Existence of infinitely many stationary layered solutions in $\mathbf{R}^2$ for a class of periodic Allen-Cahn equations

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Abstract. We consider a class of periodic Allen-Cahn equations

$$-\Delta u(x,y) + a(x,y)W'(u(x,y)) = 0, \quad (x,y) \in \mathbf{R}^2 \tag{1}$$

where $a : \mathbf{R}^2 \to \mathbf{R}^2$ is an even, periodic, positive function and $W : \mathbf{R} \to \mathbf{R}$ is modeled on the classical two well Ginzburg-Landau potential $W(s) = (s^2 - b^2)^2$. We show, via variational methods, that there exist infinitely many solutions, distinct up to periodic translations, of (1) asymptotic as $x \to \pm\infty$ to the pure states $\pm b$, i.e., solutions satisfying the boundary conditions

$$\lim_{x\to\pm\infty} u(x,y) = \pm b, \quad \text{uniformly in } y \in \mathbf{R}. \tag{2}$$

In fact, we prove the existence of solutions of (1)-(2) which are periodic in the y variable and if such solutions are finite modulo periodic translations, we can prove the existence of infinitely many (modulo periodic translations) solutions of (1)-(2) asymptotic to different periodic solutions as $y \to \pm\infty$.

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1 Introduction

In this paper we deal with a class of semilinear elliptic equations of the form

$$-\Delta u(x,y) + a(x,y)W'(u(x,y)) = 0, \qquad (x,y) \in \mathbf{R}^2, \tag{E}$$

where we assume

(H_1) a is Hölder continuous on $\mathbf{R}^2$, strictly positive and

(i) there exist $T_1, T_2 > 0$ such that $a(x+T_1, y+T_2) = a(x,y)$ for all $(x,y) \in \mathbf{R}^2$,

(ii) $a(x,y) = a(x,-y)$ for all $(x,y) \in \mathbf{R}^2$.

(H_2) $W \in \mathcal{C}^2(\mathbf{R})$ satisfies

(i) there exists $b > 0$ such that $W(\pm b) = 0$, $W''(\pm b) > 0$ and $W(s) > 0$ for any $s \in \mathbf{R}\backslash\{-b, b\}$,

(ii) there exists $R_0 > b$ such that $W(s) > W(R_0)$ for any $s > R_0$ and $W(s) > W(-R_0)$ for any $s < -R_0$.

In the sequel, without loss of generality, we will assume $T_1 = T_2 = 1$.

This kind of equations arises in various fields of Mathematical Physics and our assumptions on W are modeled on the classical two well Ginzburg-Landau potential $W(s) = (s^2-1)^2$. In fact (E) can be viewed as a generalization of the stationary Allen-Cahn equation introduced in 1979 by S.M. Allen and J.W. Cahn (see [6]) as a model for phase transitions in binary metallic alloys. In these models the function u is an order parameter representing pointwise the state of the material. The global minima $\pm b$ of W are called the *pure phases* and different values of u represent a mixed configuration.

In this paper we look for two phases layered solutions of (E). Namely, we look for solutions of (E) asymptotic as $x \to \pm\infty$ to the pure states $\pm b$, i.e., solutions of the boundary value problem

$$\begin{cases} -\Delta u(x,y) + a(x,y)W'(u(x,y)) = 0, & (x,y) \in \mathbf{R}^2 \\ \lim\limits_{x\to\pm\infty} u(x,y) = \pm b, & \text{uniformly in } y \in \mathbf{R}. \end{cases} \tag{P}$$

Apart from its physical aspects, problem (P) presents interesting mathematical features. In [12], N. Ghoussoub and C. Gui proved a De Giorgi's conjecture (see [10]) related to (P). They obtain, in particular, the following result.

Theorem 1.1 *If $a(x,y) = a_0 > 0$ for any $(x,y) \in \mathbf{R}^2$ and if $u \in \mathcal{C}^2(\mathbf{R}^2)$ is a solution of (P) then $u(x,y) = q(x)$ for any $(x,y) \in \mathbf{R}^2$ for some $q \in \mathcal{C}^2(\mathbf{R})$ solution of the problem*

$$\begin{cases} -\ddot{q}(x) + a_0 W'(q(x)) = 0, & x \in \mathbf{R} \\ \lim\limits_{x\to\pm\infty} q(x) = \pm b. \end{cases}$$

In other words, by Theorem 1.1, if a is constant any solution of (P) depends only on the x variable and is solution of the corresponding ordinary differential equation. For other works on this kind of problems as well as extensions of this result in dimension greater than 2 and to more general settings, we refer to [8], [9], [11], [2], [7] and the references therein.

All these results show that if (P) is autonomous then the set of its solutions is in a certain sense trivial, the problem being in fact one dimensional. For completeness we recall that this is not the case, in general, if instead of a single equation one consider systems of autonomous Allen Cahn equations ([1]).

In the present paper we pursue the work started in [5] to analyze the set of solutions of (P) in cases where the function a is non constant. In [5], we proved that, even if a depends only on the x variable (being constant in y), then (P) admits, generically with respect to a, *two dimensional* solutions, i.e. solutions depending on both the variables x and y. More precisely, we proved the existence of a dense subset (with respect to the L^∞-norm) in $\{a \in \mathcal{C}(\mathbf{R}) \mid a \text{ is periodic and positive}\}$ for which (P) possesses two dimensional solutions. Now we consider a fully non autonomous case and we show :

Theorem 1.2 *Assume (H_1)-(H_2). Then, (P) has infinitely many solutions, distinct up to periodic translations.*

To prove Theorem 1.2, we first consider in section 2, the periodic problem associated to (P). More precisely, we look for solutions of (P) which are periodic in the y variable, i.e. solutions of

$$\begin{cases} -\Delta u(x,y) + a(x,y)W'(u(x,y)) = 0, & (x,y) \in \mathbf{R}^2 \\ u(x,y) = u(x,y+1), & (x,y) \in \mathbf{R}^2 \\ \lim\limits_{x\to\pm\infty} u(x,y) = \pm b, & \text{uniformly in } y \in \mathbf{R}. \end{cases} \tag{1.1}$$

The main feature (and difficulty) of this problem is that it has mixed boundary conditions, requiring the solution to be periodic in the y variable and of the heteroclinic type in the x variable.

Letting $S_0 = \mathbf{R} \times [0,1]$, we look for minima of the action functional

$$\varphi(u) = \int_{S_0} \tfrac{1}{2}|\nabla u(x,y)|^2 + a(x,y)W(u(x,y))\, dx\, dy.$$

on the class

$$\Gamma = \{u \in H^1_{loc}(S_0) \mid \int_0^1 |u(x,y) \mp b|^2 dy \to 0 \text{ as } x \to \pm\infty\}.$$

As we shall see these minima satisfy the equation and the right boundary conditions in the x variable. In addition they satisfy Neumann boundary conditions in the y variable.

To prove that these minima are also solutions of (1.1) we follow P.H. Rabinowitz ([14], ([15]), observing that, thanks to the reversibility assumption (H_1)-(ii), they belong to the smaller class

$$\Gamma_p = \{u \in H^1_{loc}(S_0) \mid u(x,0) = u(x,1) \text{ for a.e. } x \in \mathbf{R},\ \int_0^1 |u(x,y) \mp b|^2 dy \to 0 \text{ as } x \to \pm\infty\}.$$

To get a minimum of φ on Γ we use a reduction procedure related to the one we introduced in [5]. Roughly speaking we consider the elements of Γ as paths $x \in \mathbf{R} \to u(x,\cdot) \in H^1(0,1)$ and we look for minimal trajectories, for the reduced functional, connecting as $x \to \pm\infty$ the constant functions $\pm b$. The reduced minimization problem is still characterized by a lack of compactness due to the unboundedness of the domain but now it can be overcome by using concentration compactness type arguments.

We want to point out that this same method can be used to obtain directly a solution of (1.1) by minimizing the functional φ on the class Γ_p. Hence, to prove the existence of such minima one does not need the reversibility assumption (H_1)-(ii). In fact, we use assumption (H_1)-(ii) to prove the existence of solutions to (1.1) which are also minima of φ on the class Γ.

We set

$$c = \inf_\Gamma \varphi \quad \text{and} \quad \mathcal{K} = \{u \in \Gamma \mid \varphi(u) = c\}$$

and say that two solutions z_1 and $z_2 \in \mathcal{K}$ are equivalent, if there exists $j \in Z$ such that $z_1(x+j,y) = z_2(x,y)$ for any $(x,y) \in \mathbf{R}^2$. We then write $z_1 \sim_x z_2$.

If the set $\mathcal{K}/\sim_x$ is infinite then Theorem 1.2 is true. Thus we may assume the opposite. In section 3, assuming that $\mathcal{K}/\sim_x$ is finite, or equivalently (see Remark 3.1) that $\mathcal{K}$ is constituted by isolated points with respect to the $H^1(S_0)$ metric, we prove that there exist heteroclinic type solutions of (P) connecting different elements of $\mathcal{K}$ as $y\to\pm\infty$. This is done adapting to our setting a method proposed in [14], [15], [16] in the study of the existence of heteroclinic connections between periodic solutions for some class of Lagrangian systems or of semilinear elliptic equations.

Precisely, denoting by z_ξ, $\xi\in\mathbf{Z}$, the elements of $\mathcal{K}$, we will look for solutions of (P) which satisfies the boundary conditions

$$\lim_{j\to-\infty}\|u-z_0\|_{H^1(S_j)}=\lim_{j\to+\infty}\|u-z_\xi\|_{H^1(S_j)}=0$$

where $S_j=\mathbf{R}\times[j,j+1]$. Such solutions are obtained as local minima of the functional

$$\Phi(u)=\sum_{j\in\mathbf{Z}}\varphi_j(u),$$

where

$$\varphi_j(u)=\int_{S_j}\tfrac{1}{2}|\nabla u(x,y)|^2+a(x)W(u(x,y))\,dx\,dy\ \ -c.$$

Since, as discussed above, $c=\min_\Gamma\varphi$, the functional Φ is well defined, and non negative, on

$$\Sigma=\{u\in H^1_{loc}(\mathbf{R}^2)\,|\,j*u\in\Gamma,\,\forall j\in\mathbf{Z}\}$$

where $(j*u)(x,y)=u(x,y+j)$. We set

$$\mathcal{H}_\xi=\{u\in\Sigma\,|\lim_{j\to-\infty}\|u-z_0\|_{H^1(S_j)}=\lim_{j\to+\infty}\|u-z_\xi\|_{H^1(S_j)}=0\}$$

and

$$m_\xi=\inf_{u\in\mathcal{H}_\xi}\Phi(u)\quad\text{and}\quad\mathcal{K}_\xi=\{u\in\mathcal{H}_\xi\,|\,\Phi(u)=m_\xi\},\quad\ \xi\in\mathbf{Z}.$$

We prove the existence of a $\eta\in\mathbf{Z}$ such that there exists $u\in\mathcal{K}_\eta$. Then u is proved to be in $\mathcal{C}^2(\mathbf{R}^2)$ and a classical solution of (P). We end the section studying further compactness properties of the minimizing sequences of Φ in $\mathcal{H}_\eta$.
We say that two solutions u_1 and $u_2\in\mathcal{K}_\eta$ are equivalent, and we write $u_1\sim_y u_2$, if there exists $j\in\mathbf{Z}$ such that $u_1(x,y+j)=u_2(x,y)$ for $(x,y)\in\mathbf{R}^2$.

In the last section, assuming that $\mathcal{K}_\eta/\sim_y$ is finite, namely that Theorem 1.2 is not already proved, we establish the existence of infinitely many heteroclinic solutions of multibump type. Roughly speaking, for any given $N\in\mathbf{N}$ we show the existence of solutions (infinitely many) of (P) which transit N times between z_0 and z_η while y runs between $-\infty$ and $+\infty$. More precisely, for every $\delta>0$ and for $n\in\mathbf{N}$ large enough, we set

$$\mathcal{U}_0=\{u\in\Sigma\mid\|u-z_0\|_{H^1(S_{-n})}\le\delta\quad\text{and}\quad\|u-z_\eta\|_{H^1(S_n)}\le\delta\}$$

and

$$\mathcal{U}_\eta=\{u\in\Sigma\mid\|u-z_\eta\|_{H^1(S_{-n})}\le\delta\quad\text{and}\quad\|u-z_0\|_{H^1(S_n)}\le\delta\}.$$

Then, for every $N\in\mathbf{N}$, $p=(p_1,...,p_N)\in\mathbf{Z}^N$ and $\sigma=(\sigma_1,...\sigma_N)\in\{0,\eta\}^N$ with $p_i-p_{i-1}\ge 4n$ and $\sigma_i\ne\sigma_{i-1}$ for all $i=2,..,N$, let

$$\mathcal{H}_{N,p,\sigma}=\{u\in\Sigma\,|\,u(x,y-p_i)\in\mathcal{U}_{\sigma_i},\ \text{for a.e. }(x,y)\in\mathbf{R}^2,\ i=1,...,N\}$$

and

$$m_{N,p,\sigma} = \inf_{\mathcal{H}_{N,p,\sigma}} \Phi \quad \text{and} \quad \mathcal{K}_{N,p,\sigma} = \{u \in \mathcal{H}_{N,p,\sigma} \mid \Phi(u) = m_{N,p,\sigma}\}.$$

Thanks to the compactness properties of the minimizing sequences of Φ in $\mathcal{H}_\eta$ obtained in section 3, we can adapt to our setting a constrained minimization procedure related to the ones of [3] and [4] to prove that for any $N \in \mathbf{N}$, $p \in \mathbf{Z}^N$ and $\sigma \in \{0, \eta\}^N$ as above, there exists $u \in \mathcal{K}_{N,p,\sigma}$. Moreover, $u \in \mathcal{C}^2(\mathbf{R}^2)$ and it is a solution of (P). This result completes the proof of Theorem 1.2.

Using the notation introduced above we can now state more in details our main result.

Theorem 1.3 *Assume (H_1)-(H_2). Then (P) admits infinitely many solutions distinct up to periodic translations. More precisely we have*

(i) *the set $\mathcal{K}$ of periodic solutions of (P) is not empty,*

(ii) *if the set $\mathcal{K}/\sim_x$ is finite, then there exists $\eta \in \mathbf{Z}$ such that the set $\mathcal{K}_\eta$ of heteroclinic type solutions of (P) is not empty,*

(iii) *if the sets $\mathcal{K}/\sim_x$ and $\mathcal{K}_\eta/\sim_y$ are finite, then for every $N \in \mathbf{N}$, $p = (p_1, ..., p_N) \in \mathbf{Z}^N$ and $\sigma = (\sigma_1, ...\sigma_N) \in \{0, \eta\}^N$ with $p_i - p_{i-1} \geq 4n$ and $\sigma_i \neq \sigma_{i-1}$ for all $i = 2, .., N$, the set $\mathcal{K}_{N,p,\sigma}$ of multibump type solutions of (P) is not empty.*

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2 The periodic problem

In this section, as stated in the introduction, we look for solutions of the problem

$$\begin{cases} -\Delta u(x,y) + a(x,y)W'(u(x,y)) = 0, & (x,y) \in \mathbf{R}^2 \\ u(x,y) = u(x,y+1), & (x,y) \in \mathbf{R}^2 \\ \lim_{x \to \pm\infty} u(x,y) = \pm b, & \text{uniformly in } y \in \mathbf{R}. \end{cases} \tag{2.1}$$

Letting $S_0 = \mathbf{R} \times [0,1]$ we look for minima of the Euler Lagrange functional

$$\varphi(u) = \int_{S_0} \tfrac{1}{2}|\nabla u(x,y)|^2 + a(x,y)W(u(x,y))\,dx\,dy$$

on the class

$$\Gamma = \{u \in H^1_{loc}(S_0) \,|\, \|u(x,\cdot) \mp b\|_{L^2(0,1)} \to 0 \text{ as } x \to \pm\infty\}.$$

Then we use the reversibility assumption (H_1)-(ii) to show that these minima are indeed minima of φ on

$$\Gamma_p = \{u \in H^1_{loc}(S_0) \,|\, u(x,0) = u(x,1) \text{ for a.e. } x \in \mathbf{R},\ \|u(x,\cdot) \mp b\|_{L^2(0,1)} \to 0 \text{ as } x \to \pm\infty\}$$

and so solutions of (2.1).

