1 > 0$ denotes the first eigenvalue of

$$-\Delta u + \mu_1 h^-(x)u = \nu c(x)u, \quad u \in H_0^1(\Omega), \tag{3.2}$$

with corresponding eigenfunction $\psi_1 > 0$.

Theorem 3.3. *Under condition* (A), *for any* $\Lambda_2 > \Lambda_1 > 0$, *any* $A > 0$, *there exists a constant* $M > 0$ *such that, for any* $\lambda \in [\Lambda_1,\Lambda_2]$, $a \in [0,A]$, *any solution* u *of*

$$-\Delta u = \lambda c(x)u + \mu(x)|\nabla u|^2 + h(x) + ac(x), \quad u \in H_0^1(\Omega)\cap L^\infty(\Omega), \tag{3.3}$$

satisfies

$$\|u\|_\infty < M.$$

Moreover, viewed as a function of Λ_1, $M = O_{0^+}(1/\Lambda_1)$.

In the above theorem, the notation $M = O_{0^+}(g(\Lambda_1))$ means the existence of $C > 0$ such that

$$\left|\frac{M(\Lambda_1)}{g(\Lambda_1)}\right| \leq C, \qquad \text{as } \Lambda_1 \to 0^+.$$

Remark 3.2. The above theorem is valid under less restrictive conditions. In fact it is valid if we replace the regularity c and $h \in L^p(\Omega)$ with $p > N$ by c and $h \in L^p(\Omega)$ with $p > N/2$ and $h^- \in L^q(\Omega)$ for some $q > N$. This last condition is used to prove that the first eigenfunction $\psi_1 > 0$ of (3.2) satisfies $\psi_1 \geq d\delta(x)$ for some constant $d > 0$ where $\delta(x)$ denotes the distance from x to $\partial\Omega$. This is needed to insure that the conclusion of Lemma 3.5 holds. Following the proof of [4, Lemma 6.3] it is possible to prove that this condition on ψ_1 holds under this stronger regularity.

In the proof of the Theorem 3.3 the following technical lemmas will be used.

Lemma 3.4. *Let* $p > \frac{N}{2}$ *and* $\theta \in]0,1[$. *There exist* $r \in]0,1[$ *and* $\alpha \in]0, \frac{p-1}{2p-1}[$ *such that if we define*

$$q = 1 + r + \frac{1+\theta\alpha}{1-\alpha}, \quad \tau = \frac{1}{q}\,\frac{\alpha}{1-\alpha}, \tag{3.4}$$

then it holds

$$\frac{1}{p} \leq q \leq \frac{2N(p-1)}{p(N-2+2\tau)} \tag{3.5}$$

and

$$1 - \alpha < \frac{2}{q}. \tag{3.6}$$

Proof. See [4, Lemma 6.2]. □

Lemma 3.5. *Let* $b \in L^p(\Omega)$ *with* $p > \frac{N}{2}$. *For any* p, $q \geq 1$ *and* $\tau \in [0,1]$ *satisfying* (3.5), *there exists* $C > 0$ *such that, for all* $w \in H_0^1(\Omega)$,

$$\left\|\frac{b^{1/q}w}{\psi_1^\tau}\right\|_q \leq C\|b\|_p\|\nabla w\|_2,$$

where $\psi_1 > 0$ *denotes the first eigenfunction* (3.2).

Proof. See [4, Lemma 6.3] or [10]. □

Proof of Theorem 3.3. Let $\lambda \in [\Lambda_1, \Lambda_2]$, $a \in [0, A]$ and u be a solution of (3.3). Assume without loss of generality that $\Lambda_1 \leq 1 \leq \Lambda_2$. We define

$$w_i(x) = \frac{1}{\mu_i}(e^{\mu_i u(x)} - 1) \text{ and } g_i(s) = \frac{1}{\mu_i}\ln(1 + \mu_i s) \quad i = 1, 2.$$

Then we have

$$u = g_1(w_1) = g_2(w_2), \tag{3.7}$$

$$e^{\mu_i u} = 1 + \mu_i w_i, \quad i = 1, 2. \tag{3.8}$$

Direct calculations give us

$$\begin{aligned} -\Delta w_i &= e^{\mu_i u}(\lambda c(x) u + h(x) + ac(x)) + e^{\mu_i u}(\mu(x) - \mu_i)|\nabla u|^2 \\ &= (1 + \mu_i w_i)(\lambda c(x) g_i(w_i) + h(x) + ac(x)) + (1 + \mu_i w_i)(\mu(x) - \mu_i)|\nabla u|^2. \end{aligned}$$

Since $\mu_1 \le \mu(x) \le \mu_2$, we have

$$-\Delta w_1 \ge (1 + \mu_1 w_1)[\lambda c(x) g_1(w_1) + h(x) + ac(x)], \tag{3.9}$$

$$-\Delta w_2 \le (1 + \mu_2 w_2)[\lambda c(x) g_2(w_2) + h(x) + ac(x)], \tag{3.10}$$

in a weak sense.

From the inequalities (3.9) and (3.10), we shall deduce that w_2 is uniformly bounded in $H_0^1(\Omega)$. This will lead to the proof of the theorem by Lemma 2.5. We shall denote by C a generic constant independent of Λ_1 and by $C(\Lambda_1)$, a generic constant depending on Λ_1. We then precise its dependence on Λ_1.

We divide the proof into three steps.

Step 1. *Let $\theta = (\mu_2 - \mu_1)/\mu_2 \in]0, 1[$. Then there exists $D = D(\Lambda_1) > 0$ independent of $\lambda \in [\Lambda_1, \Lambda_2]$ and of $a \in [0, A]$ such that*

$$\int_\Omega (1 + \mu_1 w_1^+)[cg_1(w_1^+) + h^+ + ac]\psi_1 \, dx \le D(\Lambda_1), \tag{3.11}$$

$$\int_\Omega (1 + \mu_2 w_2^+)^{1-\theta}[cg_2(w_2^+) + h^+ + ac]\psi_1 \, dx \le D(\Lambda_1). \tag{3.12}$$

Moreover $D(\Lambda_1) = O_{0^+}(e^{1/\Lambda_1})$.

Indeed, using $\psi_1 > 0$ (defined in (3.2)) as a test function in (3.9) and integrating we have

$$\int_\Omega [\tilde{\nu}_1 c - \mu_1 h^-] w_1 \psi_1 \, dx \ge \int_\Omega (1 + \mu_1 w_1)[\lambda c g_1(w_1) + h + ac]\psi_1 \, dx.$$

Recording that $\lambda \le \Lambda_2$ and then, by Lemma 3.1, that $g_1(w_1^-) = u^-$ is uniformly bounded we then obtain

$$\begin{aligned} \tilde{\nu}_1 \int_\Omega c w_1 \psi_1 \, dx &\ge \int_\Omega (1 + \mu_1 w_1)[\lambda c g_1(w_1) + h + ac]\psi_1 \, dx + \mu_1 \int_\Omega h^- w_1 \psi_1 \, dx \\ &= \int_\Omega (1 + \mu_1 w_1)[\lambda c g_1(w_1) + h^+ + ac]\psi_1 \, dx - \int_\Omega h^- \psi_1 \, dx \\ &\ge \int_\Omega (1 + \mu_1 w_1^+)[\lambda c g_1(w_1^+) + h^+ + ac]\psi_1 \, dx - C. \end{aligned}$$

Then, since $\lambda \geq \Lambda_1$ and $\Lambda_1 \leq 1$, we deduce that

$$\tilde{\nu}_1 \int_\Omega c w_1 \psi_1 \, dx \geq \Lambda_1 \int_\Omega (1 + \mu_1 w_1^+)[c g_1(w_1^+) + h^+ + ac]\psi_1 \, dx - C. \tag{3.13}$$

Note that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for all $t > 0$,

$$t \leq \varepsilon(1 + \mu_1 t) g_1(t) + C_\varepsilon. \tag{3.14}$$

A direct calculation shows that we can assume that $C_\varepsilon = O_{0^+}(\varepsilon e^{1/\varepsilon})$. Using (3.14) with $\varepsilon = \dfrac{\Lambda_1}{2\tilde{\nu}_1}$, we get that

$$\begin{aligned} \tilde{\nu}_1 \int_\Omega c w_1 \psi_1 \, dx &\leq \tilde{\nu}_1 \int_\Omega c w_1^+ \psi_1 \, dx \\ &\leq \frac{\Lambda_1}{2} \int_\Omega (1 + \mu_1 w_1^+)[c g_1(w_1^+) + h^+ + ac]\psi_1 \, dx + C_{\Lambda_1}. \end{aligned} \tag{3.15}$$

We then obtain (3.11) from (3.13) and (3.15). Now observe that by (3.8),

$$1 + \mu_1 w_1 = e^{\mu_1 u} = (e^{\mu_2 u})^{1-\theta} = (1 + \mu_2 w_2)^{1-\theta}.$$

Thus from (3.7) we see that (3.12) is nothing but (3.11).

Step 2. *There exists a constant $D = D(\Lambda_1) > 0$ independent of $a \in [0, A]$ and $\lambda \in [\Lambda_1, \Lambda_2]$ such that*

$$\|\nabla w_2^+\|_2 \leq D(\Lambda_1). \tag{3.16}$$

Moreover $D(\Lambda_1) = O_{0^+}(e^{\beta/\Lambda_1})$ with $\beta = \frac{\alpha}{2 - q(1-\alpha)}$.

First we use Lemma 3.4 to choose $r \in]0, 1[$ and $\alpha \in]0, \frac{p-1}{2p-1}[$ such that q and τ defined by (3.4) satisfy (3.5) and (3.6).