We start this study letting $E_0 = H^1_{loc}(S_0)$ and noticing that the regularity assumptions on the functions a and W are sufficient to prove that φ is lower semicontinuous with respect to the weak convergence in E_0, that is, if $u_n \to u$ weakly in $H^1(\Omega)$ for any Ω relatively compact in S_0, then $\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n)$. Moreover we have

Lemma 2.1 *If $(u_n) \subset E_0$ is such that $u_n \to u$ weakly in E_0 and $\varphi(u_n) \to \varphi(u)$ then*

$$\int_{S_0} |\nabla u_n|^2\, dx\, dy \to \int_{S_0} |\nabla u|^2\, dx\, dy \text{ and } \int_{S_0} a(x,y)W(u_n)\, dx\, dy \to \int_{S_0} a(x,y)W(u)\, dx\, dy. \quad (2.2)$$

Proof. Indeed

$$\begin{aligned}
\int_{S_0} a(x,y)W(u)\, dx\, dy &\leq \liminf \int_{S_0} a(x,y)W(u_n)\, dx\, dy \leq \limsup \int_{S_0} a(x,y)W(u_n)\, dx\, dy \\
&= \limsup(\varphi(u_n) - \int_{S_0} |\nabla u_n|^2\, dx\, dy) = \varphi(u) - \liminf \int_{S_0} |\nabla u_n|^2\, dx\, dy \\
&\leq \int_{S_0} a(x,y)W(u)\, dx\, dy.
\end{aligned}$$

Then $\int_{S_0} a(x,y)W(u_n)\, dx\, dy \to \int_{S_0} a(x,y)W(u)\, dx\, dy$ and, since $\varphi(u_n) \to \varphi(u)$, (2.2) follows. $\square$

Remark 2.1 Letting $S^L = \{(x,y) \in S_0 \,|\, |x| \leq L\}$, the two functionals $\int_{S_0} |\nabla u|^2\, dx\, dy$ and $\int_{S_0} a(x,y)W(u)\, dx\, dy$ can be written for any $L > 0$ as sum of two lower semicontinuous functionals as follows

$$\begin{aligned}
\int_{S_0} |\nabla u|^2\, dx\, dy &= \int_{S^L} |\nabla u|^2\, dy\, dx + \int_{S_0 \setminus S^L} |\nabla u|^2\, dy\, dx, \\
\int_{S_0} a(x,y)W(u)\, dx\, dy &= \int_{S^L} a(x,y)W(u)\, dy\, dx + \int_{S_0 \setminus S^L} a(x,y)W(u)\, dy\, dx.
\end{aligned}$$

Then, arguing as in the preceding lemma, we derive that if $(u_n) \subset E_0$ is such that $u_n \to u$ weakly in E_0 and $\varphi(u_n) \to \varphi(u)$ then

$$\begin{aligned}
&\int_{S^L} |\nabla u_n|^2\, dy\, dx \to \int_{S^L} |\nabla u|^2\, dy\, dx, \qquad \int_{S^L} aW(u_n)\, dy\, dx \to \int_{S^L} aW(u)\, dy\, dx \\
&\int_{S_0 \setminus S^L} |\nabla u_n|^2\, dy\, dx \to \int_{S_0 \setminus S^L} |\nabla u|^2\, dy\, dx, \qquad \int_{S_0 \setminus S^L} aW(u_n)\, dy\, dx \to \int_{S_0 \setminus S^L} aW(u)\, dy\, dx.
\end{aligned}$$

We shall prove that any $u \in E_0$ with $\varphi(u) < +\infty$ has some definite asymptotic behavior as $x \to \pm\infty$. To this aim we establish some preliminary results.

Note that, by Fubini Theorem, if $u \in E_0$ then $u(\cdot, y) \in H^1_{loc}(\mathbf{R})$ for a.e. $y \in (0,1)$. So, if $u \in E_0$, for all $x_1, x_2 \in \mathbf{R}$ we obtain

$$\begin{aligned}
\int_0^1 |u(x_2,y) - u(x_1,y)|^2\, dy &= \int_0^1 |\int_{x_1}^{x_2} \partial_x u(x,y)\, dx\, |^2 dy \\
&\leq |x_2 - x_1| \int_0^1 [\int_{x_1}^{x_2} |\partial_x u(x,y)|^2\, dx]\, dy \\
&\leq 2\varphi(u)|x_2 - x_1|.
\end{aligned}$$

Given $u \in E_0$, still by Fubini Theorem, the function $u(x,\cdot) \in L^2(0,1)$ for a.e. $x \in \mathbf{R}$. If $\varphi(u) < +\infty$, by the previous estimate, the function $x \to u(x,\cdot)$ is Hölder continuous from a dense subset of $\mathbf{R}$ with values in $L^2(0,1)$ and so it can be extended to a continuous function on $\mathbf{R}$. According to that, any function $u \in E_0 \cap \{\varphi < +\infty\}$ defines a continuous trajectory in $L^2(0,1)$ verifying

$$\mathsf{d}(u(x_2,\cdot), u(x_1,\cdot))^2 \equiv \int_0^1 |u(x_2,y) - u(x_1,y)|^2\, dy \leq 2\varphi(u)|x_2 - x_1|, \quad \forall\, x_1, x_2 \in \mathbf{R}. \quad (2.3)$$

Then, we obtain

Lemma 2.2 *For all $r > 0$ there exists $\mu_r > 0$ such that if $u \in E_0$ satisfies $\min\{\|u(x,\cdot) - b\|_{H^1(0,1)}, \|u(x,\cdot) + b\|_{H^1(0,1)}\} \geq r$ for a.e. $x \in (x_1, x_2)$ then*

$$\begin{aligned}\int_{x_1}^{x_2} [\int_0^1 \tfrac{1}{2}|\nabla u(x,y)|^2 + a(x,y)W(u(x,y))dy]\,dx &\geq \tfrac{1}{2(x_2-x_1)}\mathsf{d}(u(x_2,\cdot),u(x_1,\cdot))^2 + \tfrac{1}{2}\mu_r^2(x_2-x_1)\\ &\geq \mu_r \mathsf{d}(u(x_1,\cdot),u(x_2,\cdot)). \end{aligned} \tag{2.4}$$

Proof. We define on $H^1(0,1)$ the functional

$$F(q) = \int_0^1 \tfrac{1}{2}|\dot q(y)|^2 + \underline{a}W(q(y))\,dy$$

where $\underline{a} = \min_{\mathbf{R}^2} a(x,y) > 0$.
Observe that if $(x_1,x_2) \subset \mathbf{R}$ and $u \in E_0$ are such that $F(u(x,\cdot)) \geq \mu$, for a.e. $x \in (x_1,x_2)$ then

$$\begin{aligned}\int_{x_1}^{x_2} [\tfrac{1}{2}\int_0^1 |\partial_x u(x,y)|^2\,dy + F(u(x,\cdot))]\,dx &\geq \int_{x_1}^{x_2}[\tfrac{1}{2}\int_0^1 |\partial_x u(x,y)|^2\,dy\,]\,dx + \mu(x_2-x_1)\\ &\geq \tfrac{1}{2(x_2-x_1)}\int_0^1 [\int_{x_1}^{x_2} |\partial_x u(x,y)|\,dx]^2\,dy + \mu(x_2-x_1)\\ &\geq \tfrac{1}{2(x_2-x_1)}\mathsf{d}(u(x_2,\cdot),u(x_1,\cdot))^2 + \mu(x_2-x_1)\\ &\geq \sqrt{2\mu}\,\mathsf{d}(u(x_2,\cdot),u(x_1,\cdot)). \end{aligned} \tag{2.5}$$

By (2.5), to prove the lemma it is sufficient to show that for any $r > 0$, there exists $\mu_r > 0$ such that if $q \in H^1(0,1)$ is such that $\min\{\|q - b\|_{H^1(0,1)}, \|q + b\|_{H^1(0,1)}\} \geq r$ then $F(q) \geq \frac{1}{2}\mu_r^2$.
To this aim, we will prove that for any sequence $(q_n) \subset H^1(0,1)$ satisfying $F(q_n) \to 0$, there is a subsequence (q_{n_j}), such that $q_{n_j} \to -b$ or $q_{n_j} \to b$ strongly in $H^1(0,1)$.

So, let $(q_n) \subset H^1(0,1)$ be such that $F(q_n) \to 0$. If we prove that $\min\{\|q_n - b\|_{L^\infty(0,1)}, \|q_n + b\|_{L^\infty(0,1)}\} \to 0$ as $n \to \infty$ we have done. Indeed, in such a case there exists a subsequence (q_{n_j}) of (q_n) which converge in $L^\infty(0,1)$ to b or $-b$. Assuming e.g. $\|q_{n_j} - b\|_{L^\infty(0,1)} \to 0$, since $\int_0^1 |\dot q_n|^2\,dy \to 0$, we conclude that $q_{n_j} \to b$ strongly in $H^1(0,1)$.
Then, assume by contradiction, that there exists a subsequence of (q_n) (still denoted q_n) such that $\min\{\|q_n - b\|_{L^\infty(0,1)}, \|q_n + b\|_{L^\infty(0,1)}\} \geq \varepsilon > 0$. Then, there exists a sequence $(y_n^1) \subset [0,1]$ such that $\min\{|q_n(y_n^1) - b|, |q_n(y_n^1) + b|\} \geq \varepsilon$. Also, by (H_2), since $\int_0^1 W(q_n)dy \to 0$, there exists a sequence $(y_n^2) \subset [0,1]$ such that $\min\{|q_n(y_n^2) - b|, |q_n(y_n^2) + b|\} \to 0$. Thus, for $n \in \mathbf{N}$ large enough we get

$$\frac{\varepsilon}{2} \leq |q_n(y_n^2) - q_n(y_n^1)| \leq |\int_{y_n^1}^{y_n^2} |\dot q_n(t)|\,dt| \leq |y_n^2 - y_n^1|^{\frac{1}{2}}[\int_0^1 |\dot q_n(t)|^2\,dt]^{\frac{1}{2}} \leq \sqrt{2}F(q_n)^{\frac{1}{2}}$$

contradicting the assumption $F(q_n) \to 0$. □

The next lemma is a direct consequence of Lemma 2.2.

Lemma 2.3 *If $u \in E_0 \cap \{\varphi < +\infty\}$ then there exist $\xi_\pm \in \{\pm b\}$ such that*

$$\mathsf{d}(u(x,\cdot),\xi_\pm) \to 0 \quad \textit{as} \quad x \to \pm\infty.$$

Proof. First of all note that, by Lemma 2.2, there exist $\xi_\pm \in \{\pm b\}$ such that

$$\liminf_{x\to\pm\infty} \mathsf{d}(u(x,\cdot),\xi_\pm) = 0.$$

Considering the case $x \to +\infty$ we argue by contradiction. We assume that there exists $r \in (0, \frac{b}{4})$ such that $\limsup_{x\to+\infty} \mathsf{d}(u(x,\cdot), \xi_+) > 2r$. Then, by (2.3), there exists a sequence of disjoint intervals $(p_i, s_i), i \in \mathbf{N}$, such that $\mathsf{d}(u(p_i,\cdot), \xi_+) = r$, $\mathsf{d}(u(s_i,\cdot), \xi_+) = 2r$ and $r \le \mathsf{d}(u(x,\cdot), \xi_+) \le 2r$ for $x \in \cup_i(p_i, s_i)$. Using (2.4) this implies $\varphi(u) = +\infty$, a contradiction.
Similarly one proves that $\mathsf{d}(u(x,\cdot), \xi_-) \to 0$ as $x \to -\infty$. $\square$

We define the class

$$\Gamma = \{u \in E_0 \cap \{\varphi < +\infty\} \,|\, \mathsf{d}(u(x,\cdot), \pm b) \to 0 \text{ as } x \to \pm\infty\}$$

and let

$$c = \inf_{\Gamma} \varphi \quad \text{and} \quad \mathcal{K} = \{u \in \Gamma \,|\, \varphi(u) = c\}.$$

Moreover, for any $\delta > 0$ we set

$$\lambda_\delta = \tfrac{1}{2}\delta^2 + (\max_{(x,y)\in\mathbf{R}^2} a(x,y)) \cdot (\max_{|s\pm b|\le\sqrt{2}\delta} W(s)). \tag{2.6}$$

We shall prove that the minimal set $\mathcal{K}$ is not empty. In this aim we start noticing that the trajectories in Γ with action close to the infimum satisfy suitable concentration properties.

Lemma 2.4 *There exists $\bar\delta_0 \in (0, \frac{b}{2})$ such that for any $\delta \in (0, \bar\delta_0)$ there exist $\rho_\delta > 0$ and $\ell_\delta > 0$ for which, if $u \in \Gamma$ and $\varphi(u) \le c + \lambda_\delta$, then*

(i) if $\min\{\|u(x,\cdot) - b\|_{H^1(0,1)}, \|u(x,\cdot) + b\|_{H^1(0,1)}\} \ge \delta$ for a.e. $x \in (s,p)$, then $p - s \le \ell_\delta$,

(ii) if $\|u(x_-,\cdot)+b\|_{H^1(0,1)} \le \delta$, then $\mathsf{d}(u(x,\cdot), -b) \le \rho_\delta$ for any $x \le x_-$ and if $\|u(x_+,\cdot)-b\|_{H^1(0,1)} \le \delta$, then $\mathsf{d}(u(x,\cdot), b) \le \rho_\delta$ for any $x \ge x_+$.

Moreover $\rho_\delta \to 0$ as $\delta \to 0$.