Using w_2^+ as a test function in (3.10) it follows that

$$\|\nabla w_2^+\|_2^2 \leq \int_\Omega (1 + \mu_2 w_2^+)[\lambda c g_2(w_2^+) + h^+ + ac] w_2^+ \, dx.$$

Setting $H = h^+ + Ac$, we have

$$\|\nabla w_2^+\|_2^2 \leq \Lambda_2 \int_\Omega (1 + \mu_2 w_2^+)[c g_2(w_2^+) + H] w_2^+ \, dx.$$

Now, using Hölder's inequality and since $w_2^+ \le (1+\mu_2 w_2^+)/\mu_2^{-1}$, we obtain, using (3.12) of Step 1 and for a $D(\Lambda_1) = O_{0^+}(e^{1/\Lambda_1})$,

$$\begin{aligned}
\|\nabla w_2^+\|_2^2 &\le \frac{\Lambda_2}{\mu_2}\int_\Omega (1+\mu_2 w_2^+)[cg_2(w_2^+)+H)]\frac{\psi_1^\alpha}{(1+\mu_2 w_2^+)^{\theta\alpha}}\frac{(1+\mu_2 w_2^+)^{1+\theta\alpha}}{\psi_1^\alpha}dx \\
&\le \frac{\Lambda_2}{\mu_2}\left(\int_\Omega (1+\mu_2 w_2^+)[cg_2(w_2^+)+H]\frac{\psi_1}{(1+\mu_2 w_2^+)^{\theta}}\,dx\right)^\alpha \\
&\qquad\times\left(\int_\Omega (1+\mu_2 w_2^+)[cg_2(w_2^+)+H]\frac{(1+\mu_2 w_2^+)^{\frac{1+\theta\alpha}{1-\alpha}}}{\psi_1^{\frac{\alpha}{1-\alpha}}}\,dx\right)^{1-\alpha} \\
&\le \frac{\Lambda_2}{\mu_2}D(\Lambda_1)^\alpha\left(\int_\Omega (1+\mu_2 w_2^+)[cg_2(w_2^+)+H]\frac{(1+\mu_2 w_2^+)^{\frac{1+\theta\alpha}{1-\alpha}}}{\psi_1^{\frac{\alpha}{1-\alpha}}}\,dx\right)^{1-\alpha}.
\end{aligned}$$

We note that, for $r>0$ given by Lemma 3.4, there exists $C>0$ such that

$$g_2(t) \le t^r + C \quad \text{for all } t \ge 0.$$

Thus, direct calculations shows that

$$(1+\mu_2 w_2^+)[cg_2(w_2^+)+H](1+\mu_2 w_2^+)^{\frac{1+\theta\alpha}{1-\alpha}} \le (c+H)(w_2^{+q}+C),$$

where q is given in (3.4). Therefore for some $D(\Lambda_1) = O_{0^+}(e^{1/\Lambda_1})$,

$$\|\nabla w_2^+\|_2^2 \le D(\Lambda_1)^\alpha\left[\left(\int_\Omega\left(\frac{(c+H)^{1/q}w_2^+}{\psi_1^\tau}\right)^q dx\right)^{1-\alpha}+1\right],$$

with q and τ given in (3.4). Applying Lemma 3.5, we then obtain

$$\|\nabla w_2^+\|_2^2 \le D(\Lambda_1)^\alpha\left[\|c+H\|_p^{q(1-\alpha)}\|\nabla w_2^+\|_2^{q(1-\alpha)}+1\right].$$

By (3.6), we have $q(1-\alpha)<2$ and this concludes the proof of Step 2.

Step 3. *Conclusion.*

By Lemma 3.1 we already know that $u > -M$ for some $M>0$. Hence we just have to show that the estimate (3.16) derived in Step 2 gives an estimate in the $L^\infty(\Omega)$ norm of w_2^+. Since w_2 satisfies (3.10) we can use Lemma 2.5 with

$$d = (1+\mu_2 w_2)\lambda c\frac{\ln(1+\mu_2 w_2)}{\mu_2 w_2} + \mu_2\,(h + A\,c)$$

and

$$f = h + A\,c.$$

Observe that, for any $r\in\,]0,1[$, there exists $C>0$ such that, for all $x\in\Omega$ and all $\lambda \le \Lambda_2$,

$$\lambda c\left|(1+\mu_2 w_2)\frac{\ln(1+\mu_2 w_2)}{\mu_2 w_2}\right| \le Cc(|w_2|^r+1),$$

where C depends on Λ_2, r, μ_2.

Thus, since $c(x) \in L^p(\Omega)$ with $p > \frac{N}{2}$ and w_2 is bounded in $L^{\frac{2N}{N-2}}(\Omega)$, taking $r > 0$ sufficiently small we see, using Hölder's inequality, that $c(x)|w_2(x)|^r \in L^{p_1}(\Omega)$ for some $p_1 > \frac{N}{2}$. Now as $h \in L^p(\Omega)$ for some $p > \frac{N}{2}$, clearly all the assumptions of Lemma 2.5 are satisfied. From (3.16) we then deduce that there exists a constant $D(\Lambda_1) > 0$ with $D(\Lambda_1) = O_{0^+}(e^{\beta/\Lambda_1})$ and β given by Step 2, such that

$$\|w_2^+\|_\infty \leq D(\Lambda_1).$$

Now since $u^+ = g_2(w_2^+)$ we deduce that

$$\|u^+\|_\infty \leq M(\Lambda_1),$$

for some $M(\Lambda_1) = O_{0^+}(1/\Lambda_1)$. □

Lemma 3.6. *For every $\Lambda_2 > 0$, there exists $A_1 > 0$, independent of $\lambda \in [0, \Lambda_2]$, such that the problem* (3.3) *has no solution for $a \geq A_1$.*

Proof. Let $\phi \in C_0^\infty(\Omega)$ such that $\int_\Omega c(x)\phi^2\,dx > 0$ and use ϕ^2 as test function in (3.3). Then we obtain

$$\begin{aligned}
\int_\Omega \frac{1}{|\mu(x)|}|\nabla\phi|^2\,dx &\geq 2\int_\Omega \phi\nabla u\nabla\phi\,dx - \int_\Omega |\mu(x)||\nabla u|^2\phi^2\,dx \\
&= \lambda\int_\Omega c\,u\,\phi^2\,dx + \int_\Omega h\,\phi^2\,dx + a\int_\Omega c\,\phi^2\,dx \\
&\geq \lambda\min u\int_\Omega c\,\phi^2\,dx + \int_\Omega h\,\phi^2\,dx + a\int_\Omega c\,\phi^2\,dx.
\end{aligned}$$

Since, by Lemma 3.1, there exists $M > 0$ such that, for all $a \geq 0$, any solution u satisfies $u > -M$, this gives a contradiction for $a > 0$ large enough. □

4. Results

This section is devoted to the proof of our main results.

Proof of Theorem 1.2. Let $\mathcal{C} \subset \Sigma$ be the continuum obtained in Theorem 1.1. Either its projection $\mathrm{Proj}_{\mathbb{R}}\mathcal{C}$ on the λ-axis is $\mathbb{R}$ or its projection on the λ-axis is $]-\infty, \overline{\lambda}]$ with $0 < \overline{\lambda} < +\infty$. In the first case, the result is proved. In the second case, as by Theorem 1.1 we know that $\mathcal{C} \cap ([0, +\infty[\times C(\overline{\Omega}))$ is unbounded, its projection on $C(\overline{\Omega})$ has to be unbounded.

By Theorem 3.3 we know that for every $0 < \Lambda_1 < \Lambda_2$, there is an a priori bound on the solutions for $\lambda \in [\Lambda_1, \Lambda_2]$. This means that the projection of $\mathcal{C} \cap ([\Lambda_1, \Lambda_2] \times C(\overline{\Omega}))$ on $C(\overline{\Omega})$ is bounded. Now by Lemma 3.1 there is a lower bound on the solutions for $\lambda \leq \Lambda_2$. Thus $\mathcal{C}$ must emanate from infinity to the right of $\lambda = 0$ with the positive part of the corresponding solution blowing up to infinity. □

Corollary 4.1. *Under assumption* (A) *and assuming that* (P_0) *has a solution, let* $\varphi_1 > 0$ *the first eigenfunction of* (1.1)*. If*

$$\int_\Omega h(x)\varphi_1(x)\,dx \geq 0,$$

then we are in case (i) of Theorem 1.2 and $\max \mathrm{Proj}_{\mathbb{R}}\mathcal{C} < \gamma_1$.

Proof. Let u be a solution of (P_λ). Multiplying by $\varphi_1 > 0$ and integrating we have

$$(\gamma_1 - \lambda)\int_\Omega c(x)u\varphi_1\,dx = \int_\Omega \mu(x)|\nabla u|^2\varphi_1\,dx + \int_\Omega h(x)\varphi_1\,dx > 0,$$

which is a contradiction for $\lambda = \gamma_1$. Hence (P_λ) has no solution for $\lambda = \gamma_1$ which proves that we are in the first situation in Theorem 1.2. □

In order to consider the situation where (P_0) has a solution with $\min u < 0$, we need the following lemmas.

Lemma 4.2. *Under assumption* (A)*, for every* $\lambda \geq 0$*, there exists a strict lower solution* v_λ *of* (P_λ) *such that, every upper solution* β *of* (P_λ) *satisfies* $v_\lambda \leq \beta$.

Proof. Let $M > 0$ be given by Corollary 3.2 such that, for every upper solution β of

$$\begin{aligned} -\Delta u &= \lambda c(x)u + \mu(x)|\nabla u|^2 - h^-(x) - 1, && \text{in } \Omega,\\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

we have $\beta \geq -M$.

Let $k > M$ and consider α_k the solution of

$$\begin{aligned} -\Delta v &= -\lambda k c(x) - h^-(x) - 1, && \text{in } \Omega,\\ v &= 0, && \text{on } \partial\Omega. \end{aligned}$$

As $-\lambda k c(x) - h^-(x) - 1 < 0$, we have $\alpha_k \ll 0$ by the strong maximum principle (see [25, Theorem 3.27]).

Claim 1: Every upper solution β *of* (P_λ) *satisfies* $\beta \geq \alpha_k$. In fact, $\beta = \min\{\beta_j \mid 1 \leq j \leq l\}$ where $\beta_1, \dots, \beta_l$ are regular upper solutions of (P_λ). Setting $w = \beta_j - \alpha_k$ for some $1 \leq j \leq l$ we have

$$\begin{aligned} -\Delta w &\geq \lambda c(x)(\beta_j + k) \geq 0, && \text{in } \Omega,\\ w &= 0, && \text{on } \partial\Omega. \end{aligned}$$

By the maximum principle $w \geq 0$ i.e. $\beta_j \geq \alpha_k$. This proves the Claim.

Consider then the problem

$$\begin{aligned} -\Delta v &= \lambda c(x)T_k(v) + \mu(x)|\nabla v|^2 - h^-(x) - 1, && \text{in } \Omega,\\ v &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} T_k(v) &= -k, \quad \text{if } v \le -k, \\ &= v, \quad\;\; \text{if } v > -k. \end{aligned}$$

It is easy to prove that α_k and β are lower and upper solutions of (4.2) and hence, by Theorem 2.1, this problem has a minimal solution v_k with $\alpha_k \le v_k \le \beta$.