Proof. Note that (i) plainly follows from Lemma 2.2. To prove (ii), we need some additional definition. Since $\lambda_\delta \searrow 0$ as $\delta \to 0^+$, for all δ small enough we can choose r_δ such that $\mu_{r_\delta} \ge \sqrt{\lambda_\delta}$, where μ_r is given by Lemma 2.2. Moreover, r_δ can be choose such that $r_\delta \searrow 0$ as $\delta \to 0^+$. Let $\rho_\delta = \max\{\delta, r_\delta\} + \frac{3\lambda_\delta}{\mu_{r_\delta}}$. Since by definition, $\mu_{r_\delta} \ge \sqrt{\lambda_\delta}$, we have that $\rho_\delta \to 0$ as $\delta \to 0^+$. Let $\bar\delta_0 \in (0, \frac{b}{2})$ be such that $\rho_\delta < \frac{b}{2}$ for all $\delta \in (0, \bar\delta_0)$.
Now let $\delta \in (0, \bar\delta_0)$, $u \in \Gamma$ and $x_- \in \mathbf{R}$ be such that $\|u(x_-,\cdot) + b\|_{H^1(0,1)} \le \delta$ and $\varphi(u) \le c + \lambda_\delta$. We define

$$u_-(x,y) = \begin{cases} -b & \text{if } x < x_- - 1, \\ (x - x_-)b + (x - x_- + 1)u(x_-, y) & \text{if } x_- - 1 \le x \le x_- \\ u(x,y) & \text{if } x \ge x_- \end{cases}$$

and note that since $u_- \in \Gamma$, $\varphi(u_-) \ge c$. Moreover, by definition (2.6) of λ_δ, recalling that $\|q\|_{L^\infty(0,1)} \le \sqrt{2}\|q\|_{H^1(0,1)}$ for any $q \in H^1(0,1)$, a direct estimate gives

$$\int_{x_- -1}^{x_-} [\int_0^1 \tfrac{1}{2}|\nabla u_-|^2 + aW(u_-)\, dy]\, dx \le \lambda_\delta$$

from which, since

$$\varphi(u_-) = \varphi(u) - \int_{-\infty}^{x_-} [\int_0^1 \tfrac{1}{2}|\nabla u|^2 + aW(u)\, dy]\, dx \quad + \int_{x_- -1}^{x_-} [\int_0^1 \tfrac{1}{2}|\nabla u_-|^2 + aW(u_-)\, dy]\, dx,$$

we obtain

$$\int_{-\infty}^{x_-}[\int_0^1 \tfrac{1}{2}|\nabla u|^2 + aW(u)\,dy]\,dx \le 2\lambda_\delta. \tag{2.7}$$

Now, assume by contradiction that there exists $x_1 < x_-$ such that $\mathsf{d}(u(x_1,\cdot),-b) \ge \rho_\delta$. Then there exists $x_2 \in (x_1, x_-]$ such that $\mathsf{d}(u(x,\cdot),-b) \ge \max\{\delta, r_\delta\}$ for $x \in (x_1,x_2)$ and $\mathsf{d}(u(x_1,\cdot),u(x_2,\cdot)) \ge \rho_\delta - \max\{\delta, r_\delta\}$. By (2.4), we get

$$\int_{-\infty}^{x_-}[\int_0^1 \tfrac{1}{2}|\nabla u|^2 + aW(u)\,dy]\,dx \ge \mu_{r_\delta}(\rho_\delta - \max\{\delta, r_d\}) \ge 3\lambda_\delta$$

which contradicts (2.7). This proves that $\mathsf{d}(u(x,\cdot),-b) < \rho_\delta$ for any $x \le x_-$.
Analogously, if $\|u(x_+,\cdot) - b\|_{H^1(0,1)} \le \delta$ then $\mathsf{d}(u(x,\cdot),b) < \rho_\delta$ for any $x \ge x_+$. □

According to Lemma 2.4, we fix $\bar\delta \in (0,\bar\delta_0)$ such that $\bar\rho = \rho_{\bar\delta} \le \frac{b}{6}$ and we denote $\bar\lambda = \lambda_{\bar\delta}$ and $\bar\ell = \ell_{\bar\delta}$.

To exploit compactness properties of φ in Γ, it is useful to introduce the function $X : E_0 \to \mathbf{R} \cup \{+\infty\}$ given by

$$X(u) = \sup\{x \in \mathbf{R} \,|\, \mathsf{d}(u(x,\cdot),b) \ge \frac{b}{2}\}.$$

We can now describe a first compactness property of the minimizing sequences of φ in Γ.

Lemma 2.5 *If $(u_n) \subset \Gamma$ is such that $\varphi(u_n) \to c$ and $X(u_n) \to X_0 \in \mathbf{R}$, then there exists $u_0 \in \mathcal{K}$ such that, along a subsequence, $\|u_n - u_0\|_{H^1(S_0)} \to 0$.*

Proof. To prove the lemma we need some preliminary estimates.
(1) For all $\lambda > 0$ there exists $R > 0$ such that if $u \in \Gamma \cap \{\varphi \le c + \lambda\}$ then

$$\|u(x,\cdot)\|_{L^2(0,1)} \le R \quad \forall x \in \mathbf{R}. \tag{2.8}$$

Indeed, let $B_r = \{q \in L^2(0,1) \,|\, \|q\|_{L^2(0,1)} \le r\}$ and note that B_{2b} contains as interior points the functions $\pm b$. To prove (2.8), assume by contradiction that there exists $\lambda > 0$ such that for any $R > 2b$ there exists $u \in \Gamma \cap \{\varphi \le c+\lambda\}$ such that $u(\bar x,\cdot) \notin B_R$ for some $\bar x \in \mathbf{R}$. Since $\mathsf{d}(u(x,\cdot),\pm b) \to 0$ as $x \to \pm\infty$, by continuity (see (2.3)), there exists $x^\pm$ such that $u(x^-,\cdot) \in \partial B_{2b}$, $u(x^+,\cdot) \in \partial B_R$ and $2b < \|u(x,\cdot)\|_{L^2(0,1)} < R$ for all $x \in (x^-,x^+)$. Then $\min\{\mathsf{d}(u(x,\cdot),-b), \mathsf{d}(u(x,\cdot),b)\} \ge b$ for all $x \in (x^-,x^+)$ and by Lemma 2.2 we obtain

$$c + \lambda \ge \varphi(u) \ge \mu_b \mathsf{d}(u(x^-,\cdot),u(x^+,\cdot)) \ge \mu_b(R - 2b)$$

which is a contradiction for R large enough.
(2) For any $\varepsilon > 0$ there exist $\lambda(\varepsilon) \in (0,\bar\lambda)$ and $\ell(\varepsilon) > \bar\ell$ such that if $u \in \Gamma \cap \{\varphi \le c + \lambda(\varepsilon)\}$ then

$$\int_{|x-X(u)|\ge\ell(\varepsilon)}[\int_0^1 |\nabla u|^2 dy]dx \le \varepsilon \text{ and } \int_{|x-X(u)|\ge\ell(\varepsilon)}[\int_0^1 W(u(x))dy]dx \le \varepsilon. \tag{2.9}$$

Indeed, by Lemma 2.4, let $\delta < \bar\delta$ be such that $\lambda_\delta \le \frac{\varepsilon}{3}\min\{\frac{1}{2},\bar a\}$, where $\bar a = \max_{\mathbf{R}^2} a$. Then, given any $u \in \Gamma \cap \{\varphi \le c + \lambda_\delta\}$, by Lemma 2.4, there exist $x_- \in (X(u) - 2\ell_\delta, X(u))$ and $x_+ \in (X(u), X(u) + 2\ell_\delta)$ such that $\|u(x_-,\cdot) + b\|_{H^1(0,1)} \le \delta$ and $\|u(x_+,\cdot) - b\|_{H^1(0,1)} \le \delta$. We define the function

$$\bar u(x,y) = \begin{cases} -b & \text{if } x < x_- - 1, \\ (x - x_-)b + (x - x_- + 1)u(x_-,y) & \text{if } x_- - 1 \le x \le x_- \\ u(x,y) & \text{if } x_- \le x \le x_+ \\ (x_+ + 1 - x)u(x_+,y) + (x - x_+)b & \text{if } x_+ \le x \le x_+ + 1 \\ b & \text{if } x > x_+ + 1 \end{cases}$$

and since $\varphi(\bar{u}) \geq c$, as in the proof of Lemma 2.4, one obtains that

$$\int_{|x-X(u)|\geq 2\ell_\delta+1}[\int_0^1 \tfrac{1}{2}|\nabla u|^2 + aW(u)\,dy]\,dx \leq \varphi(u) - \varphi(\bar{u}) + 2\lambda_\delta \leq 3\lambda_\delta.$$

Then (2.9) follows setting $\ell(\varepsilon) = 2\max\{\ell_\delta, \bar{\ell}\} + 1$ and $\lambda(\varepsilon) = \lambda_\delta$.

(3) There exists $\gamma > 0$ such that for all $\varepsilon > 0$, for all $u \in \Gamma \cap \{\varphi \leq c + \lambda(\varepsilon)\}$ we have

$$\int_{|x-X(u)|\geq \ell(\varepsilon)}[\int_0^1 |u(x,y) - U(x - X(u), y)|^2\,dy]\,dx \leq \gamma\varepsilon \tag{2.10}$$

where

$$U(x,y) = \begin{cases} -b & \text{if } x < 0, \\ (x-1)b + xb & \text{if } 0 \leq x \leq 1 \\ b & \text{if } x > 1. \end{cases}$$

First, by (H_2), note that both $\pm b$ are non degenerate minima of W and around each of these points W behaves quadratically. Thus, setting $d(s) = \min\{|s-b|, |s+b|\}$, we have

$$\text{there exists } w_0 > 0 \text{ such that } W(s) \geq w_0\, d^2(s) \text{ when } d(s) \leq \frac{b}{2}. \tag{2.11}$$

Now, given $u \in \Gamma \cap \{\varphi \leq c + \lambda(\varepsilon)\}$, let $S^- = (-\infty, X(u) - \ell(\varepsilon)] \times [0,1]$ and $A^- = \{(x,y) \in S^- \,|\, |u(x,y) + b| > \frac{b}{2}\}$. Then, since $d(u(x,y)) = |u(x,y) + b|$ for all $(x,y) \in S^- \setminus A^-$, by (2.11) and (2.9) it follows

$$\int_{S^-\setminus A^-} |u(x,y) + b|^2\,dx\,dy \leq \frac{1}{w_0}\int_{S^-\setminus A^-} W(u(x,y))\,dx\,dy \leq \frac{\varepsilon}{w_0}. \tag{2.12}$$

To estimate $\int_{S^-\setminus A^-} |u(x,y) + b|^2\,dx\,dy$, we note that since $u \in H^1_{loc}(S_0)$ there exists $C \subset (0,1)$ with $\mathrm{meas}(C) = 1$ and such that the function $x \to u(x,y)$ is continuous on $\mathbf{R}$ for any $y \in C$. Then, setting $A_y = \{x < X(u) - \ell(\varepsilon) \,|\, |u(x,y) + b| > \frac{b}{2}\}$, we have that A_y is open for any $y \in C$ and so $\pi A^- = \cup_{y\in C} A_y$ is open too. Hence, since $\mathrm{meas}(A^- \setminus (\pi A^- \times (0,1))) = 0$, we have $\int_{A^-} |u(x,y) + b|^2\,dx\,dy \leq \int_{\pi A^-}\int_0^1 |u(x,y) + b|^2\,dy\,dx$.

We then observe that, for any $x \in \pi A^-$, since $x \leq X(u) - \ell(\varepsilon) \leq X(u) - \bar{\ell}$, by Lemma 2.4 we have $\mathsf{d}(u(x,\cdot), -b) \leq \bar{\rho} \leq \frac{b}{4}$ and so there exists $y_x \in (0,1)$ such that $|u(x, y_x) + b| \leq \frac{b}{4}$. Moreover, for all $x \in \pi A^-$, by definition, there exists y^x such that $(x, y^x) \in A^-$. Since, by Fubini's Theorem $u(x,\cdot) \in H^1(0,1)$ for a.e. $x \in \pi A^-$, we obtain that for a.e. $x \in \pi A^-$

$$\frac{b}{4} \leq |u(x,y^x) - u(x,y_x)| \leq |\int_{y_x}^{y^x} |\partial_y u(x,y)|\,dy| \leq |y^x - y_x|^{\frac{1}{2}}[\int_0^1 |\partial_y u(x,y)|^2 dy]^{\frac{1}{2}}$$

and thus

$$\int_0^1 |\partial_y u(x,y)|^2 dy \geq \frac{b^2}{16}.$$

Now since, by (2.9),

$$\int_{\pi A^-}[\int_0^1 |\partial_y u|^2\,dy]\,dx \leq \int_{S^-} |\nabla u|^2\,dx\,dy \leq \varepsilon$$

we deduce that $\mathrm{meas}(\pi A^-) \leq \frac{16}{b^2}\varepsilon$. Then, it follows

$$\int_{A^-} |u(x,y) + b|^2\,dx\,dy \leq \int_{\pi A^-}[\int_0^1 |u(x,y) + b|^2\,dy]\,dx \leq \mathrm{meas}(\pi A^-)\mathsf{d}(u(x,\cdot), -b)^2 \leq \varepsilon.$$

Taking into account (2.12) we obtain

$$\int_{S^-} |u(x,y)+b|^2\,dx\,dy \le \tfrac{\gamma}{2}\varepsilon, \tag{2.13}$$

where $\gamma = 2(\frac{1}{w_0}+1)$. In the same way we can prove that

$$\int_{S^+} |u(x,y)-b|^2\,dx\,dy \le \tfrac{\gamma}{2}\varepsilon \tag{2.14}$$

where $S^+ = [X(u)+\ell(\varepsilon), +\infty)\times[0,1]$. Gathering (2.13) and (2.14), (2.10) follows.

We can now prove the lemma. Let $(u_n)\subset\Gamma$ be such that $\varphi(u_n)\to c$ and $X(u_n)\to X_0\in\mathbf{R}$. Using (2.8) we have that there exists $R>0$ such that $\|u_n(x,\cdot)\|_{L^2(0,1)}\le R$ for all $x\in\mathbf{R}$, $n\in\mathbf{N}$. Since we also have $\int_{S_0}|\nabla u_n|^2 dx\,dy \le 2\varphi(u_n)$ we conclude that (u_n) is bounded in $H^1_{loc}(S_0)$. Thus, there exists $u_0\in H^1_{loc}(S_0)$ such that, up to a subsequence, $u_n\to u_0$ weakly in $H^1_{loc}(S_0)$.
We shall now prove that $u_0\in\Gamma$ and $\|u_n-u_0\|_{H^1(S_0)}\to 0$.
By (2.9) and (2.10) we derive that for any $\varepsilon>0$ there exists $\ell(\varepsilon)>\bar\ell$ for which

$$\int_{|x-X_0|\ge\ell(\varepsilon)}[\int_0^1 |u_n(x,y)-U(x-X_0,y)|^2\,dy]\,dx \le \gamma\varepsilon \tag{2.15}$$

for any $n\in\mathbf{N}$ sufficiently large. Then, by semicontinuity, we deduce

$$\int_{|x-X_0|\ge\ell(\varepsilon)}[\int_0^1 |u_0(x,y)-U(x-X_0,y)|^2\,dy]\,dx \le \gamma\varepsilon. \tag{2.16}$$

The inequality (2.16) implies, by Lemma 2.3, that $u_0\in\Gamma$ and then $\varphi(u_0)\ge c$. By semicontinuity again we obtain $\varphi(u_0)=c$, i.e. $u_0\in\mathcal{K}$.
To prove $\|u_n-u_0\|_{H^1(S_0)}\to 0$ we note that since $u_n-u_0\to 0$ strongly in $L^2_{loc}(S_0)$, the arbitrariness of ε in (2.15) and (2.16) implies that $u_n-u_0\to 0$ strongly in $L^2(S_0)$. Moreover by Lemma 2.1, since $\varphi(u_n)\to c=\varphi(u_0)$, we obtain $\int_{S_0}|\nabla u_n|^2\,dx\,dy\to\int_{S_0}|\nabla u_0|^2\,dx\,dy$ which, together with the fact that $\nabla u_n\to\nabla u_0$ weakly in $L^2(S_0)$, implies $\|u_n-u_0\|_{H^1(S_0)}\to 0$. $\square$

As direct consequence of Lemma 2.5, we obtain the following estimate which will be useful in the next section.