Claim 2: Every upper solution β of (P_λ) satisfies $\beta \ge v_k$. Observe that, by construction of (4.2), every upper solution β of (P_λ) is also an upper solution of (4.2). As, by Claim 1, we have $\beta \ge \alpha_k$, the minimality of v_k implies that $v_k \le \beta$.

Claim 3: v_k is a lower solution of (P_λ). Observe that v_k is an upper solution of (4.1). Hence $v_k \ge -M > -k$ and v_k satisfies

$$\begin{aligned} -\Delta v = \lambda c(x) v + \mu(x)|\nabla v|^2 - h^-(x) - 1, &\quad \text{in } \Omega, \\ v = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

This implies that v_k is a lower solution of (P_λ).

Claim 4: v_k is a strict lower solution of (P_λ). Let u be a solution of (P_λ) with $u \ge v_k$. Then $w = u - v_k$ satisfies

$$\begin{aligned} -\Delta w - \mu(x)\langle \nabla u + \nabla v_k \mid \nabla w\rangle \ge \lambda c(x) w + h^+(x) + 1 \ge 1, &\quad \text{in } \Omega, \\ w = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

By the strong maximum principle (see [25, Theorem 3.27]), we deduce that $w \gg 0$ i.e. $u \gg v_k$. □

Remark 4.1. Lemma 4.2 shows that, for (P_0), to have an upper solution is equivalent to have a solution.

Proof of Theorem 1.3. We proceed in several steps.

Step 1: For all $\varepsilon > 0$, there exists $R > 0$ such that $\deg(I - \mathcal{M}_0, \mathcal{S}) = 1$ with

$$\mathcal{S} = \{u \in C_0^1(\overline{\Omega}) \mid u_0 - \varepsilon \ll u \ll u_0 + \varepsilon,\ \|u\|_{C^1} < R\}.$$

It is easy to prove that $u_0 - \varepsilon$ and $u_0 + \varepsilon$ are lower and upper solutions of (P_0). Moreover, as u_0 is the unique solution of (P_0), we deduce that $u_0 - \varepsilon$ and $u_0 + \varepsilon$ are strict lower and upper solutions of (P_0). The result then follows by Theorem 2.1.

Step 2: There exists a $\lambda_0 > 0$ such that $\deg(I - \mathcal{M}_\lambda, \mathcal{S}) = 1$ for $\lambda \in]0, \lambda_0[$ with $\mathcal{S}$ defined in Step 1.

Let us prove first the existence of $\lambda_0 > 0$ such that, for $\lambda \in]0, \lambda_0[$, (P_λ) has no solution on $\partial\mathcal{S}$. Otherwise, there exist a sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ and a corresponding sequence of solution $\{u_n\} \subset W^{2,p}(\Omega)$ of (P_λ) with $u_n \in \partial\mathcal{S}$. Increasing R if necessary, this means that $u_0 - \varepsilon \le u_n \le u_0 + \varepsilon$ and either $\max(u_n - u_0) = \varepsilon$ or $\min(u_n - u_0) = -\varepsilon$. By Lemma 2.4, there exists a $R > 0$

such that, for all $n \in \mathbb{N}$, $\|u_n\|_{W^{2,p}} < R$. Hence, up to a subsequence, $u_n \to u$ in $C_0^1(\overline{\Omega})$. From this strong convergence we easily observe that

$$\begin{aligned} -\Delta u &= \mu(x)|\nabla u|^2 + h(x), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned}$$

and either $\max(u - u_0) = \varepsilon$ or $\min(u - u_0) = -\varepsilon$ i.e. u is a solution of (P_0) with $u \in \partial\mathcal{S}$ which contradicts Step 1.

We conclude by the invariance by homotopy of the degree that

$$\deg(I - \mathcal{M}_\lambda, \mathcal{S}) = \deg(I - \mathcal{M}_0, \mathcal{S}) = 1.$$

Step 3: (P_λ) has two solutions when $\lambda \in]0, \lambda_0[$. By Step 2, the existence of a first solution $u_0 - \varepsilon \ll u_{\lambda,1} \ll u_0 + \varepsilon$ is proved.

Also, using Lemma 3.6, there exists $A_1 > 0$ large enough such that (3.3) has no solution for $a \geq A_1$. By Theorem 3.3 and Lemma 2.4 there exists a $R_0 > R > 0$ such that, for all $a \in [0, A_1]$, every solution of (3.3) satisfies $\|u\|_{C^1} < R_0$. Hence, by homotopy invariance of the degree, we have

$$\deg(I - \mathcal{M}_\lambda, B(0, R_0)) = \deg(I - \mathcal{M}_\lambda - \mathcal{L}^{-1}(A_1 c), B(0, R_0)).$$

As for $a = A_1$, the problem (3.3) has no solution, we have

$$\deg(I - \mathcal{M}_\lambda - \mathcal{L}^{-1}(A_1 c), B(0, R_0)) = 0.$$

We then conclude that

$$\deg(I - \mathcal{M}_\lambda, B(0, R_0) \setminus \mathcal{S}) = \deg(I - \mathcal{M}_\lambda, B(0, R_0)) - \deg(I - \mathcal{M}_\lambda, \mathcal{S}) = -1.$$

This proves the existence of a second solution $u_{\lambda,2}$ of (P_λ) with $u_{\lambda,2} \in B(0, R_0) \setminus \mathcal{S}$.

Step 4: Existence of $\overline{\lambda}$ such that, for all $\lambda \in]0, \overline{\lambda}[$, the problem (P_λ) has at least two solutions with $u_{\lambda,1} \ll u_{\lambda,2}$. Define

$$\overline{\lambda} = \sup\{\mu \mid \forall \lambda \in]0, \mu[,\ (P_\lambda) \text{ has at least two solutions}\}.$$

For $\lambda \in]0, \overline{\lambda}[$, (P_λ) has at least two solutions $u_{\lambda,1}$ and $u_{\lambda,2}$. Let us consider the strict lower solution α given by Lemma 4.2. As $\alpha \leq u$ for all u solution of (P_λ), we can choose $u_{\lambda,1}$ as the minimal solution with $u_{\lambda,1} \geq \alpha$. Hence we have $u_{\lambda,1} \lneqq u_{\lambda,2}$ as otherwise there exists a solution u with $\alpha \leq u \leq \min(u_{\lambda,1}, u_{\lambda,2})$ which contradicts the minimality of $u_{\lambda,1}$.

Now observe that, by convexity of $y \mapsto |y|^2$, the function $\beta = \frac{1}{2}(u_{\lambda,1} + u_{\lambda,2})$ is an upper solution of (P_λ) which is not a solution. Let us prove that β is a strict upper solution of (P_λ). Let u be a solution of (P_λ) with $u \leq \beta$. Then $v := \beta - u$ satisfies

$$\begin{aligned} -\Delta v - \mu(x)\langle \nabla\beta + \nabla u \mid \nabla v\rangle &\geq \lambda c(x) v \geq 0, && \text{in } \Omega, \\ v &\geq 0, && \text{in } \Omega. \end{aligned}$$

By the strong maximum principle, we deduce that either $v \gg 0$ or $v \equiv 0$. If $v \equiv 0$, then $\beta = u$ is solution which contradicts the construction of β. As

$u_{\lambda,1} \lneqq \beta \lneqq u_{\lambda,2}$, we deduce from the fact that β is strict that $u_{\lambda,1} \ll \beta \lneqq u_{\lambda,2}$ and hence we have proved the step.

Step 5: In case $\overline{\lambda} < +\infty$, the problem $(P_{\overline{\lambda}})$ has at least one solution u. Let $\{\lambda_n\} \subset]0, \overline{\lambda}[$ be a sequence such that $\lambda_n \to \overline{\lambda}$ and $\{u_n\} \subset W^{2,p}(\Omega)$ be a sequence of corresponding solutions. By Theorem 3.3, there exists a constant $M > 0$ such that, for all $n \in \mathbb{N}$, $\|u_n\|_\infty < M$ and, by Lemma 2.4, we have $R > 0$ such that, for all $n \in \mathbb{N}$, $\|u_n\|_{W^{2,p}} < R$. Hence, up to a subsequence, $u_n \to u$ in $C_0^1(\overline{\Omega})$. From this strong convergence we easily observe that

$$\begin{aligned} -\Delta u = \overline{\lambda} c(x) u + \mu(x)|\nabla u|^2 + h(x), &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega, \end{aligned}$$

namely $u \in W^{2,p}(\Omega)$ is a solution of $(P_{\overline{\lambda}})$.

Step 6: Uniqueness of the solution of $(P_{\overline{\lambda}})$ in case $\overline{\lambda} < +\infty$. Otherwise, if we have two distincts solutions u_1 and u_2 of $(P_{\overline{\lambda}})$, then, as in Step 4, we prove that $\beta = \frac{1}{2}(u_1 + u_2)$ is a strict upper solution of $(P_{\overline{\lambda}})$. Let us consider the strict lower solution $\alpha \ll \beta$ of $(P_{\overline{\lambda}})$ given by Lemma 4.2. By Theorem 2.1, we then have $R > 0$ such that

$$\deg(I - \mathcal{M}_{\overline{\lambda}}, \tilde{\mathcal{S}}) = 1,$$

where

$$\tilde{\mathcal{S}} = \{u \in C_0^1(\overline{\Omega}) \mid \alpha \ll u \ll \beta, \quad \|u\|_{C^1} < R\}.$$

Arguing as in Step 2, we prove the existence of $\varepsilon > 0$ such that, for all $\lambda \in [\overline{\lambda} - \varepsilon, \overline{\lambda} + \varepsilon]$, $\deg(I - \mathcal{M}_\lambda, \tilde{\mathcal{S}}) = 1$ and, as in Step 3, we prove that (P_λ) has at least two solutions for $\lambda \in [\overline{\lambda}, \overline{\lambda} + \varepsilon]$ which contradicts the definition of $\overline{\lambda}$.

Step 7: Behaviour of the solutions for $\lambda \to 0$. Let $\{\lambda_n\} \subset]0, \overline{\lambda}[$ be a decreasing sequence such that $\lambda_n \to 0$. Without loss of generality, we suppose $\lambda_n \in]0, \lambda_0[$. Then, by Steps 2 and 4, the corresponding solutions $u_{\lambda_n,1}$ satisfy $u_{\lambda_n,1} \leq u_0 + \varepsilon$. Recall that, by Corollary 3.2, there exists $M > 0$ such that, for all n, $u_{\lambda_n,1} \geq -M$. This implies that the sequence $\{u_{\lambda_n,1}\}$ is bounded in $C(\overline{\Omega})$. We argue then as in Step 5 to prove that $u_{\lambda_n,1} \to u$ in $C_0^1(\overline{\Omega})$ with u solution of (P_0). By uniqueness of the solution of (P_0), we deduce that $u = u_0$.

Now let us consider the sequence $\{u_{\lambda_n,2}\}$. If $\{u_{\lambda_n,2}\}$ is bounded, then as in Step 5, we have that $u_{\lambda_n,2} \to u$ in $C_0^1(\overline{\Omega})$ with u solution of (P_0). By Step 3 and the facts that $u_{\lambda_n,2} \notin \mathcal{S}$, $u_{\lambda_n,2} \gg u_{\lambda_n,1}$ and $u_{\lambda_n,1} \to u_0$, we know that $\max\{u_{\lambda_n,2} - u_0\} > \varepsilon$. This implies that $u \neq u_0$ which contradicts the uniqueness of the solution of (P_0). □

Remark 4.2. Observe that, by the above proof, we see that the set of λ for which the problem (P_λ) has at least two solutions is open in $]0, +\infty[$.

Proof of Theorem 1.4. We proceed in several steps.

Step 1: Every non-negative upper solution of (P_λ) *satisfies* $u \gg u_0$. If u is a non-negative upper solution of (P_λ) then u is an upper solution of (P_0). By Proposition 2.3 we deduce that $u \geq u_0$ and hence u is not a solution of (P_0). As in Step 4 of the proof of Theorem 1.3, we prove that $u \gg u_0$.

Step 2: The problem (P_λ) *has no non-negative solution for* λ *large.* Let $\varphi_1 > 0$ the first eigenfunction of (1.1). If (P_λ) has a non-negative solution, multiplying (P_λ) by $\varphi_1 > 0$ and integrating we obtain

$$\begin{aligned}\gamma_1 \int_\Omega c(x) u \varphi_1 \, dx &= -\int_\Omega \Delta u \varphi_1 \, dx \\ &= \lambda \int_\Omega c(x) u \varphi_1 \, dx + \int_\Omega \mu(x) |\nabla u|^2 \varphi_1 \, dx + \int_\Omega h(x) \varphi_1 \, dx,\end{aligned}$$

and hence, for $\lambda > \gamma_1$, as $u \geq u_0$, we have

$$\begin{aligned}0 &\geq (\lambda - \gamma_1) \int_\Omega c(x) u \varphi_1 \, dx + \int_\Omega \mu(x) |\nabla u|^2 \varphi_1 \, dx + \int_\Omega h(x) \varphi_1 \, dx \\ &\geq (\lambda - \gamma_1) \int_\Omega c(x) u_0 \varphi_1 \, dx + \int_\Omega \mu(x) |\nabla u|^2 \varphi_1 \, dx + \int_\Omega h(x) \varphi_1 \, dx,\end{aligned}$$

which gives a contradiction for λ large enough.

Step 3: Define $\overline{\lambda} = \sup\{\lambda \mid (P_\lambda) \text{ has a solution } u_\lambda \geq 0\}$*, then,* $\overline{\lambda} < +\infty$ *and, for all* $\lambda > \overline{\lambda}$*,* (P_λ) *has no non-negative solution.* This is obvious by definition of $\overline{\lambda}$ and Step 2.

Step 4: For all $0 < \lambda < \overline{\lambda}$*,* (P_λ) *has well ordered strict lower and upper solutions.* Observe that u_0 is a lower solution of (P_λ) which is not a solution. By definition of $\overline{\lambda}$, we can find $\tilde{\lambda} \in]\lambda, \overline{\lambda}[$ and a non-negative solution $u_{\tilde{\lambda}}$ of $(P_{\tilde{\lambda}})$. Then $u_{\tilde{\lambda}}$ is an upper solution of (P_λ) and satisfies $u_{\tilde{\lambda}} \gg u_0$ by Step 1. At this point following the arguments of Step 4 of the proof of Theorem 1.3, we prove that u_0 and $u_{\tilde{\lambda}}$ are strict lower and upper solutions of (P_λ).

Step 5: For all $\lambda \in]0, \overline{\lambda}[$*,* (P_λ) *has at least two positive solutions with* $u_0 \ll u_{\lambda,1} \ll u_{\lambda,2}$. By Step 4, Theorem 2.1 and Remark 2.2, we have $R_0 > 0$ such that $\deg(I - \mathcal{M}_\lambda, \mathcal{S}) = 1$ with

$$\mathcal{S} = \{u \in C_0^1(\overline{\Omega}) \mid u_0 \ll u \ll u_{\tilde{\lambda}}, \ \|u\|_{C^1} < R_0\},$$

and we have the existence of a first solution $u_{\lambda,1}$ of (P_λ) with $u_0 \leq u_{\lambda,1} \leq u_{\tilde{\lambda}}$. Let us choose $u_{\lambda,1}$ as the minimal solution between u_0 and $u_{\tilde{\lambda}}$.

Now, using Lemma 3.6, there exists $A_1 > 0$ large enough such that (3.3) has no solution for $a \geq A_1$. By Theorem 3.3 and Lemma 2.4 there exists $R_1 > R_0 >$

0 such that, for any $a \in [0, A_1]$, every solution of (3.3) with $u \geq u_0$ satisfies $\|u\|_{C^1} < R_1$. Hence, by homotopy invariance of the degree we have

$$\deg(I - \mathcal{M}_\lambda, \mathcal{D}) = \deg(I - \mathcal{M}_\lambda - \mathcal{L}^{-1}(A_1 c), \mathcal{D}),$$

where

$$\mathcal{D} = \{u \in C_0^1(\overline{\Omega}) \mid u_0 \ll u,\ \|u\|_{C^1} < R_1\}.$$

As for $a = A_1$, (3.3) has no solution, $\deg(I - \mathcal{M}_\lambda - \mathcal{L}^{-1}(A_1 c), \mathcal{D}) = 0$ and we obtain

$$\deg(I - \mathcal{M}_\lambda, \mathcal{D} \setminus \mathcal{S}) = \deg(I - \mathcal{M}_\lambda, \mathcal{D}) - \deg(I - \mathcal{M}_\lambda, \mathcal{S}) = 0 - 1 = -1.$$

This proves the existence of a second solution $u_{\lambda,2}$ of (P_λ) with $u_{\lambda,2} \gg u_0$. As $u_{\lambda,1}$ is the minimal solution between u_0 and $u_{\tilde{\lambda}}$, we have $u_{\lambda,1} \lneqq u_{\lambda,2}$ as otherwise, by Theorem 2.1, we have a solution u with $u_0 \leq u \leq \min\{u_{\lambda,1}, u_{\lambda,2}, u_{\tilde{\lambda}}\}$ which contradicts the minimality of $u_{\lambda,1}$. We proceed as in Step 4 of the proof of Theorem 1.3 to conclude that $u_{\lambda,1} \ll u_{\lambda,2}$.

Step 6: For $\lambda_1 < \lambda_2$, we have $u_{\lambda_1,1} \ll u_{\lambda_2,1}$. As $u_{\lambda,1}$ is the minimal solution above u_0 and, as in Step 4, $u_{\lambda_2,1}$ is a strict upper solution of (P_{λ_1}) with $u_{\lambda_2,1} \geq u_0$, we deduce that $u_{\lambda_1,1} \ll u_{\lambda_2,1}$.

Step 7: The problem $(P_{\overline{\lambda}})$ has at least one solution. Let $\{\lambda_n\} \subset]0, \overline{\lambda}[$ be a sequence such that $\lambda_n \to \overline{\lambda}$ and $\{u_n\} \subset W^{2,p}(\Omega)$ be a sequence of corresponding non negative solutions. We argue as in Step 5 of the proof of Theorem 1.3 to obtain that, up to a subsequence, $u_n \to u$ in $C_0^1(\overline{\Omega})$ with $u \in W^{2,p}(\Omega)$ solution of $(P_{\overline{\lambda}})$.

Step 8: Uniqueness of the non-negative solution of $(P_{\overline{\lambda}})$. The proof follows the lines of Step 6 of the proof of Theorem 1.3.

Step 9: Behaviour of the solutions for $\lambda \to 0$. This can be proved as in Step 7 of the proof of Theorem 1.3. □

Proposition 4.3. *Under assumption (A), assume that (P_0) has a solution $u_0 \leq 0$ with $cu_0 \lneqq 0$. Then, for all $\lambda \geq 0$, problem (P_λ) has at most one solution $u \leq 0$.*

Proof. The proof is divided in three steps.

Step 1: If u is a lower solution of (P_λ) with $u \leq 0$, then $u \ll u_0$. In fact, u is a lower solution of (P_0) and, by Proposition 2.3, we have $u \leq u_0$. In addition, for $w = u_0 - u$, as $cu \leq cu_0 \lneqq 0$, we have

$$\begin{aligned} -\Delta w - \mu(x)\langle \nabla u + \nabla u_0 \mid \nabla w \rangle = -\lambda c(x) u \gneqq 0, &\quad \text{in } \Omega, \\ w = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

This implies that $w \gg 0$ i.e. $u \ll u_0 \leq 0$.

Step 2: If we have two solutions $u_1 \leq 0$ and $u_2 \leq 0$ of (P_λ) then we have two ordered solutions $\tilde{u}_1 \lneqq \tilde{u}_2 \leq u_0$. By Step 1, we have $u_1 \ll u_0$ and $u_2 \ll u_0$. In case u_1 and u_2 are not ordered, as u_0 is an upper solution of (P_λ), applying

Theorem 2.1, there exists a solution u_3 of (P_λ) with $\max\{u_1, u_2\} \leq u_3 \leq u_0$. This proves the step by choosing $\tilde{u}_1 = u_1$ and $\tilde{u}_2 = u_3$.