Lemma 2.6 *For all $r>0$ there exists $h_r>0$ such that if $u\in\Gamma$ and $\varphi(u)\le c+h_r$ then* $\inf_{z\in\mathcal{K}}\|u-z\|_{H^1(S_0)}\le r$.

Proof. Arguing by contradiction, let $r>0$ and $(u_n)\subset\Gamma$ be such that $\varphi(u_n)\to c$ and $\inf_{z\in\mathcal{K}}\|u_n-z\|_{H^1(S_0)}>r$ for all $n\in\mathbf{N}$. Then, setting $v_n(x,y)=u_n(x+[X(u_n)],y)$ (where $[\cdot]$ denote the integer part), by periodicity we have $(v_n)\subset\Gamma$, $\varphi(v_n)\to c$ and $\inf_{z\in\mathcal{K}}\|v_n-z\|_{H^1(S_0)}>r$ for all $n\in\mathbf{N}$. This is in contradiction with Lemma 2.5 since $(X(v_n))\subset[0,1]$. $\square$

Moreover, we can prove the following existence result

Proposition 2.1 *$\mathcal{K}\neq\emptyset$ and any $u\in\mathcal{K}$ satisfies $u\in\mathcal{C}^2(\mathbf{R}^2)$, is a classical solution of $-\Delta u+aW'(u)=0$ on S_0 with $\partial_y u(x,0)=\partial_y u(x,1)=0$ for all $x\in\mathbf{R}$ and $\|u\|_{L^\infty(S_0)}\le R_0$. Finally, $u(x,y)\to\pm b$ as $x\to\pm\infty$, uniformly in $y\in[0,1]$.*

Proof. We note that by periodicity, there always exists a minimizing sequence $(u_n)\subset\Gamma$ for φ such that $X(u_n)\in[0,1]$ for any $n\in\mathbf{N}$. Then, by Lemma 2.5, the set $\mathcal{K}$ is not empty. To complete

the proof we show that any $u \in \mathcal{K}$ is a classical solution of $-\Delta u + aW'(u) = 0$ on S_0 satisfying the above specified properties.
First, note that if $u \in \mathcal{K}$, then, by (H_2), $\|u\|_{L^\infty(S_0)} \le R_0$. Indeed, suppose that $u \in \Gamma$ satisfies $|u| > R_0$ on a set of non zero measure. Then, setting $\tilde{u} = \max\{\min\{u, R_0\}, -R_0\}$ we have $W(\tilde{u}) < W(u)$ on this set and, using Stampacchia inequality, we deduce that $\varphi(\tilde{u}) < \varphi(u)$. Clearly $\tilde{u} \in \Gamma$ and so this contradict the definition of c.
Let $h \in C_0^\infty(S_0)$ and $\delta \in \mathbf{R}$. Then, $u + \delta h \in \Gamma$ and since $u \in \mathcal{K}$, $\varphi(u + \delta h)$ is a C^1 function of δ with a local minimum at $\delta = 0$. Therefore

$$\varphi'(u)h = \int_{S_0} \nabla u \nabla h + aW'(u)h \, dx \, dy = 0$$

for all such h, namely u is a weak solution of $-\Delta u + aW'(u) = 0$ on $\mathbf{R}\times]0,1[$. Standard regularity arguments, then show u is a classical solution and satisfies the Neumann boundary conditions. Moreover, since $\|u\|_{L^\infty(S_0)} \le R_0$, there exists $C > 0$ such that $\|u\|_{\mathcal{C}^2(S_0)} \le C$. This C^2 estimate implies that u satisfies the remaining boundary conditions. Indeed, assume by contradiction that u does not verify $u(x,y) \to -b$ as $x \to -\infty$ uniformly with respect to $y \in [0,1]$. Then, there exist $\delta > 0$ and a sequence $(x_n, y_n) \in S_0$ with $x_n \to -\infty$ and $|u(x_n, y_n) + b| \ge 2\delta$ for all $n \in \mathbf{N}$. The C^2 estimate above implies that there exists $\rho > 0$ such that

$$|u(x,y) + b| \ge \delta \quad \forall (x,y) \in B_\rho(x_n, y_n),\ n \in \mathbf{N}. \tag{2.17}$$

Along a subsequence, $y_n \to y_0$ and thus $|u(x,y)+b| \ge \delta$ for all $(x,y) \in B_{\frac{\rho}{2}}(x_n, y_0)$, n large enough. But this is not possible since $u \in \Gamma$ implies that $\mathsf{d}(u(x,\cdot), -b) \to 0$ as $x \to -\infty$.
One argues analogously to prove the other case. □

We shall now explore the consequence of the reversibility condition (H_1)-(ii), and in particular we will prove that the solution that we find in Proposition 2.1 is in fact a solution of (2.1).

Let $c_p = \inf_{\Gamma_p} \varphi$ where

$$\Gamma_p = \{u \in \Gamma \cap \{\varphi < +\infty\} \mid u(x,0) = u(x,1) \text{ for a.e. } x \in \mathbf{R}\ \}.$$

Then we have

Lemma 2.7 $c_p = c$.

Proof. Since $\Gamma_p \subset \Gamma$, $c_p \ge c$. To prove the equality suppose $c_p > c$. Then, there exists $u \in \Gamma$ such that $\varphi(u) < c_p$. Writing

$$\varphi(u) = \int_{\mathbf{R}} [\int_0^{\frac{1}{2}} \tfrac{1}{2}|\nabla u|^2 + a\,W(u)\,dy]\,dx + \int_{\mathbf{R}} [\int_{\frac{1}{2}}^1 \tfrac{1}{2}|\nabla u|^2 + a\,W(u)\,dy]\,dx = I_1 + I_2$$

it follows that $\min\{I_1, I_2\} < \frac{c_p}{2}$. Suppose e.g. $I_1 < \frac{c_p}{2}$. Define

$$v(x,y) = \begin{cases} u(x,y) & \text{if } x \in \mathbf{R} \text{ and } 0 \le y \le \frac{1}{2}, \\ u(x, 1-y) & \text{if } x \in \mathbf{R} \text{ and } \frac{1}{2} \le y \le 1. \end{cases} \tag{2.18}$$

Then $v \in \Gamma_p$ and, because of (H_1)-(ii), $\varphi(v) = 2I_1 < c_p$, contradicting the definition of c_p. □

We shall now prove, following [15], that any $u \in \mathcal{K}$ is periodic in y.

Lemma 2.8 *If $u \in \mathcal{K}$ then $u(x,0) = u(x,1)$ for all $x \in \mathbf{R}$.*

Proof. Suppose $u \in \mathcal{K}$ and define v as in (2.18). Then, $v \in \Gamma_p$ and it satisfies $v(x,y) = v(x,1-y)$ for all $(x,y) \in S_0$. Moreover, by (H_1)-(ii), $\varphi(u) = \varphi(v) = c = c_p$ and so $v \in \mathcal{K}$. Then, both u and v are local minimum of φ and hence both are classical solution of

$$\begin{cases} -\Delta u + aW'(u) = 0, & \text{on } S_0 \\ \partial_y u(x,0) = \partial_y u(x,1) = 0, & \text{for all } x \in \mathbf{R}. \end{cases} \tag{2.19}$$

Then, letting $w = u - v$, we obtain

$$\begin{cases} -\Delta w(x,y) = b(x,y)w(x,y), & \text{on } S_0 \\ \partial_y w(x,0) = \partial_y w(x,1) = 0, & \text{for all } x \in \mathbf{R} \end{cases}$$

where

$$b(x,y) = \begin{cases} \frac{a(x,y)[W(v(x,y)) - W(u(x,y))]}{u(x,y)-v(x,y)} & \text{if } u(x,y) \neq v(x,y), \\ -a(x,y)W'(u(x,y)) & \text{if } u(x,y) = v(x,y). \end{cases}$$

Note that b is continuous and $w(x,y) = 0$ for $(x,y) \in \mathbf{R} \times [0, \frac{1}{2}]$. But then a local unique continuation theorem for elliptic equations (see e.g. Nirenberg [13]) and a continuation argument imply $w(x,y) = 0$ for $(x,y) \in \mathbf{R} \times [0,1]$, that is $u = v$ in S_0. $\square$

¿From the above considerations it is immediate to derive the existence of solutions of (2.1). Indeed we know, by Proposition 2.1, that the set $\mathcal{K}$ is non empty and that given $u \in \mathcal{K}$, by Proposition 2.1 and Lemma 2.8, we have $u(x,0) = u(x,1)$ and $\partial_y u(x,0) = \partial_y u(x,1) = 0$ for all $x \in \mathbf{R}$. Hence we can extend u by periodicity on $\mathbf{R}^2$, obtaining that u is a classical solution of (2.1). In the sequel, the extension of $\mathcal{K}$ by periodicity on $\mathbf{R}^2$ will be still denoted by $\mathcal{K}$.

We end this section fixing some notation. First note that by the definition of Γ and the proof of Lemma 2.5 the following metric is well defined on Γ

$$\mathsf{D}(u,v)_0 = \|u - v\|_{H^1(S_0)}, \quad \forall u, v \in \Gamma.$$

Moreover, given $u \in \Gamma$ and $A \subset \Gamma$, we set $\mathsf{Dist}(u,A)_0 = \inf\{\mathsf{D}(u,v)_0 \,|\, v \in A\}$. Analogously, setting $S_j = \mathbf{R} \times [j, j+1]$, $j \in \mathbf{Z}$, for $u, v \in H^1_{loc}(\mathbf{R}^2)$ and for $A \subset H^1_{loc}(\mathbf{R}^2)$, we set

$$\mathsf{D}(u,v)_j = \|u - v\|_{H^1(S_j)} \text{ and } \mathsf{D}(u,v)_{j,k} = \|u - v\|_{H^1(S_j \cup \ldots \cup S_k)},$$

and

$$\mathsf{Dist}(u,A)_j = \inf_{v \in A} \mathsf{D}(u,v)_j \text{ and } \mathsf{Dist}(u,A)_{j,k} = \inf_{v \in A} \mathsf{D}(u,v)_{j,k},$$

when the quantities make sense. Finally, for $u \in H^1_{loc}(\mathbf{R}^2)$ and $j \in \mathbf{Z}$ we denote $(j * u)(x,y) = u(x, y+j)$.

3 Heteroclinic type solutions

In the previous section we proved the existence of solutions of (P) which are periodic in the y variable. We say that two solutions z_1 and $z_2 \in \mathcal{K}$ are equivalent, and we write $z_1 \sim_x z_2$, when there exists $j \in Z$ such that $z_1(x+j,y) = z_2(x,y)$ for any $(x,y) \in \mathbf{R}^2$. If the set $\mathcal{K}/\sim_x$ is infinite then Theorem 1.2 holds true. Thus in the rest of the paper we analyze the case where

$$(*) \qquad \mathcal{K}/\sim_x \text{ is finite.}$$

In this section, adapting a variational approach introduced in [14] and [15], we prove that when $(*)$ hold (P) admits heteroclinic type solutions connecting different periodic solutions as $y \to \pm\infty$.

By $(*)$, $\mathcal{K}$ is a countable set of isolated points that we will enumerate by z_ξ, $\xi \in \mathbf{Z}$. Also, setting

$$\alpha = \inf_{\xi \neq \eta} \mathsf{D}(z_\xi, z_\eta)_0$$

we have $\alpha > 0$ since, by $(*)$, $\mathcal{K}$ is locally finite.

Remark 3.1 Assuming $(*)$ is equivalent to ask $\mathcal{K}$ to be constituted by isolated points with respect to the $H^1(S_0)$ metric. Indeed if $\mathcal{K}/\sim_x$ is not finite, there exists a sequence $(u_n) \subset \mathcal{K}$ such that $u_n \neq u_m$ if $n \neq m$ and $X(u_n) \in [0,1)$ for any $n \in \mathbf{N}$. By Lemma 2.5 there exists $u_0 \in \mathcal{K}$ such that along a subsequence $u_n \to u_0$ with respect to the $H^1(S_0)$ distance and so u_0 is not isolated in $\mathcal{K}$.

Let

$$\Sigma = \{u \in H^1_{loc}(\mathbf{R}^2) \mid j * u \in \Gamma, \forall j \in \mathbf{Z}\}$$

and for $u \in \Sigma$, $j \in \mathbf{Z}$, define

$$\varphi_j(u) = \int_{S_j} \tfrac{1}{2}|\nabla u(x,y)|^2 + a(x)W(u(x,y))\,dx\,dy \; - c.$$

We observe that, $\varphi_j(u) = \varphi_0(j*u) = \varphi(j*u) - c$. Also, by definition of Σ, $\varphi_j(u) \geq 0$ for all $u \in \Sigma$, $j \in \mathbf{Z}$ and $\varphi_j(u) = 0$ if and only if $j * u \in \mathcal{K}$.

We consider on Σ the functional

$$\Phi(u) = \sum_{j \in \mathbf{Z}} \varphi_j(u)$$

and following [15], we will look for non periodic solutions of (E) as minima of Φ on suitable subsets of Σ.

We first note that since each φ_j is non negative and lower semicontinuous along weakly convergent sequences in $H^1_{loc}(\mathbf{R}^2)$, the same holds true for the functional Φ.

Lemma 3.1 *Let $(u_n) \subset \Sigma$ and $u \in \Sigma$ be such that $u_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$, then $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$.*

Proof. For any fixed $k \in \mathbf{N}$ we have

$$\liminf_{n\to\infty} \sum_{j=-k}^{k} \varphi_j(u_n) \geq \sum_{j=-k}^{k} \varphi_j(u).$$

Then, if $\Phi(u) = +\infty$ we have also $\liminf_{n\to\infty} \Phi(u_n) = +\infty$. If otherwise $\Phi(u) < +\infty$ then, given any $\varepsilon > 0$ and fixed $k \in \mathbf{N}$ such that $\sum_{|j|>k} \varphi_j(u) < \varepsilon$, we obtain

$$\liminf_{n\to\infty} \Phi(u_n) \geq \liminf_{n\to\infty} \sum_{j=-k}^{k} \varphi_j(u_n) \geq \sum_{j=-k}^{k} \varphi_j(u) \geq \Phi(u) - \varepsilon$$

and the semicontinuity follows since ε is arbitrary. $\square$

Concerning the coerciveness of Φ, we first establish some estimates which will be useful to characterize the compactness properties of sublevels of Φ.