Step 3: Conclusion. Let us assume by contradiction that we have two solutions $u_1 \leq 0$ and $u_2 \leq 0$. By Step 2, we can suppose $u_1 \lneqq u_2$. As $|u_2| \gg 0$, the set $\{v \in C_0^1(\overline{\Omega}) \mid v \leq |u_2|\}$ is an open neighborhood of 0 and hence the set $\{\varepsilon > 0 \mid u_2 - u_1 \leq \varepsilon |u_2|\}$ is not empty. Then defining

$$\bar{\varepsilon} = \inf\{\varepsilon > 0 \mid u_2 - u_1 \leq \varepsilon |u_2|\},$$

we have that $0 < \bar{\varepsilon} < \infty$ and

$$\bar{\varepsilon} = \min\{\varepsilon > 0 \mid u_2 - u_1 \leq \varepsilon |u_2|\}. \tag{4.3}$$

Letting

$$w_{\bar{\varepsilon}} = \frac{(1+\bar{\varepsilon})u_2 - u_1}{\bar{\varepsilon}},$$

we can write

$$\nabla u_2 = (\frac{\bar{\varepsilon}}{1+\bar{\varepsilon}})\nabla w_{\bar{\varepsilon}} + (\frac{1}{1+\bar{\varepsilon}})\nabla u_1,$$

and by convexity

$$|\nabla u_2|^2 \leq (\frac{\bar{\varepsilon}}{1+\bar{\varepsilon}})|\nabla w_{\bar{\varepsilon}}|^2 + (\frac{1}{1+\bar{\varepsilon}})|\nabla u_1|^2.$$

We then obtain

$$-\Delta w_{\bar{\varepsilon}} \leq \lambda c(x) w_{\bar{\varepsilon}} + \mu(x)|\nabla w_{\bar{\varepsilon}}|^2 + h(x).$$

By the choice of $\bar{\varepsilon} > 0$, $w_{\bar{\varepsilon}} \leq 0$ and, by Step 1, $w_{\bar{\varepsilon}} \ll u_0 \leq 0$. At this point, we have a contradiction with the definition of $\bar{\varepsilon}$ given in (4.3). □

Our next result can be viewed as a generalization of [3, Theorem 3.12].

Corollary 4.4. *Under assumption (A), assume that $h \lneqq 0$. Then, for all $\lambda > 0$, the problem (P_λ) has exactly one solution $u \leq 0$.*

Proof. Clearly $u \equiv 0$ is an upper solution of (P_λ) for all $\lambda \geq 0$. By Lemma 4.2, for all $\lambda \geq 0$, (P_λ) has a lower solution $\alpha_\lambda \leq 0$. From Theorem 2.1 it follows that (P_λ) has a solution u_λ with $\alpha_\lambda \leq u_\lambda \leq 0$. Now, as u_0 satisfies

$$-\Delta u_0 = \mu(x)|\nabla u_0|^2 + h(x),$$

the strong maximum principle and $h \lneqq 0$, implies that $u_0 \ll 0$ and in particular $cu_0 \lneqq 0$. We now conclude with Proposition 4.3. □

Proof of Theorem 1.5. We proceed in several steps.

Step 1: For all $\lambda > 0$, u_0 is a strict upper solution of (P_λ). Clearly u_0 is an upper solution of (P_λ) which is not a solution. To prove that it is a strict upper solution, we argue as in Step 4 of the proof of Theorem 1.3.

Step 2: For all $\lambda > 0$, (P_λ) has a strict lower solution α with $\alpha \leq \beta$ for all upper solution β of (P_λ). This is Lemma 4.2.

Step 3: For all $\lambda > 0$, (P_λ) has at least two solutions with

$$u_{\lambda,1} \ll u_0, \quad u_{\lambda,1} \ll u_{\lambda,2} \quad \text{and} \quad \max u_{\lambda,2} > 0.$$

By Steps 1, 2 and Theorem 2.1, there exists a $R > 0$ such that $\deg(I - \mathcal{M}_\lambda, \mathcal{S}) = 1$ with

$$\mathcal{S} = \{u \in C_0^1(\overline{\Omega}) \mid \alpha \ll u \ll u_0, \|u\|_{C^1} < R\}.$$

In particular the existence of a first solution $u_{\lambda,1} \ll u_0$ is proved.

The proof of the existence of a second solution $u_{\lambda,2}$ with $u_{\lambda,1} \ll u_{\lambda,2}$ is derived exactly as in Step 3 and 4 of the proof of Theorem 1.3. By Proposition 4.3, we have $\max u_{\lambda,2} > 0$.

Step 4: If $\lambda_1 < \lambda_2$, then $u_{\lambda_1,1} \gg u_{\lambda_2,1}$. As $u_{\lambda_1,1}$ is a strict upper solution of (P_{λ_2}) and $u_{\lambda_2,1}$ is the minimal solution of (P_{λ_2}), we have $u_{\lambda_1,1} \gg u_{\lambda_2,1}$.

Step 5: Behaviour of the solutions for $\lambda \to 0$. This can be proved as in Step 7 of the proof of Theorem 1.3. □

Proof of Corollary 1.6. By the proof of Corollary 4.4, as $h \lneqq 0$, we have the existence of a solution u_0 of (P_0) with $u_0 \ll 0$ and hence the result follows by Theorem 1.5. □

Proof of Theorem 1.7. First observe that if (P_λ) has an upper solution $\beta_\lambda \leq 0$, then β_λ satisfies also $c\beta_\lambda \lneqq 0$ as otherwise, it is also an upper solution of (P_0), which contradicts the assumption (a) by Lemma 4.2 and Theorem 2.1.

Let us define

$$\underline{\lambda} = \inf\{\lambda \geq 0 \mid (P_\lambda) \text{ has an upper solution } \beta_\lambda \leq 0 \text{ with } c\beta_\lambda \lneqq 0\}.$$

Let $\lambda > \underline{\lambda}$. By definition of $\underline{\lambda}$, there exists $\tilde{\lambda} \in]\underline{\lambda}, \lambda[$ such that $(P_{\tilde{\lambda}})$ has an upper solution $\beta_{\tilde{\lambda}} \leq 0$ with $c\beta_{\tilde{\lambda}} \lneqq 0$. Clearly $\beta_{\tilde{\lambda}}$ is an upper solution of (P_λ) which is not a solution and hence, as in Step 4 of the proof of Theorem 1.3, β is a strict upper solution of (P_λ).

By Lemma 4.2, (P_λ) has a strict lower solution $\alpha \leq \beta_{\tilde{\lambda}}$ and $\alpha \leq u$ for all solution u of (P_λ). Using Theorem 2.1 there exists $R > 0$ such that $\deg(I - \mathcal{M}_\lambda, \mathcal{S}) = 1$ with

$$\mathcal{S} = \{u \in C_0^1(\overline{\Omega}) \mid \alpha \ll u \ll \beta_{\tilde{\lambda}}, \|u\|_{C^1} < R\}.$$

In particular the existence of a first solution $u_{\lambda,1} \ll 0$ follows.

To obtain a second solution $u_{\lambda,2}$ satisfying $u_{\lambda,1} \ll u_{\lambda,2}$ we now just repeat the arguments of Steps 3 and 4 of the proof of Theorem 1.3.

Again, following the arguments of Step 4 of the proof of Theorem 1.5, we prove that if $\lambda_1 < \lambda_2$, then $u_{\lambda_1,1} \gg u_{\lambda_2,1}$.

To show that $(P_{\underline{\lambda}})$ has at least one solution with $u \leq 0$, let $\{\lambda_n\} \subset]\underline{\lambda}, +\infty[$ be a decreasing sequence such that $\lambda_n \to \underline{\lambda}$ and $\{u_n\} \subset W^{2,p}(\Omega)$ be a sequence of corresponding solutions with $u_n \leq u_{n+1} \leq 0$. As $\{u_n\}$ is increasing and bounded above, there exists $M > 0$ such that, for all $n \in \mathbb{N}$, $\|u_n\|_\infty < M$ and hence, arguing as in Step 5 of the proof of Theorem 1.3, we prove that $(P_{\underline{\lambda}})$ has at least one solution with $u \leq 0$.

By assumption (a), we have that $\underline{\lambda} > 0$ as we just proved that $(P_{\underline{\lambda}})$ has at least one solution with $u \leq 0$. The proof of the uniqueness of the non-positive solution of $(P_{\underline{\lambda}})$ follows then as in Step 6 of the proof of Theorem 1.3. Finally (iii) follows by definition of $\underline{\lambda} > 0$ and the first part of the proof. $\square$

Proof of Theorem 1.8. Let $\lambda > \nu_1$. We proceed in several steps.

Step 1: For $k > 0$ small, $(Q_{\lambda,k})$ admits a solution. In view of Lemma 4.2 and of Theorem 2.1 it suffices to show that $(Q_{\lambda,k})$ admits an upper solution.

Let $\varepsilon_0 > 0$ be given by Proposition 2.6 corresponding to $\bar{c} = c$, $\bar{d} = \mu_2 \tilde{h}^-$ and $\bar{h} = \mu_2 \tilde{h}^+$ and choose $\lambda_0 \in]\nu_1, \min(\nu_1 + \varepsilon_0, \nu_1 + \frac{\lambda - \nu_1}{2})]$. Then let $w \ll 0$ be the solution of

$$\begin{aligned} -\Delta u + \mu_2 \tilde{h}^-(x) u &= \lambda_0 c(x) u + \mu_2 \tilde{h}^+(x), && \text{in } \Omega,\\ u &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Also, taking $\delta > 0$ small enough, we have that

$$\lambda_0 s \geq (1 + \lambda s) \ln(1 + \lambda s),$$

for all $s \in [-\delta, 0]$. Thus, defining $\tilde{\beta}_k = \frac{k}{\lambda} w$ for $k > 0$ small enough, it follows that $\tilde{\beta}_k \in [-\delta, 0]$ and

$$\begin{aligned} -\Delta \tilde{\beta}_k + \mu_2 \tilde{h}^-(x) \tilde{\beta}_k &\geq c(x)(1 + \lambda \tilde{\beta}_k) \ln(1 + \lambda \tilde{\beta}_k) + k \tfrac{\mu_2}{\lambda} \tilde{h}^+(x), && \text{in } \Omega,\\ \tilde{\beta}_k &= 0, && \text{on } \partial\Omega. \end{aligned}$$

At this point defining $\beta_k = \frac{1}{\mu_2} \ln(\lambda \tilde{\beta}_k + 1)$ we see, after some standard calculations, that $\beta_k \ll 0$ is an upper solution for $(Q_{\lambda,k})$.