Lemma 3.2 *For any $r > 0$ there exists $\beta_r > 0$ such that if $u \in \Sigma$ and $\varphi_j(u) + \varphi_{j+1}(u) \le \beta_r$, $j \in \mathbf{Z}$, then $\mathsf{Dist}(u, \mathcal{K})_{j,j+1} \le r$.*

Proof. The properties of the functional $\varphi_j + \varphi_{j+1}$ are very similar to the ones of the functional φ and to prove that $\varphi_j + \varphi_{j+1}$ attains its minimum on Σ we can repeat the reasoning of the previous section.
Moreover, as in Lemmas 2.7, 2.8, one can show that for any $j \in \mathbf{Z}$ the infimum of the functional $\varphi_j + \varphi_{j+1}$ on Σ is reached on $\mathcal{K}$. More precisely, $\varphi_j(u) + \varphi_{j+1}(u) = \inf_{v\in\Sigma} \varphi_j(v) + \varphi_{j+1}(v)$ if and only if $u \in \mathcal{K}$. Then, the proof of Lemma 2.6 can be adapted to yield the result. □

According to the previous lemma, we fix $\bar\beta > 0$ such that

$$\text{if } u \in \Sigma \text{ and } j \in \mathbf{Z} \text{ verify } \varphi_j(u) + \varphi_{j+1}(u) \le \bar\beta \text{ then } \mathsf{Dist}(u, \mathcal{K})_{j,j+1} \le \tfrac{\alpha}{4}.$$

We say that $\mathcal{Z}$ is a set of consecutive elements of $\mathbf{Z}$ if and only if $\mathcal{Z}$ is of the form $\{h \in \mathbf{Z} \,|\, j \le h < j + k\}$ or $\{h \in \mathbf{Z} \,|\, j - k < h \le j\}$ for a $j \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{+\infty\}$. Note that such sets may contain an infinite number of elements.

Lemma 3.3 *For any $u \in \Sigma$ and any set $\mathcal{Z}$ of consecutive elements of $\mathbf{Z}$ which satisfy $\varphi_j(u) \le \frac{\bar\beta}{2}$ for any $j \in \mathcal{Z}$, there exists $\xi \in \mathbf{Z}$ such that $\mathsf{D}(u, z_\xi)_j \le \frac{\alpha}{4}$, for all $j \in \mathcal{Z}$.*

Proof. Let $j \in \mathcal{Z}$ be such that $j + 1 \in \mathcal{Z}$. Then $\varphi_j(u) + \varphi_{j+1}(u) \le \bar\beta$, and thus, by definition of $\bar\beta$, $\mathsf{Dist}(u, \mathcal{K})_{j,j+1} \le \frac{\alpha}{4}$. In particular, by definition of α, there exists $\xi_j \in \mathbf{Z}$ such that $\mathsf{D}(u, z_{\xi_j})_{j,j+1} \le \frac{\alpha}{4}$ and therefore $\mathsf{D}(u, z_{\xi_j})_j \le \frac{\alpha}{4}$ and $\mathsf{D}(u, z_{\xi_j})_{j+1} \le \frac{\alpha}{4}$. Clearly, if $j + 2 \in \mathcal{Z}$, then $\xi_j = \xi_{j+1}$ and the lemma follows. □

Lemma 3.3 allows us to get a first boundedness property of the functions in the sublevels of Φ.

Lemma 3.4 *For any $\lambda > 0$ there exists $\Lambda > 0$ such that if $u \in \{\Phi \le \lambda\}$ then $\|j * u - k * u\|_{L^2(S_0)} \le \Lambda$ for any $j, k \in \mathbf{Z}$.*

Proof. First note that if $\Phi(u) \le \lambda$ then $\|j * u - (j + 1) * u\|_{L^2(S_0)} \le (2(\lambda + 2c))^{\frac{1}{2}}$ for any $j \in \mathbf{Z}$. Indeed we have

$$\|j * u - (j + 1) * u\|^2_{L^2(S_0)} = \int_{S_j} |u(x, y + 1) - u(x, y)|^2 dy\, dx$$

and so there exists $\bar y$ in $(j, j + 1)$ such that

$$\int_{\mathbf{R}} |u(x, \bar y + 1) - u(x, \bar y)|^2 dx \ge \|j * u - (j + 1) * u\|^2_{L^2(S_0)}.$$

On the other hand

$$\begin{aligned} 2(\Phi(u) + 2c) &\ge \int_j^{j+2} [\int_{\mathbf{R}} |\partial_y u(x, y)|^2 dx]\, dy \ge \int_{\mathbf{R}} [\int_{\bar y}^{\bar y+1} |\partial_y u(x, y)|^2 dy]\, dx \\ &\ge \int_{\mathbf{R}} [\int_{\bar y}^{\bar y+1} |\partial_y u(x, y)| dy]^2\, dx \ge \int_{\mathbf{R}} |u(x, \bar y + 1) - u(x, \bar y)|^2 dx \end{aligned}$$

and thus, by the choice of $\bar y$, we get $\|j * u - (j + 1) * u\|^2_{L^2(S_0)} \le 2(\Phi(u) + 2c)$.
Now, let $u \in \Sigma$ be such that $\Phi(u) \le \lambda$. We denote by $L(u)$ the subset of $\mathbf{Z}$ for which $\varphi_j(u) \ge \frac{\bar\beta}{2}$

if $j \in L(u)$. Plainly, the number $l(u)$ of elements of $L(u)$ is at most $[\frac{2\lambda}{\beta}]+1$ where $[\cdot]$ denote the integer part. Then, the set $\mathbf{Z}\backslash L(u)$ is constituted of $\bar{l}(u)$ sets of consecutive elements of $\mathbf{Z}$, $\mathcal{Z}_i(u)$, with $\bar{l}(u) \leq l(u)+1$. We have, for any $k, j \in \mathbf{Z}$,

$$\begin{aligned} \|k*u - j*u\|_{L^2(S_0)} \quad &\leq \quad (l(u)+1)\sup_{j\in\mathbf{Z}} \|j*u-(j+1)*u\|_{L^2(S_0)} \\ &\quad + \sum_{i=1}^{\bar{l}(u)} \sup_{j,k\in\mathcal{Z}_i(u)} \|j*u-k*u\|_{L^2(S_0)}. \end{aligned} \tag{3.1}$$

Then, the first term in the right side of (3.1) is bounded by $(l(u)+1)(2(\lambda+2c))^{\frac{1}{2}}$ and by Lemma 3.3 the sum by $\bar{l}(u)\frac{\alpha}{2}$. Therefore the lemma follows. □

The following lemma describes a first compactness property of the functional Φ.

Lemma 3.5 *Let $(u_n) \subset \Sigma \cap \{\Phi \leq \lambda\}$ be such that there exist $\bar{\Lambda} < +\infty$ and $\xi \in \mathbf{Z}$ for which, for all $n \in \mathbf{N}$, $\mathsf{D}(u_n, z_\xi)_{j_n} \leq \bar{\Lambda}$, for a $j_n \in \mathbf{Z}$. Then, there exists $u \in \Sigma \cap \{\Phi \leq \lambda\}$ such that, up to a subsequence, $u_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$.*

Proof. By Lemma 3.4, there exists $\Lambda > 0$ such that if $u \in \Sigma \cap \{\Phi \leq \lambda\}$ then $\|k*u - k'*u\|_{L^2(S_0)} \leq \Lambda$ for any $k', k \in \mathbf{Z}$ and if $\mathsf{D}(u, z_\xi)_j \leq \bar{\Lambda}$, for some $j \in \mathbf{Z}$, we obtain $\|u - z_\xi\|_{L^2(S_k)} \leq \bar{\Lambda}+\Lambda$ for any $k \in \mathbf{Z}$.
Thus, given $(u_n) \subset \Sigma \cap \{\Phi \leq \lambda\}$ as in the statement, for any $L \in \mathbf{N}$, setting $T_L = \mathbf{R}\times(-L, L)$ we obtain

$$\|u_n - z_\xi\|^2_{L^2(T_L)} + \|\nabla u_n\|^2_{L^2(T_L)} \leq 2L(\Lambda+\bar{\Lambda})^2 + 2Lc + \lambda.$$

and we deduce that $(u_n - z_\xi)$ is bounded in $H^1(T_L)$ for any $L \in \mathbf{N}$. This implies that there exists a subsequence (u_{n_k}) of (u_n) and a function u such that $u - z_\xi \in \cap_{L>0} H^1(T_L)$ and $u_{n_k} - z_\xi \to u - z_\xi$ weakly in $H^1_{loc}(\mathbf{R}^2)$. Since $u - z_\xi \in \cap_{L>0} H^1(T_L)$ we have that $u - z_\xi \in H^1(S_j)$ for any $j \in \mathbf{Z}$ from which we deduce that $u \in \Sigma$ and $\Phi(u) \leq \lambda$. □

¿From Lemma 3.3 we also deduce the following result concerning the asymptotic behavior of the functions in the sublevels of Φ.

Lemma 3.6 *If $u \in \Sigma \cap \{\Phi < +\infty\}$, there exist $\xi^\pm \in \mathbf{Z}$ such that*

$$\mathsf{D}(u, z_{\xi^\pm})_j \to 0 \quad \textit{as} \quad j \to \pm\infty.$$

Proof. Since $\Phi(u) < +\infty$, $\varphi_j(u) \to 0$ as $|j| \to \infty$ and there exists j_0 such that $\varphi_j(u) \leq \frac{\bar{\beta}}{2}$ for any $|j| \geq j_0$. Thus, by Lemma 3.3, there exists $\xi^\pm \in \mathbf{Z}$ such that $\mathsf{D}(u, z_{\xi^+})_j \leq \frac{\alpha}{4}$ for $j \geq j_0$ and $\mathsf{D}(u, z_{\xi^-})_j \leq \frac{\alpha}{4}$ for $j \leq -j_0$. Using again that $\varphi_j(u) \to 0$ as $|j| \to \infty$, we can conclude because of Lemma 2.6. □

By Lemma 3.6 we can restrict ourselves to consider the elements in Σ which have prescribed limits as $j \to \pm\infty$. For $\xi \in \mathbf{Z}$, we consider the class

$$\mathcal{H}_\xi = \{u \in \Sigma \,|\, \lim_{j\to-\infty} \mathsf{D}(u, z_0)_j = \lim_{j\to+\infty} \mathsf{D}(u, z_\xi)_j = 0\}.$$

By Lemma 3.5 every sequence in $\mathcal{H}_\xi \cap \{\Phi \leq \lambda\}$ admits a subsequence which converge in the specified sense to some $u \in \Sigma$ and along which the functional is lower semicontinuous. The next step in our proof is to show that there exist particular classes $\mathcal{H}_\xi$ on which suitable sublevels of Φ satisfy additional compactness properties.

We define

$$m_\xi = \inf_{u\in\mathcal{H}_\xi} \Phi(u) \quad \text{and} \quad \mathcal{K}_\xi = \{u \in \mathcal{H}_\xi \,|\, \Phi(u) = m_\xi\}, \quad \xi \in \mathbf{Z}.$$

Using suitable test functions one can prove that $m_\xi < +\infty$ for any $\xi \in \mathbf{Z}$. Moreover

Lemma 3.7 *There results $m_\xi \geq \frac{\bar\beta}{2}$ for any $\xi \neq 0$ and $m_\xi \to +\infty$ as $|\xi| \to \infty$.*

Proof. We observe that $\mathsf{D}(z_0, z_\xi)_0 \to +\infty$ as $|\xi| \to \infty$ and thus, from the definition of $\mathcal{H}_\xi$ and Lemma 3.4, it readily follows that $m_\xi \to +\infty$ as $|\xi| \to \infty$. To prove the first estimate let $\xi \neq 0$ and $u \in \mathcal{H}_\xi$. We have $\mathsf{D}(u, z_0)_j \to 0$ as $j \to -\infty$ while $\liminf_{j\to+\infty} \mathsf{D}(u, z_0)_j \geq \mathsf{D}(z_0, z_\xi)_0 \geq \alpha$. Thus, by Lemma 3.3, there exists $k_0 \in \mathbf{Z}$ such that $\varphi_{k_0}(u) \geq \frac{\bar\beta}{2}$ and hence $\Phi(u) \geq \varphi_{k_0}(u) \geq \frac{\bar\beta}{2}$. □

By Lemma 3.7 there exists $\eta \in \mathbf{Z}$ such that

$$m_\eta = \min_{\xi\neq 0} m_\xi.$$

As we will see in the next lemma, the minimality property of η allows us to further characterize the functions in $\mathcal{H}_\eta$ whose action is close to m_η.

Remark 3.2 For $u \in \Sigma$, $\xi \in \mathbf{Z}$ and $j \in \mathbf{Z}$ we define

$$\chi^+_{\xi,j}(u)(x,y) = \begin{cases} u(x,y) & \text{if } y \in (-\infty, j], \\ u(x,y)(j+1-y) + z_\xi(x,y)(y-j) & \text{if } y \in (j, j+1), \\ z_\xi(x,y) & \text{if } y \in [j+1, +\infty) \end{cases}$$

and

$$\chi^-_{\xi,j}(u)(x,y) = \begin{cases} z_\xi(x,y) & \text{if } y \in (-\infty, j], \\ u(x,y)(y-j) + z_\xi(x,y)(j+1-y) & \text{if } y \in (j, j+1), \\ u(x,y) & \text{if } y \in [j+1, +\infty). \end{cases}$$

Note that, by continuity, for any $\delta > 0$ there exists $\tilde\lambda_\delta$, with $\tilde\lambda_\delta \to 0$ as $\delta \to 0$, such that, for any $u \in \Sigma$ and $j \in \mathbf{Z}$, if $\mathsf{D}(u, z_0)_j \leq \delta$, then $\varphi_j(u) \leq \tilde\lambda_\delta$. Then, by definition, one can easily see that the following estimate holds. If $u \in \Sigma$ is such that $\mathsf{D}(u, z_\xi)_j \leq \delta$ for some $\xi \in \mathbf{Z}$ and $j \in \mathbf{Z}$, then

$$\sum_{k=p}^{s} \varphi_k(\chi^\pm_{\xi,j}(u)) \leq \sum_{k=p}^{s} \varphi_k(u) + 2\tilde\lambda_\delta$$

for all $p < j < s - 1$.

Then we obtain

Lemma 3.8 *There exists $\tilde\delta_0 \in (0, \frac{\alpha}{4})$ such that for any $\delta \in (0, \tilde\delta_0)$ there exist $\tilde\rho_\delta > 0$ and $\tilde\ell_\delta \in \mathbf{N}$ for which, if $u \in \mathcal{H}_\eta$ and $\Phi(u) \leq m_\eta + \tilde\lambda_\delta$, then*

(i) if $\mathsf{Dist}(u, \mathcal{K})_j \geq \delta$ for all j in a set $\mathcal{Z}$ of consecutive numbers, then $Card(\mathcal{Z}) \leq \tilde\ell_\delta$,

(ii) if $\mathsf{D}(u, z_0)_{j_0} \leq \delta$, then $\mathsf{D}(u, z_0)_j \leq \tilde\rho_\delta$, $\forall j \leq j_0$,

(iii) if $\mathsf{D}(u, z_\eta)_{j_0} \leq \delta$, then $\mathsf{D}(u, z_\eta)_j \leq \tilde\rho_\delta$, $\forall j \geq j_0$,

(iv) if $\xi \in \mathbf{Z}\backslash\{0, \eta\}$, then $\mathsf{D}(u, z_\xi)_j > \delta$ for all $j \in \mathbf{Z}$.

Moreover $\tilde\rho_\delta \downarrow 0$ as $\delta \to 0^+$.