Step 2: For k large, the problem $(Q_{\lambda,k})$ has no solution. Let $\phi \in C_0^\infty(\Omega)$ such that $\phi^2 \gg 0$. Then, using ϕ^2 as test function we obtain, by Lemma 3.1,

$$\begin{aligned} \int_\Omega \frac{1}{\mu(x)} |\nabla \phi|^2 \, dx &\geq 2 \int_\Omega \phi \langle \nabla u, \nabla \phi \rangle \, dx - \int_\Omega \mu(x) |\nabla u|^2 \phi^2 \, dx \\ &= \lambda \int_\Omega c(x) u \phi^2 \, dx + k \int_\Omega \tilde{h}^+(x) \phi^2 \, dx - \int_\Omega \tilde{h}^-(x) \phi^2 \, dx \\ &\geq -\lambda M \int_\Omega c(x) \phi^2 \, dx + k \int_\Omega \tilde{h}^+(x) \phi^2 \, dx - \int_\Omega \tilde{h}^-(x) \phi^2 \, dx, \end{aligned}$$

which is a contradiction for $k > 0$ large enough.

Step 3: Define

$$\overline{k} = \sup\{k \in]0, +\infty[\mid \textit{the problem } (Q_{\lambda,k}) \textit{ has at least one solution}\},$$

then $\overline{k} \in]0, +\infty[$ *and for* $k \in]0, \overline{k}[$, *the problem* $(Q_{\lambda,k})$ *has a strict upper solution.*

By Step 1 and 2 we have easily $\overline{k} \in]0, +\infty[$.

Let $k \in]0, \overline{k}[$ and $\tilde{k} \in]k, \overline{k}[$ be such that $(Q_{\lambda,\tilde{k}})$ has a solution $\tilde{\beta}$. Then $\beta = \frac{k}{\tilde{k}}\tilde{\beta}$ is an upper solution of $(Q_{\lambda,k})$ as

$$\begin{aligned} -\Delta\beta &= \lambda c(x)\beta + \tfrac{\tilde{k}}{k}\mu(x)|\nabla\beta|^2 + k\tilde{h}^+(x) - \tfrac{k}{\tilde{k}}\tilde{h}^-(x) \\ &\geq \lambda c(x)\beta + \mu(x)|\nabla\beta|^2 + k\tilde{h}^+(x) - \tilde{h}^-(x), \qquad \text{in } \Omega, \\ &\beta \geq 0, \qquad \text{on } \partial\Omega, \end{aligned}$$

i.e. β is an upper solution of $(Q_{\lambda,k})$. Now, as in Step 4 of the proof of Theorem 1.4 we can prove that β is a strict upper solution of $(Q_{\lambda,k})$.

Step 4: Conclusion. At this point the proof follows as in the proof of Theorems 1.4 or 1.5. This is possible in view of Step 2 and of Theorem 3.3. □

Proof of Corollary 1.9. First observe that, by [4, Lemma 6.1] (see also the proof of Corollary 4.1 above), we know that (P_{γ_1}) has no solution. Hence also, for all $\lambda > 0$, (P_λ) has no solution with $cu \equiv 0$ as otherwise u is solution for every $\lambda \in \mathbb{R}$ which contradicts the non existence of a solution for $\lambda = \gamma_1$.

By Step 3 of the proof of Theorem 1.8, there exists $\tilde{k} > 0$ such that, for all $k \in]0, \tilde{k}]$, the problem $(P_{\tilde{\lambda}})$ has a strict upper solution β_0 with $\beta_0 \ll 0$. The existence of $\lambda_2 > \gamma_1$ as in (iii) can then be deduced from Theorem 1.7.

By [4, Theorem 1.1], decreasing $\tilde{k}$ if necessary, we know that for all $k \in]0, \tilde{k}]$, the problem (P_0) has a solution $u_0 \gg 0$. Hence the existence of λ_1 as in (i) can be deduced from Theorem 1.4. □

Proof of Theorem 1.10. First observe that, for all $\lambda \in \mathbb{R}$, $u \equiv 0$ is solution of (1.5).

Step 1: for all $\lambda \in]0, \gamma_1[$, *the problem* (1.5) *has a second solution* $u_{\lambda,2} \gneqq 0$. Let us prove that the problem (1.5) has a strict upper solution $\beta \gg 0$. To this end, let $\lambda < \gamma_1$ and $\varepsilon > 0$ such that, for all $v \in [0, \varepsilon]$, $\lambda\frac{(1+\mu_2 v)\ln(1+\mu_2 v)}{\mu_2} \leq \gamma_1 v$. Consider then the function $\tilde{\beta} = \varepsilon\varphi_1$ where φ_1 denotes the first eigenfunction of (1.1) with $\|\varphi_1\|_\infty = 1$ and observe that

$$\begin{aligned} -\Delta\tilde{\beta} &\gneqq \lambda c(x)\frac{(1+\mu_2\tilde{\beta})\ln(1+\mu_2\tilde{\beta})}{\mu_2}, \quad \text{a.e. in } \Omega, \\ \tilde{\beta} &= 0, \qquad \text{on } \partial\Omega. \end{aligned}$$

Hence for β being defined by $\beta = \frac{\ln(\mu_2\tilde{\beta}+1)}{\mu_2}$, we have

$$\begin{aligned} -\Delta\beta \gneqq \lambda c(x)\beta + \mu_2|\nabla\beta|^2 \geq \lambda c(x)\beta + \mu(x)|\nabla\beta|^2, &\quad \text{a.e. in } \Omega,\\ \beta = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

This implies, as in Step 4 of the proof of Theorem 1.3, that $\beta \gg 0$ is a strict upper solution of (1.5).

By [4, Lemma 6.1], we know that, every solution u of (1.5) satisfies $u \geq 0$ and by Lemma 4.2, the problem (1.5) has a strict lower solution $\alpha \lneqq 0$. Hence we conclude the proof of (i) following the same lines as in the proof of Theorem 1.4, the solution $u_{\lambda,1}$ being $u \equiv 0$.

Step 2: For $\lambda = \gamma_1$ the problem (1.5) *has only the trivial solution.* This can be proved as in Corollary 4.1.

Step 3: For $\lambda > \gamma_1$, the problem (1.5) *has a second solution $u_{\lambda,2} \ll 0$.* Let $\lambda > \gamma_1$ and $\lambda_0 \in]\gamma_1, \lambda]$ such that, by Proposition 2.6, the problem

$$-\Delta u = \lambda_0 c(x)u + 1, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{4.4}$$

has a solution $u \ll 0$. This implies that for $\varepsilon > 0$ small enough, the function $\beta_0 = \varepsilon u$ satisfies

$$-\Delta\beta_0 = \lambda_0 c(x)\beta_0 + \varepsilon \geq \lambda_0 c(x)\beta_0 + \mu\varepsilon^2|\nabla u|^2 = \lambda_0 c(x)\beta_0 + \mu|\nabla\beta_0|^2,$$

and the problem (P_{λ_0}) has an upper solution β_0 with $\beta_0 \leq 0$ and $c\beta_0 \lneqq 0$. The result follows by Theorem 1.7. □

5. Complement in case μ constant

In that case it is possible to precise the blow-up rate, as $\lambda \to 0^+$, of our solutions $u_{\lambda,2}$ obtained in Theorems 1.3, 1.4, 1.5 and 1.10.

Proposition 5.1. *Assume that (A) holds with μ a positive constant and the problem (P_0) has a solution. Moreover, assume the existence of a sequence $\{\lambda_n\} \subset]0, +\infty[$ with $\lambda_n \to 0$ and two sequences $\{u_{\lambda_n}\}$, $\{\tilde{u}_{\lambda_n}\}$ of solutions of (P_{λ_n}) such that*

$$\lambda_n\|u_{\lambda_n}\|_\infty \to 0 \quad \text{and} \quad \lambda_n\|\tilde{u}_{\lambda_n}\|_\infty \to 0,$$

as $\lambda_n \to 0$. Then, for any $n \in \mathbb{N}$ sufficiently large, $u_{\lambda_n} = \tilde{u}_{\lambda_n}$.

Proof. First we recall that if (P_0) has a solution then, by [4, Remark 3.2]

$$\inf_{\{u\in H_0^1(\Omega) \,|\, \|u\|_{H_0^1(\Omega)}=1\}} \int_\Omega \left(|\nabla u|^2 - \mu h(x)u^2\right) dx > 0.$$

Hence in particular $\xi_1(c) > 0$ where $\xi_1(c)$ is the first eigenvalue of the problem

$$\begin{aligned} -\Delta w - \mu h(x)w = \xi\, c(x)w, &\quad \text{in } \Omega,\\ w = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

Now, if u_n is a solution of (P_{λ_n}), by the change of variable $u_n = \frac{1}{\mu}\ln(v_n+1)$ we have that $v_n > -1$ is solution of

$$
(5.1) \qquad \begin{array}{rcll} -\Delta v_n - \mu h(x) v_n &=& \lambda_n\, c(x)\,(1+v_n)\ln(1+v_n) + \mu h(x), & \text{in } \Omega, \\ v_n &=& 0, & \text{on } \partial\Omega. \end{array}
$$

Setting $D(\lambda_n) := \|u_n\|_\infty$, since $v_n = e^{\mu u_n} - 1$ we deduce that $\|v_n\|_\infty \leq C(\lambda_n)$ where

$$
C(\lambda_n) = e^{\mu D(\lambda_n)} - 1.
$$

If we assume that $\lambda_n D(\lambda_n) \to 0$, then

$$
\lim_{\lambda_n \to 0} \lambda_n (\ln(1 + C(\lambda_n)) + 1) = \lim_{\lambda_n \to 0} \lambda_n D(\lambda_n) = 0.
$$

Since $\xi_1(c) > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$
\lambda_n (\ln(1 + C(\lambda_n)) + 1) < \xi_1(c).
$$

If we assume by contradiction that, for $n \geq n_0$, $u_{\lambda_n} \neq \tilde{u}_{\lambda_n}$ then (5.1) has also two distinct solutions $v_{n,1}$ and $v_{n,2}$ and $w_n = v_{n,1} - v_{n,2}$ is a solution of

$$
(5.2) \qquad \begin{array}{rcll} -\Delta w - \mu h(x) w &=& \lambda_n\, c(x) \rho_n(x) w, & \text{in } \Omega, \\ w &=& 0, & \text{on } \partial\Omega \end{array}
$$

with

$$
\begin{array}{rcll} \rho_n(x) &=& \dfrac{(1+v_{n,1}(x))\ln(1+v_{n,1}(x)) - (1+v_{n,2}(x))\ln(1+v_{n,2}(x))}{v_{n,1}(x) - v_{n,2}(x)}, & \\ & & & \text{if } v_{n,1}(x) \neq v_{n,2}(x), \\ &=& \ln(1+v_{n,1}(x)) + 1, & \text{if } v_{n,1}(x) = v_{n,2}(x), \end{array}
$$

and by assumption $0 < \lambda_n \rho_n < \xi_1(c)$.

As (5.2) has a nontrivial solution, we have $\xi_i(\lambda_n c \rho_n) = 1$ for some $i \in \mathbb{N}$. Moreover, as $\lambda_n \rho_n < \xi_1(c)$, we know by [14] that $1 = \xi_i(\lambda_n c \rho_n) > \xi_i(c \xi_1(c)) = \xi_i(c)/\xi_1(c)$. This contradicts that the sequence of eigenvalues $(\xi_i(c))_i$ is strictly increasing and proves the proposition. □

Under the assumption that μ is constant, the following lemma gives informations on the set of solutions of (P_λ) for $\lambda > 0$ small.

Corollary 5.2. *Assume that assumption (A) holds with μ a positive constant and that (P_0) has a solution u_0. Let $\{u_{\lambda_n}\}$ be a sequence of solutions of (P_{λ_n}) satisfying $\lambda_n \|u_{\lambda_n}\|_\infty \to 0$ as $\lambda_n \to 0^+$. Then we have, for any $n \in \mathbb{N}$ sufficiently large,*

(i) *$u_{\lambda_n} = u_{\lambda_n,1}$ where $u_{\lambda_n,1}$ is the minimal solution given in Theorem 1.3. In particular $u_{\lambda_n} \to u_0$ in $C_0^1(\overline{\Omega})$.*

(ii) *$(\lambda_n, u_{\lambda_n})$ belongs to $\mathcal{C}$ where $\mathcal{C}$ is defined in Theorem 1.1.*

Proof. Since $\{u_{\lambda_n,1}\}$ satisfies $\lambda_n\|u_{\lambda_n,1}\|_\infty \to 0$ as $\lambda_n \to 0$ it directly follows from Proposition 5.1 that, for any $n \in \mathbb{N}$ large enough, $u_{\lambda_n} = u_{\lambda_n,1}$. In particular it follows from Theorem 1.3 that $u_{\lambda_n} \to u_0$ in $C_0^1(\overline{\Omega})$. Now by Theorem 1.1 we know that, for $n \in \mathbb{N}$ large enough, there exists $\overline{u}_{\lambda_n}$ such that $(\lambda_n, \overline{u}_{\lambda_n}) \in \mathcal{C}$. Since, by continuity, we have that $\lambda_n\|\overline{u}_{\lambda_n}\|_\infty \to 0$ we deduce by (i) that $\overline{u}_{\lambda_n} = u_{\lambda_n,1}$. Thus $u_{\lambda_n} = \overline{u}_{\lambda_n}$. □

Also using again that $\lambda_n\|u_{\lambda_n,1}\|_\infty \to 0$ as $\lambda_n \to 0$, we immediately deduce from Proposition 5.1 the following result.

Corollary 5.3. *Assume that* (A) *holds with* μ *a positive constant.*

(i) *In Theorems 1.3, 1.4, 1.5 and 1.10 we have that*

$$\liminf_{\lambda\to 0^+} \lambda\|u_{\lambda,2}\|_\infty > 0.$$

(ii) *The bound derived in Theorem 3.3,* $\|u\|_\infty \leq M(\lambda)$ *for any solution* u *of* (3.3) *with*

$$\limsup_{\lambda\to 0^+} M(\lambda)\lambda \leq C,$$

for some $C > 0$ *is sharp.*

6. Case $N = 1$ and open problems

In case $\Omega = [-\frac{T}{2}, \frac{T}{2}]$ i.e. $N = 1$ and $\mu > 0$, $c > 0$ and $h \neq 0$ are constants, we can make a more precise study of the situation.

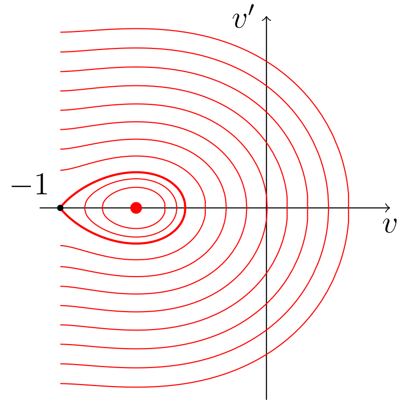
By the classical change of variable $v = e^{\mu u} - 1$, we are reduce to the problem

$$\begin{gathered} -v'' - \mu h v = \lambda(v+1)\ln(v+1) + \mu h, \quad \text{in } [-\tfrac{T}{2}, \tfrac{T}{2}] \\ v > -1, \quad \text{in } [-\tfrac{T}{2}, \tfrac{T}{2}] \\ v(-\tfrac{T}{2}) = 0, \quad v(\tfrac{T}{2}) = 0. \end{gathered} \tag{6.1}$$

It is easy to prove that in case $\lambda = 0$ this problem has a solution if and only if $\mu\, h < (\pi/T)^2$ which corresponds to the condition (1.2).

As this problem is autonomous, we can make a phase-plane analysis. There are three different situations: $h > 0$ and $\lambda > 0$ small, $h > 0$ and λ large, $h < 0$.

Case 1: $0 < \lambda < 2\mu h$**.** In that case the phase plane is given by



We then see that the only possibility is to have positive solutions. Moreover considering the time map $T_+(a)$ which gives the time for the positive part of the orbit to go from $(0, a)$ to $(0, -a)$ with $a > 0$, it is easy to prove that

$$\lim_{a \to 0} T_+(a) = 0 \quad \text{and} \quad \lim_{a \to +\infty} T_+(a) = 0.$$

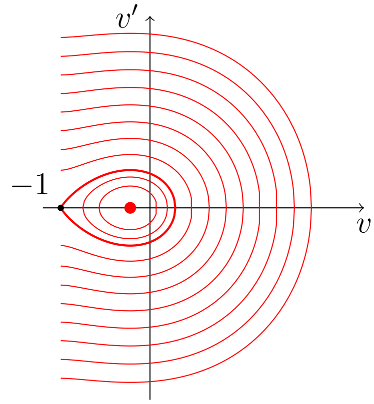
This implies the existence of $T_0 > 0$ such that, for all $T < T_0$, the problem (6.1) has two solutions and, for $T > T_0$ the problem (6.1) has no solution. Numerical experiment shows that the count is exact.

This corresponds to what we prove in Theorem 1.4 together with [4, Lemma 6.1] where it is shown that, in case $h \gneqq 0$, for all $\lambda < \gamma_1$, every solution of (P_λ) is non-negative.

Open problem 1 Can we prove that, for all $\lambda < \gamma_1$, every solution of (P_λ) is non-negative under the sole condition that (P_0) has a solution u_0 with $u_0 \geq 0$ and $cu_0 \gneqq 0$?

Open problem 2 Can we prove, under the assumptions of Theorem 1.4 or even under the assumptions of Theorem 1.3, that, for all $\lambda < \gamma_1$, we have at most two solutions?

Case 2: $\lambda > 2\mu h > 0$. In that case the phase plane is richer and is given by



We see the possibilities of positive solutions but also of negative or sign-changing ones.

We can prove that if $\mu\, h \geq (\pi/T)^2$ or $\lambda \geq (\pi/T)^2$ then the problem (6.1) has no non-negative solutions i.e. the time $T_+(a)$ for the positive part of the orbit to go from $(0, a)$ to $(0, -a)$ with $a > 0$ is too short with respect to the length of the interval we consider.

For what concerns negative or sign-changing solutions, we see that, if we denote by T_0 the time needed by the solution with $\max_{]-\frac{T}{2},\frac{T}{2}[} u = 0$ to make a turn in the phase plane, then for $T > T_0$, there is a negative solution as well as a sign-changing one. This is the situation studied in Theorem 1.7.

But for $T > kT_0$ we have also solutions making k turns in the phase plane.

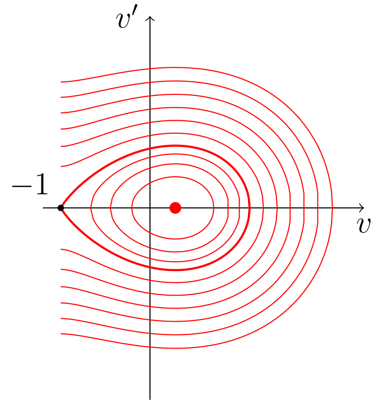
Open problem 3 Can we prove in Theorem 1.7 that the second solution changes sign?

Open problem 4 Can we prove that in a small interval below $\underline{\lambda}$ in Theorem 1.7, the problem (P_λ) has no solution and that $u_{\underline{\lambda}} \leq 0$ but $u_{\underline{\lambda}} \not\ll 0$?

Open problem 5 Can we prove the existence of more then two solutions for λ large? Is there a link with the spectrum of the problem

$$-\Delta\varphi_1 = \gamma c(x)\varphi_1, \quad \varphi_1 \in H_0^1(\Omega)? \tag{6.2}$$

Case 3: $h < 0$. In that case, the phase portrait is given by



and we see that we have always a negative solution. Moreover, if we denote by T_1 the time needed by the solution with $\min_{]-\frac{T}{2},\frac{T}{2}[} u = 0$ to make a turn in the phase plane, then, for $T < T_1$ the problem (6.1) has a positive solution (as again, considering the time map $T_+(a)$ which gives the time for the positive part of the orbit to go from $(0, a)$ to $(0, -a)$ with $a > 0$, we have $\lim_{a\to+\infty} T_+(a) = 0$) and for $T > T_1$ we have a sign-changing solution. This is the situation considered in Theorem 1.5.

Open problem 6 Can we prove in Theorem 1.5 that the second solution is positive for $\lambda > 0$ small and changes sign for λ large?

Moreover, for $T > kT_1$ we have also solutions making k turns in the phase plane.

Open problem 7 As in open problem 5, can we prove the existence of more than two solutions for λ large?