Proof. Note that (i) plainly follows from Lemma 2.6.
To prove (ii)-(iv) we shall need some additional definitions. We fix $\tilde{\delta}_0 \in (0, \frac{\alpha}{4})$ such that $\tilde{\lambda}_\delta < \frac{m_\eta}{4}$ for all $\delta \in (0, 2\tilde{\delta}_0)$. Moreover, by Lemma 2.6, it is also possible to choose a function f such that, for any $u \in \Sigma$ and $j \in \mathbf{Z}$ if $\varphi_j(u) \le \delta$ then $\mathsf{Dist}(u, \mathcal{K})_j \le f(\delta)$ with $f(\delta) \to 0$ as $\delta \to 0$. Clearly it is not restrictive to assume that f is a non decreasing function. Then, for any $\delta \in (0, \tilde{\delta}_0)$ we set $\tilde{\rho}_\delta = f(2\tilde{\lambda}_\delta)$.
Let $u \in \mathcal{H}_\eta$ be such that $\Phi(u) \le m_\eta + \tilde{\lambda}_\delta$ and assume that $j_0 \in \mathbf{Z}$ is such that $\mathsf{D}(u, z_0)_{j_0} \le \delta$. We define $\tilde{u} = \chi^-_{0,j_0}(u)$, noting that $\tilde{u} \in \mathcal{H}_\eta$. Then, we obtain

$$m_\eta \quad \le \quad \Phi(\tilde{u}) = \Phi(u) - \sum\nolimits_{k=-\infty}^{j_0} \varphi_k(u) + \varphi_{j_0}(\tilde{u}).$$

On S_{j_0} we have $\tilde{u}(x, y) - z_0(x, y) = (y - j_0)(u(x, y) - z_0(x, y))$ and thus we obtain $\mathsf{D}(\tilde{u}, z_0)_{j_0} \le \mathsf{D}(u, z_0)_{j_0} \le \delta$ from which $\varphi_{j_0}(\tilde{u}) \le \tilde{\lambda}_\delta$. Therefore

$$m_\eta \quad \le \quad \Phi(\tilde{u}) \le \Phi(u) - \sum_{k=-\infty}^{j_0} \varphi_k(u) + \tilde{\lambda}_\delta$$

from which, since $\Phi(u) \le m_\eta + \tilde{\lambda}_\delta$ we conclude that $\sum_{-\infty}^{j_0} \varphi_k(u) \le 2\tilde{\lambda}_\delta$. In particular $\varphi_j(u) \le 2\tilde{\lambda}_\delta$ for any $j \le j_0$ and thus $\mathsf{D}(u, z_0)_j \le \tilde{\rho}_\delta$ for any $j \le j_0$. This proves (ii). In a similar way one proves (iii).
To prove the last part of the lemma we argue by contradiction assuming that there exist $j_0 \in \mathbf{Z}$ and $\xi \in \mathbf{Z} \setminus \{0, \eta\}$ such that $\mathsf{D}(u, z_\xi)_{j_0} \le \delta$. Let $u_1 = \chi^+_{\xi,j_0}(u)$ and $u_2 = \chi^-_{\xi,j_0}(u)$ and note that $u_1 \in \mathcal{H}_\xi$ while $u_2(\cdot - \xi, \cdot) \in \mathcal{H}_{\eta-\xi}$. Thus, by the choice of η, we obtain $\Phi(u_1) + \Phi(u_2) \ge 2m_\eta$. Now, we have

$$\Phi(u_1) + \Phi(u_2) \le \Phi(u) + \varphi_{j_0}(u_1) + \varphi_{j_0}(u_2) \le m_\eta + \tilde{\lambda}_\delta + \varphi_{j_0}(u_1) + \varphi_{j_0}(u_2)$$

but since $\mathsf{D}(u, z_0)_{j_0} \le \delta$, arguing as above, we obtain $\varphi_{j_0}(u_1) + \varphi_{j_0}(u_2) \le 2\tilde{\lambda}_\delta$, which leads to the contradiction $3\tilde{\lambda}_\delta \ge m_\eta$. □

According to Lemma 3.8, we fix $\tilde{\delta} \in (0, \tilde{\delta}_0)$ such that $\tilde{\rho} = \tilde{\rho}_{\tilde{\delta}} < \frac{\alpha}{4}$ and we will denote $\tilde{\lambda} = \tilde{\lambda}_{\tilde{\delta}}$ and $\tilde{\ell} = \tilde{\ell}_{\tilde{\delta}}$.

We are now able to prove the following compactness property of the minimizing sequences of Φ in $\mathcal{H}_\eta$. They will be sufficient to use the direct method of the Calculus of Variation to show that the functional Φ admits a minimum in the class $\mathcal{H}_\eta$ as stated in Theorem 3.1 below.

Lemma 3.9 *Let $(u_n) \subset \mathcal{H}_\eta$ and $\delta > 0$ be such that $\Phi(u_n) \to m_\eta$ and $\mathsf{D}(u_n, \mathcal{K})_0 \ge \delta$ for any $n \in \mathbf{N}$. Then there exists $u \in \mathcal{K}_\eta$ such that, up to a subsequence, $u_n \to u$ as $n \to \infty$ weakly in $H^1_{loc}(\mathbf{R}^2)$.*

Proof. Without loss of generality we can assume that $\delta \le \tilde{\delta}$ and that $\Phi(u_n) \le m_\eta + \tilde{\lambda}_\delta$ for any $n \in \mathbf{N}$. Then $\rho_\delta < \tilde{\rho}_\delta \le \frac{\alpha}{4}$ and, since $\mathsf{D}(u_n, \mathcal{K})_0 \ge \delta$, by Lemma 3.8 we obtain that for any $n \in \mathbf{N}$,

$$\mathsf{D}(u_n, z_0)_j \le \frac{\alpha}{4},\ \forall j \le -\tilde{\ell}_\delta \quad \text{and} \quad \mathsf{D}(u_n, z_\eta)_j \le \frac{\alpha}{4},\ \forall j \ge \tilde{\ell}_\delta. \tag{3.2}$$

By Lemma 3.5 there exists $u \in \Sigma$ such that along a subsequence $u_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$. Finally, by (3.2) and the weakly lower semicontinuity of the distance we obtain

$$\mathsf{D}(u, z_0)_j \le \frac{\alpha}{4},\ \forall j \le -\tilde{\ell}_\delta \quad \text{and} \quad \mathsf{D}(u, z_\eta)_j \le \frac{\alpha}{4},\ \forall j \ge \tilde{\ell}_\delta. \tag{3.3}$$

By Lemma 3.6, we conclude that $\mathsf{D}(u, z_0)_j \to 0$ as $j \to -\infty$ and $\mathsf{D}(u, z_\eta) \to 0$ as $j \to +\infty$, that is $u \in \mathcal{H}_\eta$. Then, by semicontinuity, $\Phi(u) = m_\eta$ and the lemma follows. □

Theorem 3.1 *If $(*)$ holds there exists $u \in \mathcal{K}_\eta$. Moreover, $u \in \mathcal{C}^2(\mathbf{R}^2)$ and is a classical solution of (P) with $\|u\|_{L^\infty(\mathbf{R}^2)} \le R_0$.*

Proof. Let $(u_n) \subset \mathcal{H}_\eta$ be such that $\Phi(u_n) \to m_\eta$. By (H_2)-(ii) we can assume that $\|u_n\|_{L^\infty(\mathbf{R}^2)} \le R_0$, indeed otherwise one can consider the minimizing sequence $\tilde{u}_n = \max\{\min\{u_n, R_0\}, -R_0\}$. Moreover since, for any $n \in \mathbf{N}$, $\mathsf{D}(u_n, z_0)_j \to 0$ as $j \to -\infty$ and $\liminf_{j\to+\infty} \mathsf{D}(u_n, z_0)_j \ge \mathsf{D}(z_0, z_\eta)_0 \ge \alpha$, by Lemma 3.3, there exists $k_n \in \mathbf{Z}$ such that $\varphi_{k_n}(u_n) \ge \frac{\bar\beta}{2}$. Then, by $(*)$, there exists $\delta > 0$ such that $\mathsf{Dist}(u_n, \mathcal{K})_{k_n} \ge \delta$ for any $n \in \mathbf{N}$. Setting $v_n = u_n(\cdot, \cdot + k_n)$, we have $v_n \in \mathcal{H}_\eta$, $\Phi(v_n) = \Phi(u_n)$ and $\mathsf{Dist}(v_n, \mathcal{K})_0 \ge \delta$. By Lemma 3.9 there exists $u \in \mathcal{K}_\eta$ such that along a subsequence $v_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$ and by semicontinuity $\|u\|_{L^\infty(\mathbf{R}^2)} \le R_0$.
Standard elliptic arguments implies that $u \in \mathcal{C}^2(\mathbf{R}^2)$ and is a classical solution of (E). Moreover, since $\|u\|_{L^\infty(\mathbf{R}^2)} \le R_0$ and $\mathsf{D}(u, z_0)_j \to 0$ as $j \to -\infty$ and $\mathsf{D}(u, z_\eta) \to 0$ as $j \to +\infty$, we can prove, as in Proposition 2.1 and in [5], that u satisfies the asymptotic conditions in the x variable uniformly with respect to $y \in \mathbf{R}$. $\square$

As last step of this section we further characterize the compactness properties of the minimizing sequences of Φ in $\mathcal{H}_\eta$. Lemma 3.10 below strengthens the result obtained in Lemma 3.9 and will be used in the next section.

Lemma 3.10 *Let $(u_n) \subset \mathcal{H}_\eta$ and $\delta > 0$ be such that $\Phi(u_n) \to m_\eta$ and $\mathsf{D}(u_n, \mathcal{K})_0 \ge \delta$ for any $n \in \mathbf{N}$. Then there exists $u \in \mathcal{K}_\eta$ such that, up to a subsequence, $\mathsf{D}(u_n, u)_j \to 0$ as $n \to \infty$, for all $j \in \mathbf{Z}$.*

Proof. By Lemma 3.9 there exists a subsequence of (u_n), still denoted (u_n), and $u \in \mathcal{K}_\eta$ such that $u_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$.
We claim that $\varphi_j(u_n) \to \varphi_j(u)$ for all $j \in \mathbf{Z}$.
Indeed by semicontinuity, $\varphi_j(u) \le \liminf \varphi_j(u_n)$ for all $j \in \mathbf{Z}$. By contradiction, assume that there exists $j_0 \in \mathbf{Z}$ such that $\limsup(\varphi_{j_0}(u_n) - \varphi_{j_0}(u)) = \varepsilon > 0$ and so that there exists a subsequence (u_{n_i}) of (u_n) such that $\varphi_{j_0}(u_{n_i}) - \varphi_{j_0}(u) \to \varepsilon$ as $i \to \infty$. Since

$$\sum_{|j|\ge k} (\varphi_j(u_n) - \varphi_j(u)) \ge -\sum_{|j|\ge k} \varphi_j(u) \to 0, \quad \text{as } k \to \infty$$

we can fix $k > |j_0|$ such that $\sum_{|j|\ge k}(\varphi_j(u_n) - \varphi_j(u)) \ge -\frac{\varepsilon}{2}$. Then, setting $b_n = \Phi(u_n) - \Phi(u) = \sum_{j\in\mathbf{Z}}(\varphi_j(u_n) - \varphi_j(u))$, since $\Phi(u_n) \to m_\eta = \Phi(u)$, we have $b_n \to 0$ as $n \to \infty$, while

$$\begin{aligned} b_{n_i} &= \varphi_{j_0}(u_{n_i}) - \varphi_{j_0}(u) + \sum_{|j|<k, j\ne j_0} (\varphi_j(u_{n_i}) - \varphi_j(u)) + \sum_{|j|\ge k} (\varphi_j(u_{n_i}) - \varphi_j(u)) \\ &\ge \varphi_{j_0}(u_{n_i}) - \varphi_{j_0}(u) + \sum_{|j|<k, j\ne j_0} (\varphi_j(u_{n_i}) - \varphi_j(u)) - \frac{\varepsilon}{2}. \end{aligned}$$

Since $\liminf(\varphi_j(u_n) - \varphi_j(u)) \ge 0$ for all $j \in \mathbf{Z}$, we obtain the contradiction $\liminf b_{n_i} \ge \frac{\varepsilon}{2}$.
In the sequel we will consider only the strip S_0, the proof being the same for the other strips.
Recalling that $\varphi(u) = \varphi_0(u) + c$, we have already proved that $\varphi(u_n) \to \varphi(u)$. By Lemma 2.1 we obtain that

$$\int_{S_0} |\nabla u_n|^2\, dx\, dy \to \int_{S_0} |\nabla u|^2\, dx\, dy \text{ and } \int_{S_0} a(x,y) W(u_n)\, dx\, dy \to \int_{S_0} a(x,y) W(u)\, dx\, dy. \quad (3.4)$$

Since (∇u_n) is a bounded sequence in $L^2(S_0)$, we deduce that $\nabla u_n \to \nabla u$ weakly in $L^2(S_0)$ (recall that $u_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$). Then, since by (3.4) $\|\nabla u_n\|_{L^2(S_0)} \to \|\nabla u\|_{L^2(S_0)}$, we conclude

$$\nabla u_n \to \nabla u \text{ in } L^2(S_0) \text{ as } n \to \infty. \quad (3.5)$$

By (3.5), to prove the lemma it remains to show that $u_n - u \to 0$ in $L^2(S_0)$. For that, since $u_n - u \to 0$ in $L^2_{loc}(\mathbf{R}^2)$, it suffices that for any $\varepsilon > 0$ there exist $n(\varepsilon) \in \mathbf{N}$ and $L_\varepsilon > 0$ such that

$$\int_{|x|>L_\varepsilon} \int_0^1 |u_n - u|^2 \, dydx \le \varepsilon \qquad \text{for any } n \ge n(\varepsilon).$$

We shall prove that for any $\varepsilon > 0$ there exist $n(\varepsilon) \in \mathbf{N}$ and $L_\varepsilon > 0$ such that

$$\textstyle\int_{x<L_\varepsilon} \int_0^1 |u+b|^2 \, dydx + \int_{x>L_\varepsilon} \int_0^1 |u-b|^2 \, dydx \le \varepsilon, \tag{3.6}$$

$$\textstyle\int_{x<L_\varepsilon} \int_0^1 |u_n+b|^2 \, dydx + \int_{x>L_\varepsilon} \int_0^1 |u_n-b|^2 \, dydx \le \varepsilon \qquad \text{for any } n \ge n(\varepsilon) \tag{3.7}$$

and the proof will be complete.
So let us fix $\varepsilon > 0$. Since $\varphi(u) < +\infty$ there exists $L_\varepsilon > 0$ such that

$$\int_{|x|>L_\varepsilon} [\int_0^1 \tfrac{1}{2}|\nabla u|^2 + a(x,y)W(u)\, dy]\, dx \le \tfrac{\varepsilon}{8} \min\{1, w_0 \underline{\mathrm{a}}\}, \tag{3.8}$$

where w_0 is given by (2.11) and $\underline{\mathrm{a}} = \min_{\mathbf{R}^2} a(x,y)$.
Since $u(x,y) \to \pm b$ as $x \to \pm\infty$ uniformly with respect to $y \in [0,1]$, setting $S^- = (-\infty, -L_\varepsilon) \times [0,1]$ and $S^+ = (L_\varepsilon, +\infty) \times [0,1]$, we can assume that L_ε is so big that

$$|u(x,y) + b| \le \frac{b}{4} \text{ for any } (x,y) \in S^- \text{ and } |u(x,y) - b| \le \frac{b}{4} \text{ for any } (x,y) \in S^+. \tag{3.9}$$