In addition to the above open problems directly induced by the phase plane analysis, we also propose the following questions.

Open problem 8 Can we give a more precise characterization of the situation in case h changes sign or u_0 changes sign?

Open problem 9 In [24] some a priori bounds for non-negative solutions have been derived without assuming that $\mu(x) \geq \mu_1 > 0$. Can a similar result be obtained in the general case ?

Open problem 10 In [4], the results are obtained under less regularity assumptions (c, $h \in L^p(\Omega)$ with $p > N/2$). In [3], the regularity is even weaker. If some of our results are still valid when (A) is weakened, how dependent is the structure of the set of solutions of our regularity assumption ?

7. Appendix : Proof of Theorem 2.1.

Let us denote $\alpha := \max\{\alpha_i \mid 1 \le i \le k\}$ where $\alpha_1, \dots, \alpha_k$ are regular lower solutions of (2.1) and $\beta = \min\{\beta_j \mid 1 \le j \le l\}$ where $\beta_1, \dots, \beta_l$ are regular upper solutions of (2.1). The proof is divided into three parts.

Part 1. Existence of a solution u of (2.1) *with $\alpha \le u \le \beta$.* Observe that by Lemma 2.4, there exist $R > 0$ such that, for every function f satisfying (2.4) and every solution u of (2.1) with $\alpha \le u \le \beta$, we have

$$\|u\|_{W^{2,p}} < R \qquad \text{and} \qquad \|u\|_{C^1_0} < R. \tag{7.1}$$

Step 1. Construction of a modified problem. Take $\overline{R}$ such that

$$\overline{R} > \max\{R, \max_{1\le i\le k} \|\alpha_i\|_{C^1}, \max_{1\le j\le l} \|\beta_j\|_{C^1}\},$$

and set, for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\overline{f}(x, s, \xi) = \begin{cases} f(x, s, \xi), & \text{if } |\xi| \le \overline{R}, \\ f(x, s, \overline{R}\frac{\xi}{|\xi|}), & \text{if } |\xi| > \overline{R}. \end{cases}$$

Now we define the functions

$$p_i(x, s, \xi) = \begin{cases} \overline{f}(x, \alpha_i(x), \xi) + \omega_{1,i}(x, \alpha_i(x) - s), & \text{if } s < \alpha_i(x), \\ \overline{f}(x, s, \xi), & \text{if } s \ge \alpha_i(x), \end{cases}$$

where

$$\omega_{1,i}(x, \delta) = \max_{|\xi|\le\delta} |\overline{f}(x, \alpha_i(x), \nabla\alpha_i(x) + \xi) - \overline{f}(x, \alpha_i(x), \nabla\alpha_i(x))|,$$

and

$$q_j(x, s, \xi) = \begin{cases} \overline{f}(x, \beta_j(x), \xi) - \omega_{2,j}(x, s - \beta_j(x)), & \text{if } s > \beta_j(x), \\ \overline{f}(x, s, \xi), & \text{if } s \le \beta_j(x), \end{cases}$$

where

$$\omega_{2,j}(x, \delta) = \max_{|\xi|\le\delta} |\overline{f}(x, \beta_j(x), \nabla\beta_j(x) + \xi) - \overline{f}(x, \beta_j(x), \nabla\beta_j(x))|,$$

for $i \in \{1, ..., k\}$ and $j \in \{1, ..., l\}$. At last, we define for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$F(x, s, \xi) = \begin{cases} \max\limits_{1\le i\le k} p_i(x, s, \xi), & \text{if } s \le \alpha(x), \\ \overline{f}(x, s, \xi), & \text{if } \alpha(x) < s < \beta(x), \\ \min\limits_{1\le j\le l} q_j(x, s, \xi), & \text{if } s \ge \beta(x). \end{cases}$$

Then we consider the modified problem

$$\begin{aligned} -\Delta u &= F(x, u, \nabla u), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{7.2}$$

Notice that F is a L^p-Carathéodory function and that there exists $\gamma \in L^p(\Omega)$, such that
$$|F(x,s,\xi)| \le \gamma(x),$$
for a.e. $x \in \Omega$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Step 2. Every solution u of (7.2) *satisfies $\alpha \le u \le \beta$.* Let u be a solution of (7.2). Assume by contradiction that $\min_{\overline{\Omega}}(u-\alpha) < 0$. Let $i \in \{1,\ldots,k\}$ and $\overline{x} \in \overline{\Omega}$ such that
$$\min_{\overline{\Omega}}(u-\alpha) = \min_{\overline{\Omega}}(u-\alpha_i) = (u-\alpha_i)(\overline{x}) < 0.$$
Define $v = u - \alpha_i$. As $v \ge 0$ on $\partial\Omega$, we have $\overline{x} \in \Omega$. Therefore $\nabla v(\overline{x}) = 0$ and there is an open ball $B \subseteq \Omega$, with $\overline{x} \in B$ such that, a.e. in B,
$$|\nabla v(x)| \le |v(x)|, \qquad v(x) < 0,$$
and
$$\begin{array}{rcl} -\Delta v & \ge & F(x,u(x),\nabla u(x)) - f(x,\alpha_i(x),\nabla\alpha_i(x)) \\ & \ge & \overline{f}(x,\alpha_i(x),\nabla u(x)) + \omega_{1i}(x,\alpha_i(x)-u(x)) - \overline{f}(x,\alpha_i(x),\nabla\alpha_i(x)) \\ & \ge & -\omega_{1i}(x,|\nabla v(x)|) + \omega_{1i}(x,|v(x)|) \\ & \ge & 0, \end{array}$$
since $\omega_{1i}(x,\cdot)$ is increasing and $|v(x)| \ge |\nabla v(x)|$. This contradicts the strong maximum principle.

Similarly, one proves that $u \le \beta$.

Step 3. Every solution of (7.2) *is a solution of* (2.1) *and satisfies $\alpha \le u \le \beta$.* In Step 2, we proved that every solution u of (7.2) satisfies $\alpha \le u \le \beta$ and hence is a solution of
$$\begin{array}{rl} -\Delta u = \overline{f}(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{array}$$
As $\overline{f}$ satisfies (2.4), we have $\|u\|_{C^1(\overline{\Omega})} < R$ and hence u is a solution of (2.1).

Step 4. Problem (7.2) *has at least one solution.* Let us consider the solution operator $\overline{\mathcal{M}} : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ associated with (7.2), which sends any function $u \in C^1(\overline{\Omega})$ onto the unique solution $v \in W^{2,p}(\Omega)$ of
$$\begin{array}{rl} -\Delta v = F(x,u,\nabla u), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{array}$$
The operator $\overline{\mathcal{M}}$ is continuous, has a relatively compact range and its fixed points are the solutions of (7.2). Hence there exists a constant $\overline{R} > 0$, that we can suppose larger than R, such that, for every $u \in C^1(\overline{\Omega})$,
$$\|\overline{\mathcal{M}}u\|_{C^1(\overline{\Omega})} < \overline{R},$$

and hence (see, e.g., [27])

$$\deg(I - \overline{\mathcal{M}}, B(0, \overline{R})) = 1, \tag{7.3}$$

where I is the identity operator in $C^1(\overline{\Omega})$ and $B(0, \overline{R})$ is the open ball of center 0 and radius $\overline{R}$ in $C^1(\overline{\Omega})$. Therefore $\overline{\mathcal{M}}$ has a fixed point and problem (7.2) has at least one solution.

Step 5. Problem (2.1) *has at least one solution.* By Step 4, we get the existence of a solution u of the problem (7.2) and Step 2 implies that u is a solution of (2.1) satisfying $\alpha \leq u \leq \beta$.

Part 2. Existence of extremal solutions. We know, from Part 1, that the solutions u of (2.1), with $\alpha \leq u \leq \beta$, are precisely the fixed points of the solution operator $\overline{\mathcal{M}}$ associated with (7.2), i.e.

$$\mathcal{H} = \{u \in C^1(\overline{\Omega}) \mid u = \overline{\mathcal{M}}u\}$$

and $\mathcal{H}$ is a non-empty compact subset of $C^1(\overline{\Omega})$. Next, for each $u \in \mathcal{H}$, define the closed set $\mathcal{C}_u = \{z \in \mathcal{H} \mid z \leq u\}$. The family $\{\mathcal{C}_u \mid u \in \mathcal{H}\}$ has the finite intersection property, as it follows from Part 1 observing that if $u_1, u_2 \in \mathcal{H}$, then $\min\{u_1, u_2\}$ is an upper solution of (7.2) with $\alpha \leq \min\{u_1, u_2\}$. Hence $\mathcal{C}_{u_1} \cap \mathcal{C}_{u_2} \neq \emptyset$. By the compactness of $\mathcal{H}$ there exists $v \in \bigcap_{u \in \mathcal{H}} \mathcal{C}_u$; clearly, v is the minimum solution in $[\alpha, \beta]$ of (2.1) in $\overline{\Omega}$.

Part 3. Degree computation. Now, let us assume that α and β are strict lower and upper solutions respectively. Since there exists a solution u of (2.1), which satisfies $\alpha \leq u \leq \beta$, and every such solution satisfies $\alpha \ll u \ll \beta$, it follows that $\alpha \ll \beta$. Hence $\mathcal{S}$ is a non-empty open set in $C^1(\overline{\Omega})$ and there is no fixed point either of $\mathcal{M}$ or of $\overline{\mathcal{M}}$ on its boundary $\partial\mathcal{S}$. Moreover, by (7.1), the sets of fixed points of $\mathcal{M}$ and $\overline{\mathcal{M}}$ coincide on $\mathcal{S} \cap B(0, R)$ and we have

$$\deg(I - \mathcal{M}, \mathcal{S} \cap B(0, R)) = \deg(I - \overline{\mathcal{M}}, \mathcal{S} \cap B(0, R)).$$

Furthermore, by the excision property of the degree (see, e.g., [27]), we get from (7.1) and (7.3)

$$\deg(I - \overline{\mathcal{M}}, B(0, R)) = 1.$$

Finally, since all fixed points of $\overline{\mathcal{M}}$ are in $\mathcal{S} \cap B(0, R)$, we conclude

$$\deg(I - \mathcal{M}, \mathcal{S} \cap B(0, R)) = \deg(I - \overline{\mathcal{M}}, \mathcal{S} \cap B(0, R)) = \deg(I - \overline{\mathcal{M}}, B(0, R)) = 1.$$

This ends the proof.

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