By (2.11), (3.8) and (3.9) we obtain

$$\begin{aligned} \int_{S^-} |u+b|^2 \, dy\, dx + \int_{S^+} |u-b|^2 \, dy\, dx &\le \frac{1}{w_0} \int_{|x|>L_\varepsilon} \int_0^1 W(u)\, dx\, dy \\ &\le \frac{1}{\underline{\mathrm{a}} w_0} \int_{|x|>L_\varepsilon} \int_0^1 a(x,y)W(u)\, dx\, dy \le \tfrac{\varepsilon}{8} \end{aligned}$$

and (3.6) follows.
To prove (3.7) note that, by (3.4) and Remark 2.1, $\int_{|x|>L_\varepsilon} [\int_0^1 \frac{1}{2}|\nabla u_n|^2 + a(x,y)W(u_n)\, dy]\, dx \to \int_{|x|>L_\varepsilon} [\int_0^1 \frac{1}{2}|\nabla u|^2 + a(x,y)W(u_n)\, dy]\, dx$. Hence there exists $n(\varepsilon) \in \mathbf{N}$, such that

$$\int_{|x|>L_\varepsilon} \int_0^1 \tfrac{1}{2}|\nabla u_n|^2 + a(x,y)W(u_n)\, dy\, dx \le \frac{\varepsilon}{4} \min\{1, w_0 \underline{\mathrm{a}}\}, \quad \forall n \ge n(\varepsilon). \tag{3.10}$$

Then, we deduce

$$\mathsf{d}(u_n(x,\cdot), b) \le \frac{b}{4}, \quad \forall |x| > L_\varepsilon,\, n \ge n(\varepsilon). \tag{3.11}$$

Indeed, for any $n \in \mathbf{N}$ there exists $L_n > 0$ such that $\mathsf{d}(u_n(x,\cdot), b) \le \frac{b}{8}$ for all $x > L_n$. Arguing by contradiction assume that for some $n \ge n(\varepsilon)$ there exists $(x_1, x_2) \subset (L_\varepsilon, +\infty)$ such that $\mathsf{d}(u_n(x_1,\cdot), b) = \frac{b}{4}$, $\mathsf{d}(u_n(x_2,\cdot), b) = \frac{b}{8}$ and $\frac{b}{8} < \mathsf{d}(u_n(x,\cdot), b) < \frac{b}{4}$ for all $x \in (x_1, x_2)$. Then, by Lemma 2.2, we obtain

$$\begin{aligned} \int_{x>L_\varepsilon} [\int_0^1 \tfrac{1}{2}|\nabla u_n|^2 + a(x,y)W(u_n)\, dy]\, dx &\ge \int_{x_1}^{x_2} [\int_0^1 \tfrac{1}{2}|\nabla u_n|^2 + a(x,y)W(u_n)\, dy]\, dx \\ &\ge \mu_{\frac{b}{8}} \mathsf{d}(u_n(x_1,\cdot), u_n(x_2,\cdot)) \ge \mu_{\frac{b}{8}} \frac{b}{8} \end{aligned}$$

which in in contradiction with (3.10) for ε small enough.
Arguing as in step (3) of Lemma 2.5, letting $A_n = \{(x,y) \in S^+ \,|\, |u_n(x,y) - b| > \frac{b}{2}\}$, by (2.11), (3.10) and (3.11), for $n \geq n(\varepsilon)$, one proves that

$$\int_{S^+\setminus A_n} |u_n(x,y) - b|^2\, dx\, dy \leq \frac{1}{w_0} \int_{S^+\setminus A_n} W(u_n(x,y))\, dx\, dy \leq \frac{\varepsilon}{4}$$

and

$$\int_{A_n} |u_n(x,y) - b|^2\, dx\, dy \leq \frac{\varepsilon}{4}.$$

Then, we conclude

$$\int_{S^+} |u_n(x,y) - b|^2\, dx\, dy \leq \frac{\varepsilon}{2}, \quad \forall n \geq n(\varepsilon). \tag{3.12}$$

In the same way one obtains that for n big enough

$$\int_{S^-} |u_n(x,y) + b|^2\, dx\, dy \leq \frac{\varepsilon}{2} \tag{3.13}$$

and (3.7) follows. □

4 Multibump type solutions

In the previous section we proved that if $(*)$ holds, that is if the set $\mathcal{K}$ is finite modulo periodic translations, then $\mathcal{K}_\eta$ is not empty. We say that u and $v \in \mathcal{K}_\eta$ are equivalent and we write $u \sim_y v$ if there exists $j \in \mathbf{Z}$ such that $u(x, y + j) = v(x, y)$ for $(x, y) \in \mathbf{R}^2$. If $(*)$ holds and $\mathcal{K}_\eta/\sim_y$ is infinite then Theorem 1.2 holds true. In the sequel we will study the alternative case

$(**)$ $\qquad$ $\mathcal{K}_\eta/\sim_y$ is finite

showing that if $(*)$ and $(**)$ hold true then (P) admits, modulo periodic translations, infinitely many multibump heteroclinic type solutions. This will end the proof of Theorem 1.2.

Remark 4.1 Let $J = \{\mathsf{D}(u, z_0)_0 \,|\, u \in \mathcal{K}_\eta\}$. By $(**)$ the set J is countable and so the set $\Delta = (0, \tilde{\delta}) \setminus J$ is a non countable, dense subset of $(0, \tilde{\delta})$.

The following result easily follows by Lemma 3.10.

Lemma 4.1 *For all $\delta \in \Delta$ there exists $\Lambda \in (0, \tilde{\lambda})$ such that if $u \in \mathcal{H}_\eta$ satisfies $\mathsf{D}(u, z_0)_0 = \delta$, then $\Phi(u) \geq m_\eta + \Lambda$.*

Proof. By contradiction assume that there exists a sequence $(u_n) \subset \mathcal{H}_\eta$ such that $\Phi(u_n) \to m_\eta$ and $\mathsf{D}(u_n, z_0)_0 = \delta$. Since $\delta < \tilde{\tilde{\delta}} \leq \frac{\alpha}{4}$, we have $\mathsf{D}(u_n, \mathcal{K})_0 = \delta$ and by Lemma 3.10 there exists $u \in \mathcal{K}_\eta$ such that up to a subsequence $u_n \to u$ in $H^1(S_0)$. Then $\mathsf{D}(u, z_0)_0 = \delta$ and $u \in \mathcal{K}_\eta$ in contradiction with the assumption $\delta \in \Delta$. □

Remark 4.2 For $u \in H^1_{loc}(\mathbf{R}^2)$, let $\bar{u}(x,y) = u(x,-y)$, $(x,y) \in \mathbf{R}^2$, and set

$$\overline{\mathcal{H}}_\eta = \{u \in \Sigma \,|\, \bar{u} \in \mathcal{H}_\eta\}.$$

By (H_1)-(ii), we have $\Phi(u) = \Phi(\bar{u})$ and so that $\inf_{\overline{\mathcal{H}}_\eta} \Phi = m_\eta$. Setting $\overline{\mathcal{K}}_\eta = \{u \in \overline{\mathcal{H}}_\eta \,|\, \Phi(u) = m_\eta\} = \{u \in \Sigma \,|\, \bar{u} \in \mathcal{K}_\eta\}$, $\overline{J} = \{\mathsf{D}(u, z_\eta)_0 \,|\, u \in \overline{\mathcal{K}}_\eta\}$ and $\overline{\Delta} = (0, \tilde{\delta}) \setminus \overline{J}$, arguing as in Lemma 4.1, one proves that for all $\delta \in \overline{\Delta}$ there exists $\Lambda \in (0, \tilde{\lambda})$ such that if $u \in \overline{\mathcal{H}}_\eta$ satisfies $\mathsf{D}(u, z_\eta)_0 = \delta$, then $\Phi(u) \geq m_\eta + \Lambda$.

If $(**)$ holds, we have that $\overline{\mathcal{K}}_\eta / \sim_y$ is finite and so both the sets Δ and $\overline{\Delta}$ are uncountable and dense in $(0, \tilde{\delta})$. Then we can choose $\delta \in \Delta \cap \overline{\Delta}$ and $\Lambda \in (0, \tilde{\lambda})$ such that

$$\text{if } u \in \mathcal{H}_\eta \text{ and } \mathsf{D}(u, z_0)_0 = \delta \text{ or } u \in \overline{\mathcal{H}}_\eta \text{ and } \mathsf{D}(u, z_\eta)_0 = \delta, \text{ then } \Phi(u) \geq m_\eta + \Lambda. \tag{4.1}$$

Let us fix some constants. Let $\tilde{\Lambda} > 0$ be such that $\tilde{\Lambda} < \min(\tilde{\lambda}_\delta, \frac{\Lambda}{2})$ where Λ and δ are given by the previous remark and $\tilde{\lambda}_\delta$ by Remark 3.2. Let $\varepsilon \in (0, \delta)$ be such that $\tilde{\lambda}_\varepsilon < \frac{\tilde{\Lambda}}{8}$. Moreover, let $n \in \mathbf{N}$ be such that $n > \frac{m_\eta + \tilde{\Lambda}}{h_\varepsilon} + 2$, where h_ε is given by Lemma 2.6, and such that there exists $u_\eta \in \mathcal{K}_\eta$ with

$$\mathsf{D}(u_\eta, z_0)_{-n} \leq \delta \quad \text{and} \quad \mathsf{D}(u_\eta, z_\eta)_n \leq \delta.$$

We define,

$$\mathcal{U}_0 = \{u \in \Sigma \mid \mathsf{D}(u, z_0)_{-n} \leq \delta \quad \text{and} \quad \mathsf{D}(u, z_\eta)_n \leq \delta\}$$

and

$$\mathcal{U}_\eta = \{u \in \Sigma \mid \mathsf{D}(u, z_\eta)_{-n} \leq \delta \quad \text{and} \quad \mathsf{D}(u, z_0)_n \leq \delta\},$$

noting that $u_\eta \in \mathcal{U}_0$, while, setting $\bar{u}_\eta(x,y) = u_\eta(x,-y)$ for all $(x,y) \in \mathbf{R}^2$, $\bar{u}_\eta \in \mathcal{U}_\eta$.

Lemma 4.2 *If $u \in \mathcal{U}_\sigma$, $\sigma \in \{0, \eta\}$, satisfies*

$$\sum_{k=-2n}^{2n-1} \varphi_k(u) \leq m_\eta + \tilde{\Lambda},$$

then there exist two integers $k^- \in [-2n+1, -n-1]$ and $k^+ \in [n+1, 2n-1]$ such that

$$\mathsf{D}(u, z_0)_{k^-} \leq \varepsilon \quad \textit{and} \quad \mathsf{D}(u, z_\eta)_{k^+} \leq \varepsilon, \quad \textit{if } \sigma = 0$$

or

$$\mathsf{D}(u, z_\eta)_{k^-} \leq \varepsilon \quad \textit{and} \quad \mathsf{D}(u, z_0)_{k^+} \leq \varepsilon, \quad \textit{if } \sigma = \eta.$$

Proof. We consider the case $\sigma = 0$, the other case follows using (H_1)-(ii).
By contradiction, assume that there exists $u \in \mathcal{U}_0$ such that

$$\sum_{k=-2n}^{2n-1} \varphi_k(u) \leq m_\eta + \tilde{\Lambda},$$

and for all integer $k \in [-2n+1, -n-1]$

$$\mathsf{D}(u, z_0)_{k^-} > \varepsilon.$$

Then, by Lemma 2.6, we obtain $\varphi_k(u) > h_\varepsilon$ for all integer $k \in [-2n+1, -n-1]$ and therefore

$$m_\eta + \tilde{\Lambda} \geq \sum_{k=-2n}^{2n-1} \varphi_k(u) \geq \sum_{k=-2n+1}^{-n-1} \varphi_k(u) > (n-2) h_\varepsilon$$

which is a contradiction with the choice of n. In the same way one can argue to prove the existence of $k^+ \in [n+1, 2n-1]$ such that $\mathsf{D}(u, z_\eta)_{k^+} \leq \varepsilon$. $\square$

Then we obtain

Lemma 4.3 *If $u \in \mathcal{U}_\sigma$, $\sigma \in \{0, \eta\}$, satisfies $\sum_{k=-2n}^{2n-1} \varphi_k(u) \le m_\eta + \tilde{\Lambda}$, then*

$$\mathsf{D}(u, z_0)_{-n} < \delta \quad \textit{and} \quad \mathsf{D}(u, z_\eta)_n < \delta, \quad \textit{if } \sigma = 0$$

or

$$\mathsf{D}(u, z_\eta)_{-n} < \delta \quad \textit{and} \quad \mathsf{D}(u, z_0)_n < \delta \quad \textit{if } \sigma = \eta.$$

Proof. We consider the case $\sigma = 0$, the other case follows using (H_1)-(ii).
So, let $u \in \mathcal{U}_0$ with $\sum_{k=-2n}^{2n-1} \varphi_k(u) \le m_\eta + \tilde{\Lambda}$ and let $k^- \in [-2n+1, -n-1]$ and $k^+ \in [n+1, 2n-1]$ be given by Lemma 4.2. We set $\bar{u} = \chi^-_{0,k^-} \circ \chi^+_{\eta,k^+}(u)$. Then, $\bar{u} \in \mathcal{H}_\eta$ and, by Remark 3.2,

$$\Phi(\bar{u}) \le \sum_{k=-2n}^{2n-1} \varphi_k(u) + 2\tilde{\lambda}_\varepsilon \le m_\eta + 2\tilde{\Lambda} < m_\eta + \Lambda,$$

by the choice of ε and $\tilde{\Lambda}$. Then, setting $\tilde{u}(x, y) = \bar{u}(x, y + n)$, we obtain $\tilde{u} \in \mathcal{H}_\eta$ with $\Phi(\tilde{u}) = \Phi(\bar{u}) < m_\eta + \Lambda$ and $\mathsf{D}(\tilde{u}, z_0)_0 \le \delta$. Therefore, necessarily, by (4.1), $\mathsf{D}(\tilde{u}, z_0)_0 < \delta$ and so we conclude $\mathsf{D}(u, z_0)_{-n} < \delta$.
Analogously we can prove that $\mathsf{D}(u, z_\eta)_n < \delta$. $\square$

We now define the classes of functions in which we look for multibump solutions. Let us fix $N \in \mathbf{N}$, $p = (p_1, ..., p_N) \in \mathbf{Z}^N$ such that $p_i - p_{i-1} \ge 4n$ and $\sigma = (\sigma_1, ...\sigma_N) \in \{0, \eta\}^N$ with $\sigma_i \ne \sigma_{i-1}$ for all $i = 2, .., N$. We set

$$\mathcal{H}_{N,p,\sigma} = \{u \in \Sigma \,|\, u(x, y - p_i) \in \mathcal{U}_{\sigma_i}, \text{ for a.e. } (x, y) \in \mathbf{R}^2, \, i = 1, ..., N\}$$

and $m_{N,p,\sigma} = \inf_{\mathcal{H}_{N,p,\sigma}} \Phi$.

Remark 4.3 Given any $i \in \{1, ..., N\}$, we define

$$w_i(x, y) = \begin{cases} u_\eta(x, y - p_i) & \text{if } \sigma_i = 0, \\ u_\eta(x, p_i - y) & \text{if } \sigma_i = \eta. \end{cases}$$

Then, noting that Lemma 4.2 applies to w_i, we have that there exist two integers $k_i^- \in [p_i - 2n + 1, p_i - n - 1]$ and $k_i^+ \in [p_i + n + 1, p_i + 2n - 1]$ such that

$$\mathsf{D}(w_i, z_{\sigma_i})_{k_i^-} \le \varepsilon \quad \text{and} \quad \mathsf{D}(w_i, z_{\sigma_{i+1}})_{k_i^+} \le \varepsilon.$$

Hence, we set

$$\tilde{w}_i = \chi^-_{\sigma_i, k_i^-} \circ \chi^+_{\sigma_{i+1}, k_i^+}(w_i)$$

noting that, by Remark 3.2, $\sum_{k \in \mathbf{Z}} \varphi_k(\tilde{w}_i) \le m_\eta + 2\tilde{\lambda}_\epsilon$.

Lemma 4.4 *If $u \in \mathcal{H}_{N,p,\sigma}$ satisfies $\Phi(u) = m_{N,p,\sigma}$, then*

$$\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) \le m_\eta + \tilde{\Lambda}, \qquad \forall \, i = 1, \cdots, N. \tag{4.2}$$

Proof. We define $I = \{i \in \{1,\dots,N\} \mid \sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) > m_\eta + \tilde{\Lambda}\}$, $I^+ = \{i \in \{1,\dots,N\} \setminus I \mid i+1 \in I\}$ and $I^- = \{i \in \{1,\dots,N\} \setminus I \mid i-1 \in I\}$. Since Lemma 4.2 applies for every $i \notin I$, let $k_i^- \in [p_i - 2n+1, p_i - n - 1]$ and $k_i^+ \in [p_i + n + 1, p_i + 2n - 1]$ be the corresponding integers.
Let us consider the sets of consecutive integers $Z_1 = \{k \in \mathbf{Z} \,|\, k \le [\frac{p_1+p_2}{2}]\}$, $Z_i = \{k \in \mathbf{Z} \,|\, [\frac{p_{i-1}+p_i}{2}] < k \le [\frac{p_i+p_{i+1}}{2}]\}$, $i = 2,\dots,N-1$, $Z_N = \{k \in \mathbf{Z} \,|\, [\frac{p_{N-1}+p_N}{2}] < k\}$ where $[\cdot]$ denote the integer part and note that $k_i^\pm \in Z_i$.
For any $i \in \{1,\dots,N\}$, we replace u in $\cup_{k\in Z_i} S_i$ with the function

$$\tilde{u} = \begin{cases} \tilde{w}_i & \text{if } i \in I \\ \chi^-_{\sigma_i,k_i^-} \circ \chi^+_{\sigma_{i+1},k_i^+}(u) & \text{if } i \in I^- \cap I^+ \\ \chi_{\sigma_i,k_i^-}(u) & \text{if } i \in I^- \setminus I^+ \\ \chi_{\sigma_{i+1},k_i^+}(u) & \text{if } i \in I^+ \setminus I^- \\ u & \text{otherwise.} \end{cases}$$

Note that $\tilde{u} \in \mathcal{H}_{N,p,\sigma}$ and so $\Phi(\tilde{u}) \ge \Phi(u)$. Moreover $\tilde{u} = u$ if and only if $I = \emptyset$, i.e. if and only if (4.2) holds true.
Note now that if $i \in I$, then, by Remark 4.3,

$$\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(\tilde{u}) = \sum_{k\in\mathbf{Z}} \varphi_k(\tilde{w}_i) \le m_\eta + 2\tilde{\lambda}_\varepsilon \le \sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) + 2\tilde{\lambda}_\varepsilon - \tilde{\Lambda},$$

while, if $i \in I^- \cup I^+$,

$$\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(\tilde{u}) \le \sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) + 2\tilde{\lambda}_\varepsilon.$$

Finally, letting $Z(I) = \{k \in \mathbf{Z} \mid k \notin \cup_{i\in I\cup I^-\cup I^+}[p_i - 2n, p_i + 2n - 1]\}$, we have that for any $k \in Z(I)$, $\varphi_k(\tilde{u}) \le \varphi_k(u)$. Then,

$$\begin{aligned}
\Phi(\tilde{u}) &= \sum_{k\in Z(I)} \varphi_k(\tilde{u}) + \sum_{i\in I}\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(\tilde{u}) + \sum_{i\in I^+\cup I^-}\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(\tilde{u}) \\
&\le \sum_{k\in Z(I)} \varphi_k(u) + \sum_{i\in I}\Big(\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) + 2\tilde{\lambda}_\varepsilon - \tilde{\Lambda}\Big) + \sum_{i\in I^+\cup I^-}\Big(\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) + 2\tilde{\lambda}_\varepsilon\Big) \\
&= \sum_{k\in Z(I)} \varphi_k(u) + \sum_{i\in I}\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) + (2\tilde{\lambda}_\varepsilon - \tilde{\Lambda})Card(I) \\
&\quad + \sum_{i\in I^+\cup I^-}\sum_{k=p_i-2n}^{p_i+2n-1} \varphi_k(u) + 2\tilde{\lambda}_\varepsilon Card(I^+\cup I^-) \\
&= \Phi(u) + (2\tilde{\lambda}_\varepsilon - \tilde{\Lambda})Card(I) + 2\tilde{\lambda}_\varepsilon Card(I^+\cup I^-).
\end{aligned}$$

Now since we clearly have $Card(I^+\cup I^-) \le 2Card(I)$, we deduce that I must be empty, for otherwise (taking into account the choice of ε),

$$\Phi(\tilde{u}) \le \Phi(u) + Card(I)(6\tilde{\lambda}_\varepsilon - \tilde{\Lambda}) < \Phi(u),$$

which is impossible because u is a global minimizer. This completes the proof. $\square$

Using the previous notation, we are now able to prove the existence of multibump type solutions of (P).

Theorem 4.1 *Assume that* $(*)$ *and* $(**)$ *hold. Then, for every* $N \in \mathbf{N}$, $p = (p_1, ..., p_N) \in \mathbf{Z}^N$ *and* $\sigma = (\sigma_1, ...\sigma_N) \in \{0, \eta\}^N$ *such that* $p_i - p_{i-1} \geq 4n$ *and* $\sigma_i \neq \sigma_{i-1}$ *for all* $i = 2, .., N$, *there exists* $u \in \mathcal{H}_{N,p,\sigma}$ *such that* $\Phi(u) = m_{N,p,\sigma}$. *Moreover,* $u \in \mathcal{C}^2(\mathbf{R}^2)$ *and is a classical solution of (P) with* $\|u\|_{L^\infty(\mathbf{R}^2)} \leq R_0$.

Proof. Let $(u_n) \subset \mathcal{H}_{N,p,\sigma}$ be such that $\Phi(u_n) \to m_{N,p,\sigma}$. By Lemma 3.5 there exist a subsequence (still denoted u_n) and $u \in \Sigma$ such that $u_n \to u$ weakly in $H^1_{loc}(\mathbf{R}^2)$ and, by Lemma 3.1, $\Phi(u) \leq m_{N,p,\sigma}$. Finally, because of the weakly lower semicontinuity of the H^1-norm, we obtain that $u \in \mathcal{H}_{N,p,\sigma}$ and therefore $\Phi(u) = m_{N,p,\sigma}$.
By Lemmas 4.4 and 4.3, we obtain

$$\mathsf{D}(u, z_{\sigma_i})_{p_i - n} < \delta \quad \text{and} \quad \mathsf{D}(u, z_{\sigma_{i+1}})_{p_i + n} < \delta$$

for all $i = 1, ..., N$, and therefore, using standard regularity arguments, we can conclude that u belongs to $\mathcal{C}^2(\mathbf{R}^2)$ and that it is a classical solution of (E). Moreover, since u is a minimizer we can prove, as in Proposition 2.1, that $\|u\|_{L^\infty(\mathbf{R}^2)} \leq R_0$ and therefore that there exists $C > 0$ such that $\|u\|_{\mathcal{C}^2(\mathbf{R}^2)} \leq C$.
To prove that u satisfies the right boundary conditions, note first that since $\mathsf{D}(u, z_{\sigma_1})_{p_1 - n} < \delta < \tilde{\delta}$ and u is a minimizer, we obtain $\mathsf{D}(u, z_{\sigma_1})_j \leq \tilde{\rho}$ for all $j \leq p_1 - n$. Indeed, by Lemma 4.2, let $k^- \in [-2n + 1 + p_1, -n - 1 + p_1]$ and $k^+ \in [p_1 + n + 1, p_1 + 2n - 1]$ be such that

$$\mathsf{D}(u, z_{\sigma_1})_{k^-} \leq \varepsilon \quad \text{and} \quad \mathsf{D}(u, z_{\sigma_2})_{k^+} \leq \varepsilon.$$

Then, setting $u^+ = \chi^+_{\sigma_2, k^+}(u)$, to prove the claim, by Lemma 3.8, it is sufficient to prove that $\Phi(u^+) \leq m_\eta + \tilde{\lambda}$. Indeed, assume by contradiction that $\Phi(u^+) > m_\eta + \tilde{\lambda}$. Then, by Remark 3.2, we have

$$\sum_{j=-\infty}^{k^+} \varphi_j(u) \geq m_\eta + \tilde{\lambda} - \tilde{\lambda}_\varepsilon. \tag{4.3}$$

Hence, if we set $u^- = \chi^-_{\sigma_1, k^-}(u)$, we obtain $u^- \in \mathcal{H}_{N,p,\sigma}$ and by Remark 3.2

$$\Phi(u^-) \leq \Phi(u) - \sum_{j=-\infty}^{k^-} \varphi_j(u) + \tilde{\lambda}_\varepsilon. \tag{4.4}$$

Since by Lemma 4.4, $\sum_{j=p_1-2n}^{p_1+2n-1} \varphi_j(u) \leq m_\eta + \tilde{\Lambda}$, using (4.3), we get

$$\sum_{j=-\infty}^{k^-} \varphi_j(u) = \sum_{j=-\infty}^{k^+} \varphi_j(u) - \sum_{j=k^-+1}^{k^+} \varphi_j(u) \geq m_\eta + \tilde{\lambda} - \tilde{\lambda}_\varepsilon - (m_\eta + \tilde{\Lambda}) = \tilde{\lambda} - \tilde{\lambda}_\varepsilon - \tilde{\Lambda}$$

and therefore, by (4.4) and the choice of ε and $\tilde{\Lambda}$, we obtain the contradiction

$$\Phi(u^-) \leq \Phi(u) + 2\tilde{\lambda}_\varepsilon - \tilde{\lambda} + \tilde{\Lambda} < \Phi(u).$$

Hence, $\mathsf{D}(u, z_{\sigma_1})_j \leq \tilde{\rho}$ for all $j \leq p_1 - n$, and therefore, by Lemma 3.6,

$$\mathsf{D}(u, z_{\sigma_1})_j \to 0 \quad \text{as} \quad j \to -\infty.$$

Similarly, we can prove that

$$\mathsf{D}(u, z_{\sigma_N})_j \to 0 \quad \text{as} \quad j \to +\infty.$$

Then, arguing as in [5], we finally obtain that u satisfies the asymptotic conditions in the x variable uniformly with respect to $y \in \mathbf{R}$ and so is a solution of (P). $\square$

Remark 4.4 As one can see from the proof of the previous theorem, there exists a constant $C > 0$ such that for every $u \in \mathcal{H}_{N,p,\sigma}$ with $\Phi(u) = m_{N,p,\sigma}$ there result $\|u\|_{\mathcal{C}^2(\mathbf{R}^2)} \le C$ for all $N \in \mathbf{N}$, $p = (p_1, ..., p_N) \in \mathbf{Z}^N$ and $\sigma = (\sigma_1, ...\sigma_N) \in \{0, \eta\}^N$. Then, using the Ascoli-Arzelá Theorem, we can prove the existence of solutions with infinitely many bumps. That is, for every sequences $(p_i) \subset \mathbf{Z}$ and $(\sigma_i) \subset \{0, \eta\}^{\mathbf{Z}}$ with $p_i - p_{i-1} \ge 4n$ and $\sigma_i \ne \sigma_{i-1}$ for all $i \in \mathbf{Z}$, there exists a solution $u \in \mathcal{C}^2(\mathbf{R}^2)$ of (P) such that $u(x, y - p_i) \in \mathcal{U}_{\sigma_i}$ for all $(x, y) \in \mathbf{R}^2$ and $i \in \mathbf{Z}$.

References

[1] S. Alama, L. Bronsard and C. Gui, *Stationary layered solutions in $\mathbf{R}^2$ for an Allen-Cahn system with multiple well potential*, Calc. Var., **5** (1997), 359–390.

[2] G. Alberti, L. Ambrosio and X. Cabré *On a long-standing conjecture of E. De Giorgi: old and recent results*, Preprint, Pisa.

[3] F. Alessio, M.L. Bertotti and P. Montecchiari, *Multibump solutions to possibly degenerate equilibria for almost periodic Lagrangian systems*, Z. Angew. Math. Phys. **50**, No.6 (1999) 860-891

[4] F. Alessio, M. Calanchi and E. Serra,*Complex dynamics in a class of reversible equations*, Nonlinear analysis and its applications to differential equations (Lisbon, 1998), 147–159, Progr. Nonlinear Differential Equations Appl., **43**, Birkhîuser Boston, Boston, MA, (2001).

[5] F. Alessio, L. Jeanjean and P. Montecchiari, *Stationary layered solutions in $\mathbf{R}^2$ for a class of non autonomous Allen-Cahn equations*, Calc. of Var. and PDE's, **11** (2000), 177–202.

[6] S. M. Allen and J. W. Cahn, *A macroscopic theory for antiphase boundary motions and its application to antiphase domain coarsening*, Acta Metal., **27** (1979), 1085–1095.

[7] L. Ambrosio and X. Cabre, *Entire solutions of semilinear elliptic equations in $\mathbf{R}^3$ and a conjecture of De Giorgi*, J. Am. Math. Soc. **13**, No. 4, (2000) 725-739.

[8] M.T. Barlow, R.F. Bass and C. Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. **53**, No.8, (2000) 1007-1038.

[9] H. Berestycki, F. Hamel and R. Monneau, *One-dimensional symmetry for some bounded entire solutions of some elliptic equations*, Duke Math. J. **103**, No.3, (2000) 375-396.

[10] E. De Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis, Rome, (1978), E. De Giorgi et al. (eds).

[11] A. Farina, *Symmetry for solutions of semilinear elliptic equations in $\mathbf{R}^N$ and related conjectures*, Ricerche di Matematica (in memory of Ennio De Giorgi) **48** (1999) 129-154.

[12] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann., **311**, (1998), 481-491.

[13] Nirenberg, L. : *Lectures on Linear Partial Differential Equations*, CBMS Reg. Conf. Ser. in Math., AMS, **17** (1973).

[14] P.H. Rabinowitz, *Heteroclinic for reversible Hamiltonian system*, Ergod. Th. and Dyn. Sys., **14** (1994), 817–829.

[15] P.H. Rabinowitz, *Solutions of heteroclinic type for some classes of semilinear elliptic partial differential equations*, J. Math. Sci. Univ. Tokio, **1** (1994), 525–550.

[16] P. H. Rabinowitz, *Connecting orbits for a reversible Hamiltonian system.* Ergodic Theory Dynam. Systems **20** no. 6 (2000) 1767–1784.